JOININGS OF HIGHER RANK DIAGONALIZABLE ACTIONS ON LOCALLY HOMOGENEOUS SPACES

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ABSTRACT. We classify joinings between a fairly general class of higher rank diagonalizable actions on locally homogeneous spaces. In particular, we classify joinings of the action of a maximal $\mathbb{R}$-split torus on G/Γ, with G a simple Lie group of $\mathbb{R}$-rank ≥ 2 and $\Gamma < G$ a lattice. We deduce from this a classification of measurable factors of such actions, as well as certain equidistribution properties.

1. INTRODUCTION

In a landmark paper [9] Furstenberg introduced two fundamental ideas: 1) joinings, and how they can be used to investigate dynamical systems both in the topological and the measurable category 2) for natural multidimensional algebraic actions (Furstenberg studies the example of multiplication by multiplicatively independent integers p and by q on the one torus), it is often the case that while the action of every individual element is quite flexible, the full action may display remarkable rigidity properties. Both of these ideas have had a major impact on modern ergodic theory. Joinings and the related notion of disjointness have become an indispensable tool to ergodic theorists (to illustrate this point, one may look for example at [10]). The multidimensional algebraic action whose study Furstenberg pioneered still contain many mysteries and are far from being understood, but substantial progress has been made and this study, particularly in the context of locally homogeneous spaces such as $\mathrm{SL}(n, \mathbb{R})/\mathrm{SL}(n, \mathbb{Z})$, has found applications in number theory and the arithmetic theory of quantum chaos (which is closely connected to the theory automorphic forms); we refer the reader to the survey [21] or to [20, 6] for more details. It is interesting to note

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that the first such application was provided already by Furstenberg in his original paper [9].

In this paper we consider the following type of algebraic dynamical systems: we take G to be a connected Lie group, $\Gamma < G$ a lattice, i.e. a discrete group of finite covolume, and let $\alpha : \mathbb{R}^n \to G$ be an injective homomorphism to an $\mathbb{R}$-split torus of G (more precisely what we need is that the image of α composed with the adjoint representation of G is diagonalizable over $\mathbb{R}$). Then α gives us an action of $\mathbb{R}^n$ on G/Γ by left translations, and in this paper we will always assume that $n \geq 2$.

Such systems are natural generalizations of the system considered by Furstenberg of multiplication by p and by q on the one torus. Here again the properties of the action of any individual element is very different from that of the full group. For example, there are many invariant measures for the action of any individual element or even every one dimensional (connected) subgroup of $\mathbb{R}^n$. For the full action, however, Furstenberg, Katok and Spatzier [14] and Margulis [24] conjectured that α-invariant probability measures are rare: they are all algebraic unless there are factors for which the actions degenerates to a rank one situation, where we call a probability measure algebraic if it is an L-invariant measure on a single (necessarily closed) orbit of a closed subgroup $L \leq G$.

Progress on that conjecture has been made so far only under additional assumptions [14, 15, 5, 12]. In some cases [29, 7, 20, 6] the only remaining additional assumption is that the measure has positive entropy under some element of the action.

We combine the two themes in Furstenberg's original paper by studying joinings of two such systems. For $i = 1, 2$ let α_i be an actions of $\mathbb{R}^n$ as above on $X_i = G_i/\Gamma$ and let m_i be the normalized Haar measure on X_i. Define $\alpha^{\mathbf{t}} = \alpha_1^{\mathbf{t}} \times \alpha_2^{\mathbf{t}}$ for $\mathbf{t} \in \mathbb{R}^n$. A **joining** between α_1 and α_2 (with respect to m_1 and m_2) is an α-invariant measure μ on $X = X_1 \times X_2$ such that $(\pi_i)_*\mu = m_i$ for $i = 1, 2$.

Before we give our theorem we give some examples of joinings. The **trivial joining** $m_1 \times m_2$ always exists. A more general class of joinings are algebraic joinings: Suppose $H \subset G_1 \times G_2$ contains $\alpha(\mathbb{R}^n)$ and projects surjectively to G_i for $i = 1, 2$. If $H(x_1, x_2)$ is closed with finite volume for some $(x_1, x_2) \in X_1 \times X_2$, then its (normalized) volume measure μ gives an **algebraic joining** between α_1 and α_2.

The actions α_1 and α_2 are **disjoint** if the trivial joining is the only joining between them.

Note that if $n = 1$, the time one maps α_i^1 have positive entropy (and are even Bernoulli automorphisms [2]). Therefore, in this case α_1^1 and α_2^1 have for any small enough s common Bernoulli factors of

entropy s by Sinai's factor theorem (see [30] or [10, Ch. 20]). The well known construction of the relatively independent joining over a factor (see (6.4)) gives now uncountably many non-algebraic joinings. This explains the requirement of injectivity and $n \geq 2$.

In this paper we give an unconditional classification of joinings for such $\mathbb{R}^n$-actions in many cases. In particular, we prove the following:

Theorem 1.1. *For $i = 1, 2$ let G_i be a connected simple Lie group, let $\Gamma_i \subset G_i$ be a lattice, let m_i be the normalized Haar measure on $X_i = G_i/\Gamma_i$, and let $\alpha_i : \mathbb{R}^n \to G_i$ be an injective homomorphism to an $\mathbb{R}$-split torus of G_i. Suppose $n \geq 2$ and that μ is an ergodic joining between α_1 and α_2, then μ is algebraic. Moreover, if μ is not the trivial joining, then G_1 and G_2 are locally isomorphic and the lattices induced by Γ_1 and Γ_2 in the common universal cover G of G_1 and G_2 are commensurable up to an automorphism of G intertwining the actions corresponding to α_1 and α_2.*

Recall that two lattices Γ_1 and Γ_2 in a Lie group G are **commensurable** if $\Gamma_1 \cap \Gamma_2$ has finite index in Γ_1 and Γ_2 (or equivalently if $\Gamma_1 \cap \Gamma_2$ is still a lattice).

With respect to the assumption of ergodicity in Theorem 1.1 note that any joining μ can be decomposed into ergodic components with respect to α (see [32] or [11, Thm. 4.2.6]), and that the projection of this decomposition to X_i gives a decomposition of m_i. However, since the latter is an ergodic measure for the action α_i by Moore's ergodic criterion (see [25] or Section 2.2 for more details) it follows that a.e. ergodic component of μ is an ergodic joining. Therefore, the characterization of ergodic joinings in Theorem 1.1 gives a complete description of all possible joinings between these actions.

We remark that for the dynamical system considered by Furstenberg of multiplication by p and q or more generally actions of non virtually-cyclic groups of commuting toral endomorphisms a full classifications of joinings has not been obtained, the available results describe ergodic joinings only up to zero entropy [13, 12, 7]. In fact, for these actions the classifications of joinings is equivalent to the classifications of all invariant measures, which is a well known open problem.

From Theorem 1.1 we easily obtain using a standard argument (originally due to Furstenberg) the following consequence about equidistribution of orbits. Let α be an $\mathbb{R}^n$-action on a locally compact metric space X, let μ be an α-ergodic and invariant probability measure on X. We say that a point $x \in X$ has μ-**equidistributed** α-orbit if

$$\lim_{\ell \to \infty} \frac{1}{(2\ell)^n} \int_{[-\ell, \ell]^n} f(\alpha^{\mathbf{t}} x) \, \mathrm{d}\mathbf{t} = \int f \, \mathrm{d}\mu$$

for all $f \in C(X)$ with compact support.[1] Note that for any ergodic μ, one has that μ-a.e. point has μ-equidistributed α-orbit by the pointwise ergodic theorem.

Corollary 1.2. *Let G be a simple connected Lie group and let $\alpha : \mathbb{R}^n \to G$ be an injective homomorphism to an $\mathbb{R}$-split torus of G. For $i = 1, 2$ let $\Gamma_i \subset G$ be a lattice and let m_i be the normalized Haar measure on $X_i = G/\Gamma_i$. Suppose $n \geq 2$ and that there is no automorphism of G commuting with left translations by elements in the image of α taking Γ_2 to a lattice commensurable with Γ_1. If $x_i \in X_i$ has m_i-equidistributed α-orbit for $i = 1, 2$, then $(x_1, x_2) \in X_1 \times X_2$ has $m_1 \times m_2$-equidistributed α-orbit, where α acts on $X_1 \times X_2$ by left multiplication with $\alpha^{\mathbf{t}} \times \alpha^{\mathbf{t}}$ for $\mathbf{t} \in \mathbb{R}^n$.*

Clearly, a similar statement holds if $X_i = G_i/\Gamma_i$ for two locally non-isomorphic simple connected Lie groups G_1 and G_2.

We now describe a more general setup that allows us to show rigidity of joinings. We will not assume that G is simple, or even semisimple. Note however, that if for instance

$$X = \mathrm{SL}(2, \mathbb{R}) \times \mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z}) \times \mathrm{SL}(2, \mathbb{Z})$$

then the projection to the first coordinate is well defined and the $\mathbb{R}^2$-action on X descends to $Y = \mathrm{SL}(2, \mathbb{R})/\mathrm{SL}(2, \mathbb{Z})$. However, in this factor the subgroup $\{0\} \times \mathbb{R}$ acts trivially – Y is a (global) rank one factor. Similar to the case of $n = 1$ the action α on X allows many non-algebraic joinings. We avoid this situation with the following definition.

Definition 1.3. Let G be a Lie group and let $\alpha : \mathbb{R}^n \to G$ be a homomorphism. We say α is $\mathbb{R}$-**diagonalizable** if the associated adjoint action of α on the Lie algebra $\mathfrak{g}$ is diagonalizable over $\mathbb{R}$. If $\mathfrak{h} \subset \mathfrak{g}$ is a Lie ideal, then consider the induced action of α via the adjoint representation on $\mathfrak{g}/\mathfrak{h}$. We say that α **has no local rank one factors** if for any proper Lie ideal $\mathfrak{h}$ the weights of the adjoint action of α on $\mathfrak{g}/\mathfrak{h}$ do not lie on a one-dimensional subspace.

In particular, if α is an $\mathbb{R}^n$-action with no local rank one factors then $n > 1$. Conversely if G is a simple Lie group, $n > 1$, α is injective and $\mathbb{R}$-diagonalizable, then α has no local rank one factors. To see this note first that the only proper Lie ideal is $\mathfrak{h} = \{0\}$. However, for $\mathfrak{g}$ the Lyapunov weights of α cannot lie in a one-dimensional subspace, otherwise there exists some $\mathbf{t}$ with $\lambda(\mathbf{t}) = 0$ for all Lyapunov weights and

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[1]One can replace the cube $[-\ell, \ell]^n$ with other averaging sets, e.g. balls around 0 or $[0, \ell]^n$. The latter gives a weaker notion of equidistribution, but the above remains true also in this case.

