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No Arbitrage Condition for Positive Diffusion Price Processes

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Abstract: Using the Ray-Knight theorem we give conditions for a nonnegative diffusion without drift to reach zero or not. These results also give necessary and sufficient conditions for such a diffusion process to be a martingale (and not just a local martinagle). We apply these results in order to give necessary and sufficient conditions for nonnegative diffusiosn to have equivalent local martingale measures.

Keywords: Equivalent Martingale Measure, Arbitrage, Ray-Knight Theorem.

1 Local Martingale Analysis

First we shall consider the following Markovian local martingale model :

$$\begin{aligned}&dX_t = \sigma(X_t)\,dW_t,\ X_0 = 1,\\ &\sigma(\cdot) \text{ satisfies } \forall\epsilon > 0,\ M < \infty, +\infty > \sup_{\epsilon\le u\le M}\sigma(u) \ge \inf_{\epsilon\le u\le M}\sigma(u) > 0.\end{aligned}$$

Furthermore the process will be stopped as soon as the origin is reached. This means that we will suppose that $\sigma(x) = 0$ for all $x \le 0$. The purpose of this section is to analyse when the process will reach 0. Of course this is an easy consequence of Feller's criterion, but our presentation is different. We will use the construction of the weak solution by means of the method of time changes. Using the Ray-Knight theorem will then allow us find the necessary and sufficient condition for X to reach the origin. Afterwards we will transform the equation and we will find a criterion that allows us to see when the solution X is not only a local martingale but is a genuine martingale. These results will be used in section 2 to solve the problem when a nonnegative diffusion process admits an equivalent local martingale measure.

Let us now present the weak solution of $dX_t = \sigma(X_t)\,dW_t$. First we take a Brownian Motion W starting at 1, i.e. $W_0 = 1$. Next we define $T_0 = \inf\{t \mid W_t = 0\}$, the hitting time of 0. The increasing process A is defined as $A_t = \int_0^t \frac{du}{\sigma^2(W_u)}$. We remark that from the boundedness properties of σ and from the continuity of the Brownian motion, it follows that for all $t < T_0$ we have that $A_t < +\infty$. The value $A_{T_0} \le +\infty$ will play a special role. The inverse function of A is defined as

$$C_t = \inf\{s \mid A_s \ge t\}.$$

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If $A_{T_0} = +\infty$, then we have that C_t is defined for all times t. In case $A_{T_0} < +\infty$, we have that C_t is only defined for $t \le A_{T_0}$. For $t \le A_{T_0}$ we define

$$S_t = W_{C_t}.$$

For $t \le A_{T_0}$, $dS_t = \sigma(S_t)\, dW'_t$, where W' is some Brownian motion. This construction is known as the method of time changes and it is an easy way to find the weak solution of the stochastic equation. We refer the reader to [1], chapter IX, section 1 (the reader should also have a look at exercise 1.16.) as well as the basic reference [2]. From the results there it follows that the equation has only one weak solution. For later use we also remark that

$$dC_t = \sigma^2(X_t)\, dt,$$

which follows easily from the definiton of C as the inverse function of the strictly increasing process A.

The above construction allows us to state the following

Proposition 1.1 *The process X defined by $dX_t = \sigma(X_t)\, dW_t$ reaches 0 in finite time if and only if $A_{T_0} < \infty$.*

The next point in the programme is to find a condition under which $A_{T_0} < +\infty$. This is done through the Ray-Knight theorem, see [1].

Theorem 1.2 (Ray-Knight) *Let B be a standard Brownian motion and let l^a be its family of local times. Let T_1 be the first time B hits the level 1, i.e. $T_1 = \inf\{u \mid B_u = 1\}$. The process $Z_a; 0 \le a \le 1$ defined as $Z_a = l_{T_1}^{1-a}$ is a BESQ^2(0) process.*

Because we were working with a Brownian motion W that starts at the point 1, we have to translate a couple of things. So we will apply the Ray-Knight theorem to the process

$$B = 1 - W.$$

In doing so, we notice that B hits the level one if W hits the zero level. Furthermore the local time of B at the point $1 - a$ is the same as the local time L^a of W at the point a. So we have that

Theorem 1.3 *The process $L^a_{T_0}$, $0 \le a \le 1$ has the same law as the BESQ^2(0) process.*

Local times are especially useful when we want to calculate the image measure of a Brownian motion. More precisely, see [1], chapter VI, section 1, we have for every nonnegative Borel function f that

$$\int_0^{T_0} f(W_u)\, du = \int_0^{+\infty} f(x) L^x_{T_0}\, dx.$$

Applied to the increasing process A this gives

$$A_{T_0} = \int_0^{+\infty} \frac{1}{\sigma^2(x)}\, L^x_{T_0}\, dx = \int_0^1 \frac{1}{\sigma^2(x)}\, L^x_{T_0}\, dx + \int_1^{+\infty} \frac{1}{\sigma^2(x)}\, L^x_{T_0}\, dx.$$

But the second term is always finite and we get that

$$A_{T_0} < +\infty \text{ if and only if } \int_0^1 \frac{1}{\sigma^2(x)}\, L^x_{T_0}\, dx < +\infty.$$

Because of the Ray-Knight theorem we can replace the local times by a BESQ^2 process. So let ${}^1B$ and ${}^2B$ be two independent standard Brownian motions, defined on some probability space. We get that

$$A_{T_0} < +\infty \text{ almost surely if and only if } \int_0^1 \frac{1}{\sigma^2(x)}\, \left(({}^1B_x)^2 + ({}^2B_x)^2\right)\, dx < +\infty \text{ almost surely.}$$