$\alpha^{\mathbf{t}}$ would belong to the center of G. Therefore the following generalizes Theorem 1.1

Theorem 1.4. *For $i = 1, 2$ let G_i be a connected Lie group, let Γ_i be a lattice in G_i, and let m_i be the Haar measure on $X_i = G_i/\Gamma_i$. Furthermore, let $\alpha_i : \mathbb{R}^n \to G_i$ be injective homomorphism that are $\mathbb{R}$-diagonalizable and have no local rank one factors. Let μ be an ergodic joining between α_1 and α_2. Then there is a closed subgroup $H \leq G_1 \times G_2$ projecting on to G_1 and G_2 such that μ is the Haar measure on a closed H-orbit in $X_1 \times X_2$.*

It is a general fact due to Dave Witte Morris [34, 35] that an algebraic classification of self joinings of the $\mathbb{R}^n$-action given by α on $(G/\Gamma, m)$ can be used to classify measurable factors of this action, or equivalently countably generated α invariant σ-algebras of Borel subsets of G/Γ. A special case of this general principle was given earlier by Marina Ratner in [27]. Using this we can deduce the following from Theorem 1.4:

Theorem 1.5. *Let G be a connected Lie group, Γ a lattice in G, and m the Haar measure on $X = G/\Gamma$. Let $\alpha : \mathbb{R}^n \to G$ be an $\mathbb{R}$-diagonalizable injective homomorphism with no local rank one factors. Let $\mathcal{A}$ be an α-invariant countably generated σ-algebra of Borel subsets of G/Γ. Then there is a surjective homomorphism of Lie groups $\psi : G \to G_1$, a lattice $\Gamma_1 \geq \psi(\Gamma)$ (in particular, $\psi(\Gamma)$ is discrete), and a compact subgroup Φ of the group of affine automorphisms $\mathrm{Aff}(G_1/\Gamma_1)$ of G_1 modulo right translation by elements of Γ_1 so that $\mathcal{A}$ is equivalent mod m to $\psi^{-1}(\mathcal{B}_\Phi)$, with $\mathcal{B}_\Phi$ denoting the σ-algebra of Φ-invariant subsets of G_1/Γ_1.*

(Note that since Φ is compact, $\mathcal{B}_\Phi$ and hence $\psi^{-1}(\mathcal{B}_\Phi)$ is automatically countably generated, see Corollary 6.7.)

In particular, as we will explicate in §6, Theorem 1.5 implies the following:

Corollary 1.6. *Let G be a connected semisimple Lie group with finite center and each factor of G of $\mathbb{R}$-rank ≥ 2, let $\Gamma < G$ be an irreducible lattice, and let α be an embedding of $\mathbb{R}^n$ onto the connected component of the identity of a maximal $\mathbb{R}$-split torus[2]. Let M denote a maximal compact subgroup of $C_G(\alpha(\mathbb{R}^n))$. Consider the action from the left given by α on the double quotient $X = M \backslash G/\Gamma$; this action preserves a smooth probability measure m. Let $\mathcal{A}$ be a nontrivial[3] α-invariant*

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[2]In other words the image of α is precisely the closed connected subgroup A appearing in the Iwasawa decomposition KAN of G, equivalently the Lie algebra of A is the negative eigenspace of a Cartan involution θ of a θ-stable maximally non-compact Cartan subalgebra.

[3]i.e. not equivalent modulo m to $\{\emptyset, X\}$

countably generated σ-algebra of Borel subsets of X. Then for m-a.e. $x \in X$ the atom $[x]_{\mathcal{A}}$ is finite. Moreover, if Γ is maximal, and every automorphism of G fixing $\alpha(\mathbb{R}^n)$ pointwise is an inner automorphism, then $|[x]_{\mathcal{A}}| = 1$ a.s., in other words $\mathcal{A}$ is equivalent mod m to the full Borel σ-algebra.

For example, any automorphism of $\mathrm{PGL}(n, \mathbb{R})$ fixing the positive diagonal matrices (which are the identity component of a maximal $\mathbb{R}$-split torus) is an inner automorphism.

Previous results on joinings of semisimple actions were obtained by Kalinin and Spatzier [12], but their method of proof required an additional assumption regarding ergodicity of certain one parameter subgroups on the joining μ. Here we will use as the main tool the high entropy argument as provided in a recent work by A. Katok and the first named author [4]. Our argument also uses elements of [8], where disjointness results were obtained for actions on zero-dimensional compact abelian groups. These tools will lead us to a situation where we will be able to apply Ratner's theorem [28] for a subgroup of G generated by unipotent elements.

More generally, we expect that similar results can be proved for e.g. semisimple Lie groups even if G_i have local rank one factors, as long as the action of α_i on G_i/Γ_i have no global rank one factors. This, however, will require a totally different proof, and in particular will require a general version of the low entropy argument used in [20, 6], a general version which we are currently in the process of writing.

2. Definitions, ergodicity, and entropy

In this section we summarize some background material needed for the proof of Theorem 1.4.

2.1. **Lyapunov weights.** We recall some definitions from [4]. Let G be a connected Lie group with Lie algebra $\mathfrak{g}$ and let $\Gamma \subset G$ be a lattice. Recall that for $a \in G$ the derivative of the map $x \mapsto axa^{-1}$ defines the adjoint automorphism Ad_a of $\mathfrak{g}$. Let α be an $\mathbb{R}^n$-action on G/Γ. Then α induces an adjoint action on $\mathfrak{g}$ of G, and α is Ad-semisimple if the adjoint action is diagonalizable over $\mathbb{C}$, and $\mathbb{R}$-diagonalizable if the same holds over $\mathbb{R}$. The logarithm of the absolute value of the eigenvalues of this action can be collected to form the **Lyapunov weights**, which are linear functionals $\lambda : \mathbb{R}^n \to \mathbb{R}$. For every Lyapunov weight λ there corresponds a sum of eigenspaces $\mathfrak{u}^\lambda$ which is called the **Lyapunov space** corresponding to λ.

Note that for an $\mathbb{R}$-diagonalizable action the Lyapunov weights are the weights of the adjoint representation restricted to α and the Lyapunov weight spaces are in fact the eigenspaces for the adjoint action. Moreover, if for some $\mathbf{t} \in \mathbb{R}^n$ all weights $\lambda(\mathbf{t})$ are zero, then $\alpha^{\mathbf{t}}$ is in the center of the group.

We define an equivalence relation between the Lyapunov weights, $\lambda \sim \eta$ if there is an $c > 0$ so that $\lambda = c\eta$. For every nonzero equivalence class Λ we define the **coarse Lyapunov space $\mathfrak{u}^\Lambda$** by

$$\mathfrak{u}^\Lambda = \bigoplus_{\lambda \in \Lambda} \mathfrak{u}^\lambda.$$

An element $u \in G$ is Ad-**unipotent** if Ad_u is unipotent as an automorphism of $\mathfrak{g}$, and a subgroup $U \leq G$ is Ad-**unipotent** if all of its elements are. The coarse Lyapunov spaces $\mathfrak{u}^\Lambda$ are in fact nilpotent Lie subalgebras, and the **coarse Lyapunov subgroups $U^\Lambda = \exp \mathfrak{u}^\Lambda$** are connected, Ad-unipotent, and normalized by α.

2.2. **The Mautner phenomenon and ergodicity.** The Mautner phenomenon (see [25], [31, Thm. 2.1], or [16, Thm. 2.1.4]) implies the following proposition.

Proposition 2.1. *Let G be a connected Lie group, let $\Gamma \subset G$ be a lattice, and let α be an $\mathbb{R}^n$-action without local rank one factors on the locally homogeneous space $X = G/\Gamma$. Then there exists a two dimensional subspace in $\mathbb{R}^n$ such that the restriction $\tilde{\alpha}$ of α to that plane still has no local rank one factors, and any nonzero element $\mathbf{t} \in \mathbb{R}^2$ satisfies that $a = \tilde{\alpha}^{\mathbf{t}} \in G$ acts ergodically with respect to the Haar measure m of X.*

Proof. Let $\lambda_1, \ldots, \lambda_\ell$ be all Lyapunov weights for α. For any two dimensional subspace the Lyapunov weights $\tilde{\lambda}_1, \ldots, \tilde{\lambda}_\ell$ for the restriction $\tilde{\alpha}$ are obtained by restricting $\lambda_1, \ldots, \lambda_\ell$ to the subspace. In general these restrictions could be (positively or negatively) proportional $\mathbb{R}\tilde{\lambda}_i = \mathbb{R}\tilde{\lambda}_j$, or some of them might be trivial $\tilde{\lambda}_k = 0$, even though this was not true for the original Lyapunov weights. However, it is easy to choose the plane so that this never happens. We claim that with this choice $\tilde{\alpha}$ has also no local rank one factors. Suppose this is not true, then there exists a proper Lie ideal $\mathfrak{h}$ in the Lie algebra $\mathfrak{g}$ of G such that all weights of $\tilde{\alpha}$ on $\mathfrak{g}/\mathfrak{h}$ lie on a single line. Suppose these weights are $\lambda'_1, \ldots, \lambda'_k$ for some $k \leq \ell$. (For the remaining weights the weight spaces are contained in $\mathfrak{h}$.) Then the weights of α on $\mathfrak{g}/\mathfrak{h}$ are $\lambda_1, \ldots, \lambda_k$, which have to lie on a single line by construction of $\tilde{\alpha}$. This contradicts the assumption that α has no local rank factors.

Suppose now α is an $\mathbb{R}^2$-action without local rank one factors. The Moore normal subgroup $M \subset G$ with respect to $a = \alpha^{\mathbf{t}}$ is defined to be the smallest normal subgroup such that Ad_a on $\mathfrak{g}/\mathfrak{m}$ is semisimple and all of its eigenvalues are of absolute value one. Here $\mathfrak{m}$ is the Lie algebra of M. It follows that $\mathfrak{m} = \mathfrak{g}$. For otherwise, $\lambda(\mathbf{t}) = 0$ for all weights λ of the adjoint action of α on $\mathfrak{g}/\mathfrak{m}$. However, then $\mathfrak{g}/\mathfrak{m}$ would be a local rank one factor of the $\mathbb{R}^2$-action α.

Therefore, the Moore normal subgroup $M = G$. Mautner's phenomenon [16, Thm. 2.1.4] now states that every a-fixed vector in a unitary representation of G must be fixed under $M = G$. In particular, every a-fixed function of $L^2(X, m)$ must be constant function and a acts ergodically with respect to the Haar measure m. $\square$

2.3. **Entropy and a generating partition.** In this section we fix some $a \in G$ and consider left multiplication by a on $X = G/\Gamma$. For a countable partition $\mathcal{P} \subset \mathcal{B}_X$ we let

$$\mathcal{P}^{[k,\ell)} = \bigvee_{i=k}^{\ell-1} a^{-i}\mathcal{P}.$$

Let $U < G$ be a subgroup. We recall that a countably generated sigma algebra $\mathcal{A}$ is said to be **subordinate** to the foliation into U-orbits (subordinate to U) with respect to μ if for a.e. x there exists a relatively compact neighborhood $V_x \subset U$ of the identity element $e \in U$ such that

$$[x]_{\mathcal{A}} = V_x x. \tag{2.1}$$

Here $[x]_{\mathcal{A}}$ is the **atom** of x, i.e. the intersection of all elements A of $\mathcal{A}$ or a countable generating algebra of $\mathcal{A}$ that contain x.

Any countably generated sigma algebra $\mathcal{A}$ gives us a system of conditional measures $\mu_x^{\mathcal{A}}$ with each such measure supported on the respective atom $[x]_{\mathcal{A}}$ and the map $x \mapsto \mu_x^{\mathcal{A}}$ is $\mathcal{A}$ measurable. For any two countably generated sigma algebras $\mathcal{A}, \mathcal{A}'$ so that every atom of $\mathcal{A}$ intersects at most countably many atoms of $\mathcal{A}'$ we set

$$\begin{aligned}
I_\mu(\mathcal{A}' \mid \mathcal{A})(x) &= -\log \mu_x^{\mathcal{A}}([x]_{\mathcal{A}'}) \text{ and} \\
H_\mu(\mathcal{A}' \mid \mathcal{A}) &= \int I_\mu(\mathcal{A}' \mid \mathcal{A})(x)d\mu(x).
\end{aligned}$$

We now quote the following from [23, Prop. 9.2] (and some additional facts appearing in the proof of that proposition). These results are an adaptation of results of Ledrappier and Young [18, 19] for C^2-diffeomorphisms on compact manifolds.

Proposition 2.2. *Let G be a connected Lie group, and let $\Gamma \subset G$ be a lattice in G. Let $a \in G$ and consider the left multiplication by a on $X = G/\Gamma$. Let*

$$U^- = \left\{ g \in G : \lim_{n\to\infty} a^n g a^{-n} = e \right\}$$

*be the **stable horospherical subgroup** defined by a. Let μ be an a-invariant and ergodic probability measure on X. Then there exists a countable partition $\mathcal{P}$ of X that satisfies the following properties:*

(1) *$\mathcal{P}$ has finite entropy $\mathrm{H}_\mu(\mathcal{P}) < \infty$.*
(2) *$\mathcal{A} = \mathcal{P}^{[0,\infty)}$ is subordinate to U^-.*
(3) *$\mathcal{P}$ is a generating partition with respect to a, i.e. $\mathcal{P}^{(-\infty,\infty)} = \mathcal{B}_X$ modulo μ-null sets. In particular,*

$$\mathrm{h}_\mu(a) = \mathrm{H}_\mu\big(a\mathcal{P}|\mathcal{A}\big).$$

Corollary 2.3. *With the same notation as in Proposition 2.2, let $U \le U^-$ be normalized by a. Then there exists a σ-algebra $\mathcal{A}_U \supset \mathcal{A}$ that satisfies*

(4) *$\mathcal{A}_U$ is subordinate to U.*
(5) *$a\mathcal{A}_U = a\mathcal{P} \vee \mathcal{A}_U$ modulo μ-null sets.*
(6) *The **entropy contribution of U***

$$h_\mu(a, U) = \mathrm{H}(a\mathcal{P}|\mathcal{A}_U) \tag{2.2}$$

is at most the entropy $\mathrm{h}_\mu(a)$.

Note that by (3), $h_\mu(a) = h_\mu(a, U^-)$.

Before we start the proof, note that the map $u \mapsto ux$ for $u \in U^-$ is injective for a.e. x. Since Γ is discrete, this map is clearly locally injective in the sense that for any compact subset $K \subset X$ there exists $\epsilon > 0$ such that $u \mapsto ux$ is injective for $u \in B_\epsilon(e)$ and any fixed $x \in K$. Suppose now this map is not injective for some x, then there exists a non-trivial $u \in U^-$ with $x = ux$. Then $a^n x = (a^n u a^{-n})a^n x$, and since $a^n u a^{-n} \to e$ it follows that $a^n x \to \infty$ for $n \to \infty$. These points form a null set by Poincaré recurrence.

Proof of Corollary 2.3. Given a point $x_0 \in X$ and a sufficiently small $\epsilon > 0$ it is easy to define a countably generated σ-ring $\mathcal{R}$ with maximal element $B_\epsilon(x_0)$ that is subordinate to the foliation into U-orbits. Here the atom $[x]_\mathcal{R}$ will consist of $B^U_{2\epsilon}(e)x \cap B_\epsilon(x_0)$ for any $x \in B_\epsilon(x_0)$.

If $x_0 \in \operatorname{supp}\mu$, the construction in [23, Prop. 9.2] shows that one can choose $\mathcal{P}$ such that there exists an element $P_0 \in \mathcal{P}$ with $\mu(P_0) > 0$ and $P_0 \subset B_\epsilon(x_0)$. Moreover, the set $V_x \subset U^-$ as in (2.1) for the σ-algebra $\mathcal{A} = \mathcal{P}^{[0,\infty)}$ has diameter less than 2ϵ for all $x \in P_0$. We define $\mathcal{P}_U$ to

be the smallest σ-algebra that contains $\mathcal{P}$ and $R \cap P_0$ for any $R \in \mathcal{R}$. Let

$$\mathcal{A}_U = \mathcal{P}_U^{[0,\infty)}.$$

We claim that

$$[x]_{\mathcal{A}_U} = [x]_{\mathcal{A}} \cap Ux \text{ for } \mu\text{-a.e. } x.$$

Then (4) follows using the injectivity we noted before. Moreover, the claim also implies that $a\mathcal{A}_U$ and $a\mathcal{P} \vee \mathcal{A}_U$ have the same atoms for a.e. x, so that (5) follows as well. Then (6) follows from the definition of the entropy contribution since $\mathcal{A} \subset \mathcal{A}_U$.

For the claim note first that $[x]_{\mathcal{A}_U} \subset Ux$ is obvious for $x \in P_0$ by the definition of $\mathcal{R}$, and follows for a.e. x from that and ergodicity. Now let $y = ux \in [x]_{\mathcal{A}}$ for some $u \in U$, we need to show that $a^n x, a^n y$ belong to the same atom of $\mathcal{P}_U$ for all $n \geq 0$. If $a^n x, a^n y \notin P_0$, then this follows from the definition of $\mathcal{A}$. Otherwise, $a^n x$ and $a^n y = (a^n u a^{-n})(a^n x)$ belong to the same $\mathcal{A}$-atom, and so $a^n x$ and $a^n y = v a^n x$ differ by an element $v \in V_{a^n x} \subset B_{2\epsilon}^{U^-}(e)$ by the above mentioned property of P_0 and $\mathcal{A}$. By injectivity we must have $v = a^n u a^{-n} \in B_{2\epsilon}^U(e)$. It follows that $a^n x$ and $a^n y$ are in the same atom of $\mathcal{R}$ which proves the claim. $\square$

2.4. **Entropy contribution and conditional measures.** The entropy contributions as defined in the previous subsection are dependent of the choice of sigma-algebra $\mathcal{A}_U$. In this section we give an alternative definition of entropy contribution which does not involve such choices. Let $a \in G$ and $U < G$ a subgroup normalized by a (later on we will assume U is a subgroup of the stable horospherical subgroup U^- of a). Let μ be a probability measure on $X = G/\Gamma$ invariant under left translation by a.

The measure μ gives rise to a system of locally finite measures $\mu_{x,U}$, which we refer to as the conditional measures on U orbits. This system of conditional measures $\mu_{x,U}$ has the following characterizing property: For any σ-algebra $\mathcal{A}_U$ that is subordinate to U we have

$$\mu_x^{\mathcal{A}_U} = \frac{1}{\mu_{x,U}(V_x)} (\mu_{x,U}|_{V_x}) x \text{ for a.e. } x,$$

where V_x is as in (2.1) and $(\mu_{x,U}|_{V_x}) x$ denotes the push forward of the restriction of $\mu_{x,U}$ to V_x under the map $u \mapsto ux$. These conditional measures are transformed in a nice way under the transformation $x \mapsto ax$: write $\theta(h) = aga^{-1}$. Then for almost every x we have that

$$\mu_{x,U} \propto \theta_* \mu_{ax,U}. \tag{2.3}$$

A general construction of these conditional measures can be found in [20, Sect. 2-3].

We now assume U is a subgroup of the stable horospherical subgroup U^- (so θ, defined as above, uniformly contracts U). We define the **volume decay entropy** at x by

$$\mathrm{vol}_\mu(a, U, x) = -\lim_{n \to \infty} \frac{\log \mu_{x,U}\big(\theta^n(B_1^U(e))\big)}{n} \qquad (2.4)$$

in particular, the above limits exists almost everywhere, see [4, Sect. 9.1]. This gives a pointwise definition of the entropy contribution, and we define

$$\mathrm{h}_\mu(a, U) = \int \mathrm{vol}_\mu(a, U, x) \, \mathrm{d}\mu. \qquad (2.5)$$

By [4, Lemma 9.2] the two definitions (2.2) and (2.5) agree whenever the σ-algebra $\mathcal{A}_U$ as in Corollary 2.3 exists.

2.5. **Product structure and summation formula.** Using the product structure of the conditional measures, the entropy contributions of the coarse Lyapunov subgroups can be related to the total entropy. This summation formula, which we present below, is closely related to the Ledrappier-Young entropy formula [19].

Let μ be an α-invariant and ergodic measure on X. Let U^- be the stable horospherical subgroup for some fixed $\alpha^{\mathbf{t}}$, and let $U^{\Lambda_1}, \ldots, U^{\Lambda_\ell} \leq U^-$ be all coarse Lyapunov subgroups contained in U^-. Then $U^- = U^{\Lambda_1} \cdots U^{\Lambda_\ell}$ and the map that sends $(u_1, \ldots, u_\ell) \in U^{\Lambda_1} \times \cdots \times U^{\Lambda_\ell}$ to $u_1 \cdots u_\ell$ is a homeomorphism (but in general not a group isomorphism). Using this homeomorphism implicitly we have

$$\mu_{x,U^-} \propto \mu_{x,U^{\Lambda_1}} \times \cdots \times \mu_{x,U^{\Lambda_\ell}} \text{ for a.e. } x \qquad (2.6)$$

by [4, Thm. 8.4], where $\propto$ denotes proportionality up to a multiplicative constant. This readily implies that

$$\mathrm{h}_\mu(\alpha^{\mathbf{t}}) = \mathrm{vol}_\mu(\alpha^{\mathbf{t}}, U^-, x) = \sum_{i=1}^{\ell} \mathrm{vol}_\mu(\alpha^{\mathbf{t}}, U^{\Lambda_i}, x). \qquad (2.7)$$

More generally, the above holds for any $U \leq U^-$ that is normalized by α, i.e.

$$\mathrm{vol}_\mu(\alpha^{\mathbf{t}}, U, x) = \sum_{i=1}^{\ell} \mathrm{vol}_\mu(\alpha^{\mathbf{t}}, U \cap U^{\Lambda_i}, x) \text{ a.e.} \qquad (2.8)$$

3. AN ENTROPY INEQUALITY AND THE SUPPORT OF THE CONDITIONAL MEASURE

We start our study of joinings, and will show in this section that the conditional measures for a joining have large support. Let $G = G_1 \times G_2$ with Lie algebra $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$, let Γ_i be a lattice in G_i and let $X_i = G_i/\Gamma_i$

for $i = 1, 2$. Define $X = X_1 \times X_2$ and π_i to be the projection map from X to X_i or the projection map from G to G_i for $i = 1, 2$ depending on the context. We will not assume that α has no local rank one factors.

Proposition 3.1. *Let $a = (a_1, a_2) \in G_1 \times G_2$, and consider left multiplication by a on $X = X_1 \times X_2$ where $X_i = G_i/\Gamma_i$ for some lattice Γ_i and $i = 1, 2$. Suppose a_i acts ergodically on X_i with respect to the Haar measure m_i for $i = 1, 2$. Let $U_1 < G_1$ and $U_2 < G_2$ be connected unipotent subgroups, and suppose $U = U_1 U_2$ is normalized by a and is contained in the stable horospherical subgroup U^-. Let μ be a (not necessarily ergodic) joining between a_1 and a_2, i.e. an a-invariant measure that projects $(\pi_i)_* \mu = m_i$ to the Haar measure on X_i for $i = 1, 2$. Then*

$$h_\mu(a, U) \leq h_{m_1}(a_1, U_1) + h_\mu(a, U_2). \tag{3.1}$$

Proof. We will need the following partitions and σ-algebras. For $i = 1, 2$ we use the ergodic action of a_i on X_i and Proposition 2.2 to find generating countable partitions $\bar{\mathcal{P}}_i$ of X_i with finite m_i-entropy so that $\bar{\mathcal{P}}_i^{[0,\infty)}$ is subordinate to the stable horospherical subgroup $U_i^- < G_i$ of a_i. Let $\mathcal{P}_i = \pi_i^{-1}(\bar{\mathcal{P}}_i)$ be the corresponding partition of X, and set $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2$. Since μ is a joining, it follows that $\mathcal{P}$ has finite μ-entropy and is generating for a. Moreover, $\mathcal{P}^{[0,\infty)}$ is subordinate to the stable horospherical subgroup $U^- < G$ of a. Now apply Corollary 2.3 to find for $i = 1, 2$ the σ-algebra $\bar{\mathcal{A}}_{U_i} \subset \mathcal{B}_{X_i}$ that is subordinate to U_i. Define $\mathcal{A}_i = \pi_i^{-1} \bar{\mathcal{A}}_{U_i}$ for $i = 1, 2$ and $\mathcal{A}_U = \mathcal{A}_1 \vee \mathcal{A}_2 = \bar{\mathcal{A}}_{U_1} \times \bar{\mathcal{A}}_{U_2}$. As above, it follows that $\mathcal{A}_U$ is subordinate to U and that $\mathcal{B}_{X_1} \times \bar{\mathcal{A}}_{U_2}$ is subordinate to U_2 since μ is a joining.

The various entropy contributions are now defined by

$$
\begin{aligned}
\mathrm{h}_\mu(a, U) &= \mathrm{H}_\mu\big(a\mathcal{P} \mid \mathcal{A}_U\big), \\
\mathrm{h}_{m_1}(a_1, U_1) &= \mathrm{H}_{m_1}\big(a_1\bar{\mathcal{P}}_1 \mid \bar{\mathcal{A}}_{U_1}\big), \text{ and} \tag{3.2} \\
\mathrm{h}_\mu(a, U_2) &= \mathrm{H}_\mu\big(a\mathcal{P} \mid \mathcal{B}_{X_1} \times \bar{\mathcal{A}}_{U_2}\big). \tag{3.3}
\end{aligned}
$$

Applying Corollary 2.3 (5) for $\bar{\mathcal{A}}_{U_i}$ and $i = 1, 2$, we see that $a\mathcal{P} \vee \mathcal{A}_U = a\mathcal{A}_U$. Therefore, we get

$$
\begin{aligned}
\mathrm{h}_\mu(a, U) &= \frac{1}{n}\,\mathrm{H}_\mu\left(a^n \mathcal{A}_U \mid \mathcal{A}_U\right) = \frac{1}{n}\,\mathrm{H}_\mu\left(\mathcal{P}^{[-n,0)} \mid \mathcal{A}_U\right) \\
&= \frac{1}{n}\,\mathrm{H}_\mu\left(\mathcal{P}_1^{[-n,0)} \mid \mathcal{A}_U\right) + \frac{1}{n}\,\mathrm{H}_\mu\left(\mathcal{P}_2^{[-n,0)} \mid \mathcal{A}_U \vee \mathcal{P}_1^{[-n,0)}\right) \tag{3.4}
\end{aligned}
$$

We first deal with the first term on the right hand side of (3.4). Since $\mathcal{A}_1 = \pi_1^{-1} \bar{\mathcal{A}}_{U_1} \subset \mathcal{A}_U$, by the properties of conditional entropy and (3.2),

$$\frac{1}{n} \mathrm{H}_\mu \left(\mathcal{P}_1^{[-n,0)} \mid \mathcal{A}_U \right) \leq \frac{1}{n} \mathrm{H}_\mu \left(\mathcal{P}_1^{[-n,0)} \mid \mathcal{A}_1 \right)$$

$$= \frac{1}{n} \mathrm{H}_{m_1} \left(\bar{\mathcal{P}}_1^{[-n,0)} \mid \bar{\mathcal{A}}_{U_1} \right) = \mathrm{h}_{m_1}(a_1, U_1). \quad (3.5)$$

We now proceed to consider the second term on the right hand side of (3.4):

$$\frac{1}{n} \mathrm{H}_\mu \left(\mathcal{P}_2^{[-n,0)} \mid \mathcal{A}_U \vee \mathcal{P}_1^{[-n,0)} \right) = \frac{1}{n} \sum_{k=1}^n \mathrm{H}_\mu \left(a^k \mathcal{P}_2 \mid \mathcal{A}_U \vee \mathcal{P}_1^{[-n,0)} \vee \mathcal{P}_2^{[-k+1,0)} \right)$$

$$= \frac{1}{n} \sum_{k=1}^n \mathrm{H}_\mu \left(a^k \mathcal{P}_2 \mid a^{k-1} \mathcal{A}_U \vee \mathcal{P}_1^{[-n,-k)} \right)$$

$$= \frac{1}{n} \sum_{k=1}^n \mathrm{H}_\mu \left(\mathcal{P}_2 \mid \mathcal{A}_U \vee \mathcal{P}_1^{[-n+k,0)} \right) \quad (3.6)$$

and by the martingale convergence theorem [26, Ch. 2, Thm. 6] for entropy one sees that

$$\mathrm{H}_\mu \left(\mathcal{P}_2 \mid \mathcal{A}_U \vee \mathcal{P}_1^{[-n+k,0)} \right) \to \mathrm{H}_\mu \left(\mathcal{P}_2 \mid \mathcal{A}_U \vee \mathcal{P}_1^{(-\infty,0)} \right).$$

However, by definition $\mathcal{A}_U$ contains $\mathcal{P}_1^{[0,\infty)}$, and since $\bar{\mathcal{P}}_1$ is generating for a_1 we see that $\mathcal{A}_U \vee \mathcal{P}_1^{(-\infty,0)} = \mathcal{B}_{X_1} \times \bar{\mathcal{A}}_{U_2}$. From (3.3) it follows that

$$\mathrm{H}_\mu \left(\mathcal{P}_2 \mid \mathcal{A}_U \vee \mathcal{P}_1^{(-\infty,0)} \right) = \mathrm{h}_\mu(T, U_2)$$

and the average in (3.6) converges to $h_\mu(a, U_2)$ which establishes (3.1). $\square$

Proposition 3.2. *Let* a, X, U, μ *be as in Proposition* 3.1. *If equality holds in* (3.1) *then* U_1 *is the smallest connected subgroup that is normalized by* a *and contains* $\pi_1(\mathrm{supp}\,\mu_{x,U})$ *for a.e.* x.

Proof. Suppose by contradiction that $U_1' < U_1$ is a proper connected subgroup that contains $\pi_1(\mathrm{supp}\,\mu_{x,U})$ for a.e. x and is normalized by a_1. Then the conditional measure $\mu_{x,U}$ is supported by $U' = U_1' U_2$ and so satisfies $(\mu_{x,U})|_{U'} = \mu_{x,U'}$ for a.e. x. It follows from the pointwise definition of the entropy contribution in (2.4) and Proposition 3.1 that

$$h_\mu(a, U) = h_\mu(a, U') \leq \mathrm{h}_{m_1}(a_1, U_1') + h_\mu(a, U_2)$$

$$< h_{m_1}(a_1, U_1) + h_\mu(a, U_2),$$

where the last strict inequality follows easily from the fact that $U_1' \subset U_1$ is a proper subgroup and a uniformly contracts U_1. This is a contradiction to our assumption. $\qquad\square$

Proposition 3.3. *Let a, μ, and $X = X_1 \times X_2$ be as in Proposition 3.1. For the stable horospherical subgroup $U^- = U_1^- U_2^-$ of a we have equality in (3.1).*

Proof. The Abramov-Rohklin conditional entropy formula gives that

$$\mathrm{h}_\mu(a) = \mathrm{h}_{m_1}(a_1) + \mathrm{h}_\mu\big(a|\pi_1^{-1}\mathcal{B}_{X_1}\big).$$

By Proposition 2.2 we have $h_\mu(a, U^-) = h_\mu(a)$ and $h_{m_1}(a_1, U_1^-) = h_{m_1}(a_1)$. It remains to match the last term of the above and (3.1). By (3.3)

$$\mathrm{h}_\mu(a, U_2^-) = \mathrm{H}_\mu\big(a\mathcal{P}|\mathcal{B}_{X_1} \times \bar{\mathcal{A}}_{U_2^-}\big) = \mathrm{H}_\mu\left(a\mathcal{P}|\mathcal{B}_{X_1} \times \bar{\mathcal{P}}_2^{[0,\infty]}\right),$$

where we used that $\bar{\mathcal{P}}_2^{[0,\infty)}$ is U_2^--foliated by Proposition 2.2. However, the last expression equals the conditional entropy $\mathrm{h}_\mu(a|\pi_1^{-1}\mathcal{B}_{X_1})$ since $\mathcal{P}$ is a generating partition for a. $\qquad\square$

We can now use the product structure to deduce a similar result regarding the coarse Lyapunov foliations.

Corollary 3.4. *Let $\alpha : \mathbb{R} \to G$ be an Ad-semisimple homomorphism such that for every nontrivial $\mathbf{t}$ the transformation $\alpha_i^{\mathbf{t}}$ on X_i acts ergodically with respect to m_i for $i = 1, 2$. Let μ be a joining between α_1 and α_2. Then we also have equality in (3.1) for a coarse Lyapunov subgroup $U = U^\Lambda = U_1^\Lambda U_2^\Lambda \leq U^-$.*

Proof. Let μ be α-invariant. Let U^Λ be a coarse Lyapunov subgroup. Choose some $a = \alpha^{\mathbf{t}}$ such that U^Λ is contained in the stable horospherical subgroup U^- to a. Let $U^- = U^{\Lambda_1} \cdots U^{\Lambda_\ell}$ be the decomposition of the stable horospherical subgroup into coarse Lyapunov subgroups as in Section 2.5 with $\Lambda = \Lambda_1$. For any j Proposition 3.1 gives

$$h_\mu(a, U^{\Lambda_j}) \leq h_{m_1}(a_1, U_1^{\Lambda_j}) + h_\mu(a, U_2^{\Lambda_j}).$$

Taking the sum over j, each term here sums up the corresponding term for $U^- = U_1^- U_2^-$ by (2.7)–(2.8). However, we already showed that the latter satisfy an exact summation formula so the same has to be true for the coarse Lyapunov subgroups. $\qquad\square$

The following is now a direct consequence of Proposition 3.2 and Corollary 3.4:

Corollary 3.5. *Let $U = U_1^\Lambda U_2^\Lambda$ be a coarse Lyapunov subgroup for α, and let $P \leq U$ be the smallest connected subgroup that is normalized by α such that $\operatorname{supp} \mu_{x,U} \subset P$ for a.e. x. Then $\pi_i(P) = U_i^\Lambda$ for $i = 1, 2$.*

In the notations of Corollary 3.5, for every $x \in X$ let $P_x \leq U$ denote the smallest connected subgroup normalized by α containing $\operatorname{supp} \mu_{x,U}$. Since a connected Lie group is determined by its Lie algebra, we can give the set of subgroups of U a natural topology coming from the Grassmannians of all dimensions of $\operatorname{Lie} U$. The map $x \mapsto P_x$ is measurable (we leave the straightforward but tedious verification of this fact to the pedantic reader) and is (modulo null sets) α invariant, hence if μ is α ergodic P_x is constant almost everywhere. This constant Lie group is necessarily the group P of Corollary 3.5 and so we obtain

Corollary 3.6. *Let μ be an α-ergodic joining of X_1 and X_2 as above, and Let $U = U_1^\Lambda U_2^\Lambda$ be a coarse Lyapunov subgroup for α. For every x, let P_x denote the smallest connected subgroup $P_x \leq U$ that is normalized by α containing $\operatorname{supp} \mu_{x,U}$. Then there is a group $P \leq U$ with $\pi_i(P) = U_i^\Lambda$ for $i = 1, 2$, so that for a.e. x we have that $P_x = P$.*

4. The high entropy theorem

As mentioned before, our main tool is the high entropy case as in [4, Thm. 8.5]. We will need the following special case.

Theorem 4.1. *Let G be a Lie group, let $\Gamma \subset G$ be a discrete subgroup, and let $X = G/\Gamma$. Let $\alpha : \mathbb{R}^n \to G$ be an $\mathbb{R}$-diagonalizable homomorphism, and let U^- be the stable horospherical subgroup to some element $a = \alpha^{\mathbf{t}}$. Furthermore, let $U^- = U^{\Lambda_1} \cdots U^{\Lambda_\ell}$ be the decomposition of U^- into coarse Lyapunov subgroups. For any α-invariant probability measure μ there exist for a.e. x two subgroups*

$$H_x \subseteq P_x \subseteq U^-$$

with the following properties:

(1) *μ_{x,U^-} is supported by P_x.*
(2) *μ_{x,U^-} is left- and right-invariant under multiplication with elements of H_x.*
(3) *H_x and P_x are connected and their Lie algebras are direct sums of subspaces of the weight spaces.*
(4) *H_x is a normal subgroup of P_x and any elements $g \in P_x \cap U^{\Lambda_r}$ and $h \in P_x \cap U^{\Lambda_s}$ of different coarse Lyapunov subgroups ($r \neq s$) satisfy that gH_x and hH_x commute with each other in P_x/H_x.*
(5) *$\mu_{x,U^{\Lambda_i}}$ is left- and right-invariant under multiplication with elements of $H_x \cap U^{\Lambda_i}$ for $i = 1, \ldots, \ell$.*

We quote also (a special case of) [4, Lemma 9.6] and its short proof since it is crucial to this paper.

Lemma 4.2. *Let G be a connected Lie group, let $\Gamma \subset G$ be a lattice, and let $\alpha : \mathbb{R}^n \to G$ be a homomorphism. Let $\mathfrak{g}$ be the Lie algebra of G, and let $\mathfrak{g}^\zeta$ denote the weight space associated for any weight ζ. Let V be a subspace of the dual of $\mathbb{R}^n$. Then $\mathfrak{g}^{(V)} = \langle \mathfrak{g}^\zeta, [\mathfrak{g}^\zeta, \mathfrak{g}^\xi] : \zeta, \xi \notin V \rangle$ is a Lie ideal in $\mathfrak{g}$.*

Therefore, if there are no rank one factors, then every element of $\mathfrak{g}^\lambda$ is a sum of expressions $[v, w]$ with $v \in \mathfrak{g}^\zeta$, $w \in \mathfrak{g}^\xi$ and $\zeta, \xi \notin \mathbb{R}\lambda$.

Proof. Since $\mathrm{Ad}_a(\mathfrak{g}^\zeta) = \mathfrak{g}^\zeta$ and $\mathrm{Ad}_a([u, v]) = [\mathrm{Ad}_a(u), \mathrm{Ad}_a(v)]$ for any $a = \alpha^{\mathbf{t}}$, it follows that $\mathfrak{g}^{(V)}$ is invariant under the adjoint action induced by α. Therefore, $\mathfrak{g}^{(V)}$ splits into a sum of subspaces of the weight spaces. We need to show that $[\mathfrak{g}^\eta, \mathfrak{g}^{(V)}] \subseteq \mathfrak{g}^{(V)}$ for all η (including $\eta = 0$).

Suppose first that $u \in \mathfrak{g}^\eta$ and $v \in \mathfrak{g}^\zeta \cap \mathfrak{g}^{(V)}$ with $\zeta \notin V$. If $\eta \notin V$ then $[u, v] \in \mathfrak{g}^{(V)}$ by definition of $\mathfrak{g}^{(V)}$, and otherwise $[u, v] \in \mathfrak{g}^{\eta + \zeta} \subseteq \mathfrak{g}^{(V)}$ because $\eta + \zeta \notin V$ and again the definition.

Let $u \in \mathfrak{g}^\eta$, $v \in \mathfrak{g}^\zeta$, and $w \in \mathfrak{g}^\xi$ with $\zeta, \xi \notin V$ (so that $[v, w] \in \mathfrak{g}^{(V)}$). Again, if $\eta + \zeta + \xi \notin V$ then there is nothing to show. So assume $\eta + \zeta + \xi \in V$. If $\eta \notin V$ then $\zeta + \xi \notin V$ and we are again done. The remaining case is $\eta, \eta + \zeta + \xi \in V$. By the Jacobi identity $[u, [v, w]] = -[v, [w, u]] - [w, [u, v]]$ and the two expressions on the right belong to $\mathfrak{g}^{(V)}$ since $\zeta, \xi + \eta, \xi, \eta + \zeta \notin V$.

For the final statement let $V = \mathbb{R}\lambda$. Then $\mathfrak{g}^{(V)}$ defines a Lie ideal such that the adjoint action on $\mathfrak{g}/\mathfrak{g}^{(V)}$ has no weights outside of V. Since there are no rank one factors, we have $\mathfrak{g}^{(V)} = \mathfrak{g}$ and every $u \in \mathfrak{g}^\lambda$ belongs to $\mathfrak{g}^{(V)}$. By restricting the sum that expresses u to those terms that belong to $\mathfrak{g}^\lambda$ the lemma follows. $\square$

Note that in particular since $\mathfrak{g}^\eta$ is nilpotent for every $\eta \neq 0$, the above lemma implies that a connected Lie group with no local rank one factors is generated by unipotent one parameter subgroups.

5. Translation invariance of the conditional measures and the joining

In this section we will use the assumption that the adjoint action to α has real eigenvalues and that there are no local rank one factors by applying the main results of [4]. For $g \in G$ write $L_g(h) = gh$ for the left action of g on G. We simply say a measure is g-invariant if it is invariant under L_g.

If μ is an ergodic joining between α_1 and α_2 as in Theorem 1.4, then by Proposition 2.1 (applied to $m_1 \times m_2$) it contains a subaction $\tilde{\alpha}$ with

the same coarse Lyapunov subgroups such that any element of $\tilde{\alpha}_i$ acts ergodically on X_i with respect to m_i for $i = 1, 2$. So, the results of Section 3 apply to this situation.

Proposition 5.1. *For every coarse Lyapunov subgroup $U^\Lambda = U_1^\Lambda U_2^\Lambda$ the connected subgroup*

$$I^\Lambda = \left\{ u \in U^\Lambda : \mu \text{ is } u\text{-invariant} \right\}$$

satisfies $\pi_i(\mathfrak{i}^\Lambda) = \mathfrak{u}_i^\Lambda$ for $i = 1, 2$. Here $\mathfrak{i}^\Lambda$ is the Lie algebra of I^Λ and $\mathfrak{u}_i^\Lambda$ is the Lie algebra of U_i^Λ.

Proof. It is easy to see that I^Λ is closed and normalized by α which implies that I^Λ is also connected. Similarly it has been shown in [4, Prop. 6.2] for a.e. x that

$$I_x^\Lambda = \left\{ u \in U^\Lambda : \mu_{x,U^\Lambda} \text{ is } u\text{-invariant} \right\}$$

is connected and that its Lie algebra is a direct sum of subspaces of $\mathfrak{u}^\lambda$ for $\lambda \in \Lambda$. Since we assume that the eigenvalues of the adjoint representation of α are real, we conclude that the Lie algebra of I_x^Λ is a direct sum of eigenspaces. Therefore, I_x^Λ is normalized by α which shows together with (2.3) that $I_x^\Lambda = I_{\alpha^\mathbf{t} x}^\Lambda$ for every $\mathbf{t} \in \mathbb{R}^n$ and a.e. $x \in X$.

The map $x \mapsto I_x^\Lambda$ is measurable (we again omit the details which are straightforward but tedious) and so by ergodicity we conclude that I_x^Λ is independent of x a.e., and by [20, Prop. 4.3] we conclude that $I_x^\Lambda = I^\Lambda$ for a.e. x.

Let $\lambda \in \Lambda$ and $v \in \mathfrak{u}_1^\lambda$. Applying Lemma 4.2 (which makes use of the assumption that there are no local rank one factors) we see that v can be expressed as a finite sum of terms $[w_1, w_2]$ with $w_1 \in \mathfrak{u}_1^{\lambda_1}$, $w_2 \in \mathfrak{u}_1^{\lambda_2}$, and λ_1, λ_2 linearly independent. We will show that the all these terms $[w_1, w_2]$ belongs to the Lie algebra $\pi_1(\mathfrak{i}^\Lambda)$. Then $v \in \pi_1(\mathfrak{i}^\Lambda)$ and the proposition will follow.

So consider now $w_1 \in \mathfrak{u}_1^{\lambda_1}$, $w_2 \in \mathfrak{u}_1^{\lambda_2}$, and λ_1, λ_2 linearly independent. Since λ_i are the weights, $[w_1, w_2] \in \mathfrak{u}_1^\lambda$ and $\lambda = \lambda_1 + \lambda_2$. By the linear independence there exists some $\mathbf{t} \in \mathbb{R}^n$ such that $\lambda(\mathbf{t}), \lambda_1(\mathbf{t}), \lambda_2(\mathbf{t}) < 0$, in other words the stable horospherical subgroup U^- corresponding to $a = \alpha^\mathbf{t}$ contains U^Λ, U^{Λ_1}, and U^{Λ_2} (where $\Lambda_j = \mathbb{R}^+ \lambda_j$).

Next we apply Theorem 4.1 to U^-: for a.e. x there exist two closed connected subgroups $H_x \leq P_x \leq U^-$ such that μ_{x,U^-} is supported by P_x and is invariant under elements of H_x, and $g_j \in P_x \cap U^{\Lambda_j}$ for $j = 1, 2$ implies $[g_1, g_2] \in H_x$. Furthermore, H_x and P_x are normalized by α, since their Lie algebras are direct sums of subspaces of $\mathfrak{u}^\zeta$ for various ζ and since we assume that α is $\mathbb{R}$-diagonalizable. By Corollary 3.6 there

exists a connected subgroup $P \leq U^-$ that is normalized by α such that $\operatorname{supp} \mu_{x,U} \subset P \leq P_x$ for a.e. $x \in X$.

Corollary 3.5 implies now that $\pi_1(P_x) = U_1$ for a.e. x. Therefore, there exist for $j = 1, 2$ elements $g_j \in P_x$ such that $\pi_1(g_j) = \exp(w_j)$. Since the Lie algebra of P_x is a direct sum of subspaces of the $\mathfrak{u}^\zeta$, we can assume here that $g_j \in P_x \cap U^{\Lambda_j}$. However, then $[g_1, g_2] \in H_x$ by Theorem 4.1. Note that $\pi_1([g_1, g_2]) = \exp([w_1, w_2] + \ldots)$ by the Campbell-Hausdorff formula,

where the dots indicate elements of $\mathfrak{u}^\zeta$ for various $\zeta \neq \lambda$. Therefore and since the Lie algebra of H_x is a direct sum of subspaces of Lyapunov spaces, we conclude that H_x contains an element $g \in U^\Lambda$ with $\pi_1(g) = \exp([w_1, w_2])$. By the properties of H_x we know that μ_{x,U^-} is left invariant under g. By (2.6) we know μ_{x,U^-} is a product measure, where we can assume that $\Lambda_1 = \Lambda$ is the first coarse Lyapunov weight that appears. However, this means that μ_{x,U^Λ} is left invariant under g and so $g \in I_x^\Lambda = I^\Lambda$. Since I^Λ is connected, we have $\log g \in \mathfrak{i}^\Lambda$ and so $[w_1, w_2] \in \pi_1(\mathfrak{i}^\Lambda)$ as claimed. $\qquad \square$

Let I be the subgroup generated by all Ad-unipotent one-parameter subgroups that preserve μ, and let $\mathfrak{i}$ be its Lie algebra.

Corollary 5.2. $\pi_i(\mathfrak{i}) = \mathfrak{g}_i$ *for* $i = 1, 2$. *Moreover,* $\mathfrak{i} \cap \big(\{0\} \times \mathfrak{g}_2 \big)$ *is a Lie ideal.*

Proof. By Lemma 4.2 the Lie algebra $\mathfrak{h}_i$ generated by $\mathfrak{u}_i^\zeta$ for non-zero ζ is a Lie ideal in $\mathfrak{g}_i$, so that $\mathfrak{h}_i = \mathfrak{g}_i$ by the assumption that there are no local rank one factors for α_i. The first statement now follows from Proposition 5.1

For the second statement, suppose $(0, w) \in \mathfrak{i}$ and $(v_1, v_2) \in \mathfrak{g}$. Then there exists some v_1' with $(v_1', v_2) \in \mathfrak{i}$, and so

$$[(0, w), (v_1, v_2)] = [(0, w), (v_1', v_2)] = (0, [w, v_2]) \in \mathfrak{i}.$$

$\qquad \square$

Note that I is a subgroup for which Ratner's theorem [28] applies. So it remains to show that μ is an I-invariant and ergodic measure, which will imply that μ is algebraic.

Let $J \leq G$ be the group generated by all one-parameter subgroups that preserve μ and let $\mathfrak{j}$ be its Lie algebra. Clearly $\alpha(\mathbb{R}^n) I \subset J$ so that $\mathfrak{a} + \mathfrak{i} \subset \mathfrak{j}$ where $\mathfrak{a}$ is the Lie algebra of $\alpha(\mathbb{R}^n)$.

Further, denote by $\mathfrak{n} \subset \mathfrak{r}$ the nil radical and the radical of $\mathfrak{g}$, i.e. $\mathfrak{n}$ is the maximal nilpotent Lie ideal and $\mathfrak{r}$ is the maximal solvable ideal.

Lemma 5.3. *We have* $\mathfrak{j} \cap \mathfrak{n} \subset \mathfrak{i}$.

Proof. Let $v \in \mathfrak{j} \cap \mathfrak{n}$. Then μ is invariant under $\exp(tv)$ for $t \in \mathbb{R}$ by definition of $\mathfrak{j}$, and ad_v is a nilpotent endomorphism of $\mathfrak{g}$ since $\mathfrak{n}$ is a nilpotent ideal. It follows that $\exp(tv)$ is Ad-unipotent, and so $v \in \mathfrak{i}$ by definition. $\qquad\square$

Lemma 5.4. *If α has no local rank one factors, then $\mathfrak{n} = \mathfrak{r}$.*

Proof. By [33, Thm. 3.8.3(iii)] we have $[\mathfrak{g}, \mathfrak{r}] \subset \mathfrak{n}$. Therefore, $\mathfrak{r}' = \mathfrak{r}/\mathfrak{n}$ is in the center and simultaneously the radical of $\mathfrak{g}' = \mathfrak{g}/\mathfrak{n}$. If $\mathfrak{g}' = \mathfrak{g}'_s + \mathfrak{r}'$ is the Levi decomposition, then $\mathfrak{g}'_s$ is an ideal. Therefore, $\mathfrak{r}'$ is an abelian factor of $\mathfrak{g}$. By the assumption that there are no local rank one factors we must have $\mathfrak{r}' = 0$. $\qquad\square$

Proof of Theorem 1.4. It suffices to show that $\mathfrak{a} \subset \mathfrak{i}$. Because then $\alpha(\mathbb{R}^n) \subset I$ which means that μ is I-invariant and ergodic. By Ratner's theorem [28] this implies that μ is algebraic.

So let $v = (v_1, v_2) \in \mathfrak{a}$. By Corollary 5.2 it follows that there exists $v'_2 \in \mathfrak{g}_2$ such that $(v_1, v'_2) \in \mathfrak{i}$. Then $(0, v_2 - v'_2) = (0, w)$ belongs to $\mathfrak{j}$ – the Lie algebra of the group J generated by all one parameter subgroups that preserve μ.

Moreover, since I is normalized by α, its Lie algebra $\mathfrak{i}$ is a direct sum of subspaces of weight spaces. Since $(v_1, v'_2) \in \mathfrak{i}$ and $(v_1, v_2) \in \mathfrak{a} \subset \mathfrak{u}^0$, this shows we can choose v'_2 so that $(v_1, v'_2), (0, w) \in \mathfrak{u}^0$, i.e. w is fixed under the adjoint representation restricted to α_2. By the first statement of Corollary 5.2 it follows that $[(0, w), \mathfrak{i}] = (0, [w, \mathfrak{g}_2]) \subset \mathfrak{j}$. Since $(0, w) \in \mathfrak{u}^0$ we have $(0, [w, \mathfrak{u}_2^\lambda]) \subset \mathfrak{u}^\lambda$. For any nonzero λ any element of $\mathfrak{u}^\lambda$ gives rise to a Ad-unipotent one parameter subgroup, therefore, $(0, [w, \mathfrak{u}_2^\lambda]) = \mathfrak{i}$. By our assumption that there are no local rank one factors and by Lemma 4.2 every element of $\mathfrak{u}^0$ can be written as a linear sum of expressions $[u, u']$ with $u \in \mathfrak{u}_2^\lambda$ and $u' \in \mathfrak{u}_2^{-\lambda}$ for some nonzero λ. By the second statement of Corollary 5.2 the Lie algebra $\mathfrak{i}_2$ with $\mathfrak{i} \cap (\{0\} \times \mathfrak{g}_2) = \{0\} \times \mathfrak{i}_2$ is actually a Lie ideal in $\mathfrak{g}_2$. The Jacobi identity now implies that

$$[w, [u, u']] = -[u, [u', w]] - [u', [w, u]] \in \mathfrak{i}_2.$$

Therefore, $[w, \mathfrak{g}_2] \subset \mathfrak{i}_2$.

Since α_2 has no local rank one factors, it follows from Lemma 5.4 that the nil radical $\mathfrak{n}_2$ of $\mathfrak{g}_2$ is also the radical. Therefore $\overline{\mathfrak{g}_2} = \mathfrak{g}_2/\mathfrak{n}_2$ is semisimple and it follows that $\bar{w} = w + \mathfrak{n}_2$ belongs to the Lie ideal generated by $[\bar{w}, \overline{\mathfrak{g}_2}]$. Therefore, $w \in \mathfrak{i}_2 + \mathfrak{n}_2$ and there exists some $w' \in \mathfrak{i}_2$ with $w - w' \in \mathfrak{n}_2$. However, since both $(0, w), (0, w') \in \mathfrak{j}$ the same applies to $(0, w - w') \in \{0\} \times \mathfrak{n}_2$. By Lemma 5.3 $(0, w - w') \in \mathfrak{i}$ and so $(0, w), v \in \mathfrak{i}$ as claimed. $\qquad\square$

6. Factors of higher rank abelian actions

Let G be a connected Lie group, Γ a lattice in G, and m Haar measure on $X = G/\Gamma$. Let $\alpha : \mathbb{R}^n \to G$ be an injective homeomorphism diagonalizable over $\mathbb{R}$ with no local rank one factors, and set $A = \alpha(\mathbb{R}^n)$. For every $g \in G$ we let L_g and R_g denote the element of $\operatorname{Aff} G$ given by left (respectively right) translation, i.e. $L_g(h) = gh$ and $R_g(h) = hg$ for any $h \in G$. Furthermore, if H is a subgroup of G we let $L_H = \{L_h : h \in H\}$ and $R_H = \{R_h : h \in H\}$. We let $\operatorname{Aff}_1 G <$ $\operatorname{Aff} G$ denote those affine groups which preserve a Haar measure on G (since G has a lattice it is in particular unimodular and so we need not distinguish between left and right Haar measures). Note that both L_G and R_G are normal subgroups of $\operatorname{Aff}_1 G$.

In this section we want to classify measurable factors (also called quotients) of G/Γ, or what amounts to the same countably generated A-invariant sigma algebras $\mathcal{A}$ of Borel subsets of G/Γ. Our input to this classification of factors is the classification of self joining; we will also use the fact that G is generated by unipotent one parameter subgroups (which follows from the local rank assumption) though this is a minor technical point. Essentially everything we do in this section is contained in Dave Witte Morris papers [34, 35] and the references cited there, in particular [3]; we have tried to give here a slightly more streamlined version of his proof (on the other hand, Witte Morris' setup, particularly in [35] is more general than ours).

It would be more convenient for us to view $\mathcal{A}$ as a left A-invariant sigma algebra of right Γ-invariant subsets of G (notice the asymmetric role of A and Γ: for every set $B \in \mathcal{A}$ and $\gamma \in \Gamma$ we have that $R_\gamma B = B$ whereas regarding A we know that for every $B \in \mathcal{A}$ and $a \in A$ the set $aB = L_a B$ is in $\mathcal{A}$ but typically will not be equal to B).

Consider the group

$$H_\mathcal{A} = \{h \in G : \text{for every } B \in \mathcal{A}, \quad m_G(R_h(B) \triangle B) = 0\} . \qquad (6.1)$$

Clearly, $H_\mathcal{A} > \Gamma$. We want to reduce to the case that $H_\mathcal{A} = \Gamma$.

Lemma 6.1. $H_\mathcal{A}$ is a closed subgroup of G.

Proof. Indeed, let m' be a probability measure in the same measure class as Haar measure m_G. Then for any $B \in \mathcal{A}$ the characteristic function $1_B \in L^2(m')$, and the map $h \mapsto 1_{R_h(B)}$ is a continuous map $G \to L^2(m')$. It follows that $\{h : m_G(R_h(B) \triangle B) = 0\}$ is closed and therefore so is $H_\mathcal{A}$. $\qquad \square$

Lemma 6.2. *The connected component of the identity $H_\mathcal{A}^0 \leq H_\mathcal{A}$ is normal in G.*

Proof. From Lemma 4.2 it follows that G is generated by unipotent one parameter subgroups, indeed that there is some element $\alpha^{\mathbf{t}}$ so that G is generated by the groups U^λ with $\lambda(\mathbf{t}) \neq 0$.

By [22, Theorem II.2.5], for any continuous representation of G any Γ-invariant subspace is G-invariant. Since $\Gamma < H_{\mathcal{A}}$ it follows that the Lie algebra of $H_{\mathcal{A}}^0$ is a Γ-invariant subspace of $\mathfrak{g}$ under the adjoint representation, hence is G-invariant. From this we conclude that $H_{\mathcal{A}}^0$ is normal in G as claimed. $\qquad\square$

Now let $N_{\mathcal{A}}$ denote the maximal closed normal subgroup of G contained in $H_{\mathcal{A}}$. By replacing G with $G_1 = G/N_{\mathcal{A}}$ and Γ with the lattice $\Gamma_1 = H_{\mathcal{A}}/N_{\mathcal{A}} < G_1$ we may, **and will for the remainder of this section**, assume without loss of generality that

$$H_{\mathcal{A}} = \Gamma \quad \text{and} \quad \Gamma \text{ contains no normal subgroup of } G. \qquad (6.2)$$

Note that $H_{\mathcal{A}}/N_{\mathcal{A}}$ is discrete since $N_{\mathcal{A}} > H_{\mathcal{A}}^0$ by Lemma 6.2.

The following reformulation of (6.1) will be useful: we will say that two points $x, x' \in X$ are equivalent modulo $\mathcal{A}$ ($x \overset{\mathcal{A}}{\sim} x'$) if $[x]_{\mathcal{A}} = [x']_{\mathcal{A}}$. Then since $\mathcal{A}$ is countably generated, (6.1) is easily seen to be equivalent to

$$H_{\mathcal{A}} = \left\{ h \in G : gh \overset{\mathcal{A}}{\sim} g \quad \text{for a.e. } g \in G \right\}.$$

Note that the equivalence relation $\overset{\mathcal{A}}{\sim}$ induces an equivalence relation on G which we shall also denote by $\overset{\mathcal{A}}{\sim}$.

Now consider the group

$$\Xi = \{ \xi \in C_{\mathrm{Aff}\,G}(L_A) : \text{for every } B \in \mathcal{A}, \quad m(\xi(B) \triangle B) = 0 \} \qquad (6.3)$$
$$= \left\{ \xi \in C_{\mathrm{Aff}\,G}(L_A) : \xi(g) \overset{\mathcal{A}}{\sim} g \quad \text{a.e.} \right\},$$

where $C_{\mathrm{Aff}\,G}(L_A)$ denotes the centralizer in $\mathrm{Aff}\,G$ of the group of left translations L_A. Notice that by (6.2), the group $\Xi \cap R_G = R_\Gamma$. The same proof as in Lemma 6.1 gives that Ξ is a closed subgroup of $\mathrm{Aff}\,G$, and hence so is $\Xi_L = \Xi \cap L_G$. Notice that both Ξ_L and R_Γ are normal subgroups of Ξ. We also note the following:

Lemma 6.3. *Let $\xi = L_{g_0}\xi_0 \in \Xi$ with ξ_0 an automorphism of G. Then $\xi_0(\Gamma) = \Gamma$.*

Proof. Suppose for example $\xi_0(\Gamma) \not\leq \Gamma$. Then for almost every $g \in G$ and $\gamma \in \Gamma$ we have

$$g\xi_0(\gamma) \overset{\mathcal{A}}{\sim} g_0^{-1}\xi_0^{-1}(g)\gamma \overset{\mathcal{A}}{\sim} g_0^{-1}\xi_0^{-1}(g) \overset{\mathcal{A}}{\sim} g$$

and $H_{\mathcal{A}} \geq \xi_0(\Gamma)$ in contradiction to our assumption. Similarly, one rules out $\xi_0(\Gamma) \not\geq \Gamma$ using ξ_0^{-1} instead of ξ_0 in the above argument. $\square$

In particular, since Γ is a lattice in G we have that

$$\Xi \leq \mathrm{Aff}_1 \, G.$$

We recall the definition of the relatively independent joining $m \times_{\mathcal{A}} m$: it is the measure on $G/\Gamma \times G/\Gamma$ defined by

$$m \times_{\mathcal{A}} m(B \times C) = \int_X m_x^{\mathcal{A}}(B) \times m_x^{\mathcal{A}}(C) dm(x). \qquad (6.4)$$

Lemma 6.4. *For every $\xi \in \Xi$, the map $\mathrm{Id} \times \xi$ preserves $m \times_{\mathcal{A}} m$.*

Proof. Let $\xi \in \Xi$. By definition of Ξ, ξ preserves the atoms of $\mathcal{A}$, so m-a.s. one has $m_{\xi(x)}^{\mathcal{A}} = m_x^{\mathcal{A}}$. Also, the measure m is preserved by any element of $\mathrm{Aff}_1 \, G$, in particular by ξ, and so m-a.s. $\xi_* m_x^{\mathcal{A}} = m_{\xi(x)}^{\mathcal{A}}$. Combining this with the previous assertion we get that $\xi_* m_x^{\mathcal{A}} = m_x^{\mathcal{A}}$ a.s. Lemma 6.4 now follows directly from the definition (6.4) of $m \times_{\mathcal{A}} m$. $\square$

The results so far are true for any action; now comes the two main steps which comes from careful analysis of the ergodic decomposition of the relatively independent joining $m \times_{\mathcal{A}} m$.

Lemma 6.5. *Let $\mathcal{J}_{\mathcal{A}}$ be the set of self joinings ρ for the action corresponding to α on (X, m) (in particular, ρ is a probability measure on $X \times X$) so that*

$$\rho\left(\left\{(x, x') \in X \times X : x \overset{\mathcal{A}}{\sim} x'\right\}\right) = 1. \qquad (6.5)$$

then for every ergodic $\rho \in \mathcal{J}_{\mathcal{A}}$ there is a $\xi \in \Xi$ (unique up to an element of R_Γ), commuting with $L_{\mathcal{A}}$, so that ρ is the unique self joining of X supported on the graph $\{(g, \xi g) : g \in G\}$.

Proof. By Theorem 1.4 there is a closed connected subgroup $H \leq G \times G$, projecting onto G in each component, so that ρ is the Haar measure on a closed orbit of H. Let π_1, π_2 denote the projections of $G \times G$ to the first and second coordinate. Suppose this H orbit is $H(g_0', g_0)(\Gamma \times \Gamma) = (g_0', g_0)\tilde{H}(\Gamma \times \Gamma)$ with $\tilde{H} = (g_0'^{-1}, g_0^{-1})H(g_0', g_0)$. Since $\pi_1(H) = G$, we may as well assume g_0' is the identity $e \in G$.

Consider now the group $H_1 = \pi_2^{-1}(e) \cap \tilde{H}$. Since $\pi_1(H) = G$, and H normalizes H_1, it follows that $\pi_1(H_1) \triangleleft G$. We also claim that $H_1 \leq H_{\mathcal{A}}$; indeed, for every $h \in \pi_1(H_1)$, for almost every $(g_1, g_2) \in \tilde{H}$, both $g_1 \overset{\mathcal{A}}{\sim} g_0 g_2$ and $g_1 h \overset{\mathcal{A}}{\sim} g_0 g_2$ hence by transitivity of $\overset{\mathcal{A}}{\sim}$ and the fact that $\pi_1(H) = G$ it holds that $g_1 \overset{\mathcal{A}}{\sim} g_1 h$ for almost every g_1 and so $h \in H_{\mathcal{A}}$. By our assumptions (6.2) on $H_{\mathcal{A}}$ it follows that $\pi_1(H_1)$ being

a normal subgroup of $H_{\mathcal{A}} = R_\Gamma$ is the identity, and the same holds for H_1. Similarly, $H_2 = \pi_1^{-1}(e) \cap \tilde{H}$ is trivial.

It follows that the group $\tilde{H} < G \times G$ is the graph of an automorphism ξ_0 of G and that for a.e. g, $g \overset{\mathcal{A}}{\sim} g_0 \xi_0(g)$, and define $\xi = L_{g_0} \xi_0$. since ρ is invariant under $\{(\alpha^{\mathbf{t}}, \alpha^{\mathbf{t}}) : \mathbf{t} \in \mathbb{R}^n\}$, this group is contained in H, and working through the definitions this implies that ξ commutes with $L_{\alpha^{\mathbf{t}}}$ for every $\mathbf{t}$.

It follows that $\xi \in \Xi$ and that ρ is the unique self joining of X supported on the graph $\{(g\Gamma, \xi g\Gamma) : g \in G\}$. $\square$

Lemma 6.6.   (1) *The group $\Phi = \Xi/R_\Gamma$ has a Φ-invariant probability measure, hence is compact.*
   (2) *There is a conull set $X_1 \subset G/\Gamma$ so that for every $x, y \in X_1$ in the same atom of $\mathcal{A}$, there is an element $\phi \in \Phi$ taking x to y.*

Proof. Aff G, hence also its closed subgroup Ξ, is a locally compact group, R_Γ a countable discrete subgroup, and so $\Phi = \Xi/R_\Gamma$ is a nice locally compact topological group. Note that Φ acts on $X = G/\Gamma$ by $(\xi R_\Gamma).(g\Gamma) = \xi(g)\Gamma$ which is well defined by Lemma 6.3 and the definition of R_Γ.

The set $\mathcal{J}_{\mathcal{A}}$ is a Borel subset of the space of probability measures on $X \times X$ equipped with the weak* topology. For any $\xi \in \Xi$, the image ρ_ξ of Haar measure on G/Γ under $g\Gamma \mapsto g\Gamma, \xi(g)\Gamma$ is an element of $\mathcal{J}_{\mathcal{A}}$, and this map $\xi \mapsto \rho_\xi$ is a Borel (indeed, continuous) map $\Xi \to \mathcal{J}_{\mathcal{A}}$. Since this map is R_Γ-invariant, we actually get a Borel map $\iota : \Phi \to \mathcal{J}_{\mathcal{A}}$. This map is clearly 1-1, and it follows that the image of ι is a Borel set and ι^{-1} is a Borel map [1, Theorem 3.3.2].[4]

The ergodic decomposition of $m \times_{\mathcal{A}} m$ with respect to the diagonal embedding of the group A gives us a probability measure ν on the space of joinings of X with itself. Since by definition

$$m \times_{\mathcal{A}} m \left(\left\{ (x, x') \in X \times X : x \overset{\mathcal{A}}{\sim} x' \right\} \right) = 1$$

the same holds for almost every ergodic component of $m \times_{\mathcal{A}} m$, hence ν is supported on the ergodic joinings in $\mathcal{J}_{\mathcal{A}}$, which by Lemma 6.5 are in $\iota(\Phi)$. Using ι^{-1}, we get a Borel probability measure $\nu' = \iota_*^{-1} \nu$ on Φ.

By Lemma 6.4, for every $\xi \in \Xi$, we have that (Id, ξ) preserves $m \times_{\mathcal{A}} m$, and since ξ commutes with L_A it follows that $(\mathrm{Id}, \xi)_*$ preserves ν.

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[4]This whole discussion (which is somewhat less generally applicable than the rest of this proof) can be circumvented by defining an associative composition operation on $\mathcal{J}_{\mathcal{A}}$ which can be shown to give a group structure when restricted to the ergodic joinings, or, what amounts to the same, by using the topology of convergence in measure on Aff(G/Γ). See [35] for further discussion of this issue.

But for every $\rho_{\xi'} \in \mathcal{J}_{\mathcal{A}}$,

$$(\mathrm{Id}, \xi)_*(\rho_{\xi'}) = \rho_{\xi\xi'}$$

and so ν' is right invariant under the action of Ξ on Φ. It follows that ν' is a finite Haar measure on the locally compact group Φ, hence Φ is a compact group, proving Lemma 6.6.(1).

To see (2), we first present $m \times_{\mathcal{A}} m$ in two ways: for any bounded measurable f,

$$\int_{X \times X} f(x,y) dm \times_{\mathcal{A}} m(x,y) = \iint_X f(x,y) dm_x^{\mathcal{A}}(y) dm(x)$$
$$= \int_{X \times X} \int_{\mathcal{J}_{\mathcal{A}}} f(x,y) d\rho(x,y) d\nu(\rho)$$
$$= \int_X \int_\Phi f(x, \phi(x)) dx d\nu'(\phi).$$

It follows that m-a.s. $m_x^{\mathcal{A}} = (\tilde{R}_x)_*(\nu')$, with $\tilde{R}_x : \Phi \to X$ the map $\phi \mapsto \phi(x)$.

Let X_1 denote the set

$$X_1 = \left\{ x \in X : m_x^{\mathcal{A}} = (\tilde{R}_x)_*(\nu') \right\}.$$

Suppose $x, x' \in X_1$ with $x \overset{\mathcal{A}}{\sim} x'$, i.e. $[x]_{\mathcal{A}} = [x']_{\mathcal{A}}$ and hence $m_x^{\mathcal{A}} = m_{x'}^{\mathcal{A}}$. Then $m_x^{\mathcal{A}}$-almost every $y \in [x]_{\mathcal{A}}$ can be represented as $\eta(x)$ as well as $\phi(x')$ for $\eta, \phi \in \Xi/R_\Gamma$. Find one $y \in [x]_{\mathcal{A}}$ that can be represented in both of these ways. Then $x = \eta^{-1}\phi(x')$ with $\eta^{-1}\phi \in \Phi$ as claimed. $\quad\square$

Corollary 6.7. *The sigma algebra $\mathcal{B}_\Phi$ of Φ-invariant Borel subsets of X is countably generated and is equivalent mod m to $\mathcal{A}$.*

Proof. That $\mathcal{B}_\Phi$ is countably generated is a general property of compact group actions on standard Borel spaces, which can be easily verified as follows: take B_i to be a generating sequence of Borel sets in X. Consider the functions $f_{B_i} = \int_\Phi 1_{B_i}(\phi x) d\phi$. The level sets $\{a < f_{B_i}(x) < b\}$ for $a, b \in \mathbb{Q}$ form a countable generating set for $\mathcal{B}_\Phi$.

We now show that $\mathcal{B}_\Phi$ is equivalent mod m to $\mathcal{A}$. By the definition (6.3) of Ξ , every set $B \in \mathcal{A}$ is almost invariant under Ξ and therefore also under Φ, i.e. $m(\phi B \triangle B) = 0$ for every $\phi \in \Phi$. As is well-known (see e.g. [36, Lemma 2.2.15]) this implies that there is a Φ-invariant Borel set B' with $m(B' \triangle B) = 0$ and by definition $B' \in \mathcal{B}_\Phi$.

Conversely, let $B \in \mathcal{B}_\Phi$. Then the function

$$\mathbf{E}(1_B \mid \mathcal{A})(x) = \int_X 1_B(x') dm_x^{\mathcal{A}}(x')$$

is $\mathcal{A}$-measurable. Let X_1 be a conull set as in Lemma 6.6.(2). For every $x \in B \cap X_1$ we have that $B \supset \Phi x \supset [x]_{\mathcal{A}} \cap X_1$, hence

$$m_x^{\mathcal{A}}(B) \geq m_x^{\mathcal{A}}([x]_{\mathcal{A}} \cap X_1) = 1 \qquad \text{a.s}$$

and hence $\mathbf{E}(1_B \mid \mathcal{A}) \geq 1_B$ a.e.. Applying the same to $B^{\complement}$ we get the reverse inequality, hence $\mathbf{E}(1_B \mid \mathcal{A}) = 1_B$ a.e. and so there is a $B' \in \mathcal{A}$ with $m(B \triangle B') = 0$. $\qquad \square$

This completes the proof of Theorem 1.5. We now specialize to the case where G is semisimple with finite center, all factors have $\mathbb{R}$-rank ≥ 2, and $\Gamma < G$ is an irreducible lattice (i.e. a lattice which is not commensurable to a product of lattices in factors of G) and deduce Corollary 1.6 from the more general Theorem 1.5. As in that corollary, we let $A = \alpha(\mathbb{R}^n)$ to be the connected component of the identity in a maximal $\mathbb{R}$-split torus. Then its centralizer $C_G(A) = MA$ where $M = C_K(A)$ by [17, Lemma 7.22] and the fact that A is maximal $\mathbb{R}$-split. Here M is the unique maximal compact subgroup of $C_G(A)$, it is compact since K is compact by the assumption that G has finite center, see [17, Theorem 6.31]. Note that in this case, since Γ is irreducible, there can be no nontrivial connected $N \triangleleft G$ so that $N\Gamma$ is closed, hence $\Gamma' := H_{\mathcal{A}} \geq \Gamma$ is discrete.

Lemma 6.8. *Let G, Γ, α and $\mathcal{A}$ be as in Corollary 1.6 (with $\mathcal{A}$ identified with a σ-algebra of left M-invariant subsets of G/Γ in the obvious way[5]). Then*

(1) *there is an open neighborhood $\mathcal{U}$ of the identity in $\mathrm{Aff}_1(G)$ so that $\mathcal{U} \cap \Xi \subset L_G$,*
(2) $\Xi \cap L_G = L_M$

Proof. Claim (1) follows from the fact that Γ' is finitely generated and Lemma 6.3.

Since $\mathcal{A}$ consists of L_M-invariant sets by definition of Ξ, $L_M \leq L_G \cap \Xi$. By definition of Ξ, every $\xi \in \Xi$ commutes with left translation by elements of A, so $\Xi \cap L_G \leq L_{C_G(A)}$. Recall that $C_G(A) = MA$. If $\Xi \cap L_G > L_M$ holds strictly, then $\Xi \cap L_G$ contains a nontrivial element of L_A. However, this is impossible since $\Phi = \Xi/R_{\Gamma'} \leq \mathrm{Aff}_1(G/\Gamma)$ is compact and since the powers of a nontrivial element of L_A are not contained in a compact subset of $\mathrm{Aff}_1(G/\Gamma)$. $\qquad \square$

Proof of Corollary 1.6. By Lemma 6.8 $L_M \triangleleft \Xi$ and Ξ/L_M is discrete, so $\Xi/L_M R_{\Gamma'}$ is both discrete and compact, hence finite, and consequently

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[5]If we view $\mathcal{A}$ in this way then the assertion of Corollary 1.6 that $[y]_{\mathcal{A}}$ is finite for a.e. $y \in M\backslash G/\Gamma$ translates to $[x]_{\mathcal{A}}$ is a union of finitely many orbits of M on G/Γ.

so is $\Xi/L_M R_\Gamma$. By Lemma 6.6.(2) this shows that for a.e. $x \in G/\Gamma$ the atom $[x]_\mathcal{A}$ consists of finitely many orbits of M. This concludes the proof of the first claim in the corollary.

If Γ is maximal, $\Gamma' = \Gamma$ and furthermore the normalizer in G of Γ is equal to Γ. Any $\xi \in \Xi$ can be written as $\xi = L_h \xi_0 = \xi_1 R_{h'}$ with $\xi_0, \xi_1 \in \mathrm{Aut}\, G$; by definition of Ξ, ξ_1 fixes A_0 pointwise, and by Lemma 6.3 ξ_0 sends Γ to itself. By the assumptions on G and A in the second part of the corollary, ξ_1 and hence also ξ_0 is necessarily an inner automorphism, and by the maximality of Γ we have that $\xi_0(g) = \gamma^{-1} g \gamma$ for some $\gamma \in \Gamma$. It follows that $L_{h\gamma^{-1}} \in \Xi$ and so by Lemma 6.8.(2) $h\gamma^{-1} \in M$, and $\xi \in L_M R_\Gamma$. It follows that $[x]_\mathcal{A}$ is almost surely a single orbit of M, or translating back to $y \in M\backslash G/\Gamma$ that $[y]_\mathcal{A}$ is a single point a.s.: which implies that when viewing $\mathcal{A}$ as a σ-algebra of Borel subsets of $M\backslash G/\Gamma$ that it is equal mod m to the full Borel sigma algebra. $\qquad\square$

7. Equidistribution

As pointed out in Section 1 Theorem 1.1 follows from Theorem 1.4. We now give the proof of the corollary regarding equidistributed orbits.

Proof of Corollary 1.2. In the following write α_i for the action on $X_i = G/\Gamma_i$ for $i = 1, 2$ and α for the action on $X_1 \times X_2$. Let $x = (x_1, x_2)$ be as in the corollary. For $\ell > 0$ define the probability measure

$$\mu_\ell(B) = \frac{1}{2^n \ell^n} \int_{[-\ell,\ell]^n} 1_B(\alpha^{\mathbf{t}} x)\, \mathrm{d}\mathbf{t}$$

for any measurable $B \subset X$. Then a weak* limit μ of any subsequence of μ_ℓ for $\ell \to \infty$ is α-invariant. Moreover, we claim μ is a probability measure and a joining between α_1 and α_2. However, then the assumptions to the corollary and Theorem 1.1 show that $\mu = m_1 \times m_2$ is the only such measure. Since μ was any weak* limit, the corollary follows.

To see the claim, let $f \in C(X_1)$ be non-negative with compact support. Let $\epsilon > 0$ be arbitrary, take a $g \in C(X_2)$ with $0 \leq g \leq 1$, compact support, and $\int g\, \mathrm{d}m_1 > 1 - \epsilon$. Then for large enough ℓ we have

$$\int_{[-\ell,\ell]^n} g(\alpha_2^{\mathbf{t}}(x_2))\, \mathrm{d}\mathbf{t} > (1 - \epsilon) 2^n \ell^n$$

by the assumption that x_2 has m_2-equidistributed α_2-orbit. Therefore,

$$\frac{1}{2^n \ell^n} \int_{[-\ell,\ell]^n} f(\alpha_1^{\mathbf{t}} x_1) \, \mathrm{d}\mathbf{t} \leq \frac{1}{2^n \ell^n} \int_{[-\ell,\ell]^n} f \otimes g(\alpha^{\mathbf{t}}(x_1, x_2)) \, \mathrm{d}\mathbf{t} + \epsilon \|f\|_\infty \leq$$

$$\frac{1}{2^n \ell^n} \int_{[-\ell,\ell]^n} f(\alpha_1^{\mathbf{t}} x_1) \, \mathrm{d}\mathbf{t} + \epsilon \|f\|_\infty$$

which shows that

$$\left| \int f \otimes g \, \mathrm{d}\mu - \int f \, \mathrm{d}m_1 \right| \leq \epsilon \|f\|_\infty.$$

Since this holds for any g with compact support as above and any ϵ, we conclude that

$$\int f \, \mathrm{d}(\pi_1)_* \mu = \int f \, \mathrm{d}m_1.$$

Varying f the claim and the corollary follow. $\qquad\qquad\square$

Acknowledgements

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