By independence this is the same as (here B denotes a Brownian motion)

$$A_{T_0} < +\infty \text{ almost surely if and only if } \int_0^1 \frac{1}{\sigma^2(x)}\, (B_x)^2\, dx < +\infty \text{ almost surely.}$$

This equivalence is the basis for the following characterisation theorem

Theorem 1.4 *Let the process X be defined by the equation $dX_t = \sigma(X_t)\, dW_t$. Let us suppose that σ satisfies the boundedness condition: for all $\epsilon > 0$ and all $\epsilon \le M < \infty$, $+\infty > \sup_{\epsilon\le u\le M} \sigma(u) \ge \inf_{\epsilon\le u\le M} \sigma(u) > 0$. Then we have two alternatives*

- *either $\int_0^1 \frac{x}{\sigma^2(x)}\, dx = +\infty$ and X does not reach the origin in finite time*
- *or $\int_0^1 \frac{x}{\sigma^2(x)}\, dx < +\infty$ and with certainty X reaches the origin in finite time.*

Proof. Let B be a standard Browmian motion. If $\int_0^1 \frac{x}{\sigma^2(x)}\, dx < +\infty$, then clearly, by Fubini's theorem, we have that $E[\int_0^1 \frac{B_x^2}{\sigma^2(x)}\, dx] < +\infty$ and this implies that almost surely $\int_0^1 \frac{B_x^2}{\sigma^2(x)}\, dx < +\infty$.

Conversely suppose that $\int_0^1 \frac{B_x^2}{\sigma^2(x)}\, dx < +\infty$ on a set of strictly positive measure, then we may suppose that for some N we have that $\int_0^1 \frac{B_x^2}{\sigma^2(x)}\, dx \le N$ on a set A of measure at least $\delta > 0$. Take now $\epsilon > 0$ so that $P[B_x^2 \le \epsilon x] \le \delta/4$. We remark that this ϵ is independent of x since for all x we have that $B_x/\sqrt{x}$ has a standard normal distribution. We now have the following inequalities:

$$N \ge \int_0^1 \frac{B_x^2}{\sigma^2(x)} 1_A\, dx \ge \int_0^1 \frac{\epsilon x}{\sigma^2(x)} 1_A 1_{\{B_x^2\ge\epsilon x\}}\, dx$$

Integration gives us that

$$N \ge \int_0^1 \frac{\epsilon x}{\sigma^2(x)} P[A\cap\{B_x^2 \ge \epsilon x\}]\, dx \ge \int_0^1 \frac{\epsilon x}{\sigma^2(x)} (\delta/2)\, dx$$

This clearly implies that $\int_0^1 \frac{x}{\sigma^2(x)}\, dx < +\infty$. $\square$

Remark 1.5 *In case the process X reaches zero in finite time we have that for all $\epsilon > 0$, necessarily $P[X_\epsilon = 0] > 0$. This is a standard application of the Markov property.*

We now turn to the question whether the process X can be a martingale. In case X is a martingale we have for all t that the measure defined as $dQ = X_t dP$ is a probability measure. Of course $Q[X_u \ne 0] = 1$ for all $u \le t$. This means that for the measure Q, the process X does not become zero. A straightforward application of Itô's formula and the measure transformation (the Girsanov-Maruyama formula, see [1] where also the case of non-equivalent measures, due to Lenglart, is treated) yields that the process $Y_u = 1/X_u$ satisfies the equation

$$dY_u = Y_u^2 \sigma(1/Y_u)\, dW'_u \text{ for some } Q\text{−Brownian motion } W' \text{ and for } u \le t.$$

Since the process Y does not reach zero before time t, we may apply the previous remark and we necessarily must have that

$$\int_0^1 \frac{y}{y^4\sigma^2(1/y)}\, dy = \infty.$$

This is the same as

$$\int_1^{+\infty} \frac{x}{\sigma^2(x)}\, dx = +\infty.$$

Suppose conversely that we have $\int_1^{+\infty} \frac{x}{\sigma^2(x)}\, dx = +\infty$. Let us define the following function

$$\Psi(x) = x \text{ for } x \le 1 \tag{1.1}$$

$$\Psi(x) = 1 + \int_1^x \frac{u}{\sigma^2(u)}(x-u)\, du \text{ for } x \ge 1. \tag{1.2}$$

This also means that for $x \ge 1$ we have that $\Psi''(x) = \frac{x}{\sigma^2(x)}$. We therefore get that

$$d\Psi(X_t) = \Psi'(X_t)\, dX_t + \frac{1}{2} 1_{X_t\ge1}\, X_t\, dt$$

For every n and fixed t we now define the stopping time τ_n as

$$\tau_n = \inf\{u \mid X_u \geq n\} \wedge t.$$

The process X remains bounded by n before time τ_n. Of course we must have that $E[X_{\tau_n}] = 1$, since up to time τ_n the local martingale X is a martingale. But the boundedness of the stopped process also implies that the martingale component (coming from $\Psi'(X_t)\, dX_t$) has expectation zero. Since in any case we have $E[X_u] \leq 1$ we have that

$$E[\Psi(X_{\tau_n})] = \Psi(1) + E\left[\int_0^{\tau_n} \frac{1}{2} 1_{X_u \geq 1}\, X_u\, du\right] \leq 1 + t.$$

But as easily seen $\int_1^{+\infty} \frac{x}{\sigma^2(x)}\, dx = +\infty$ implies that the function Ψ satisfies

$$\lim_{x \to +\infty} \frac{\Psi(x)}{x} = +\infty.$$

The criterion of de la Vallée Poussin then shows that the sequence X_{τ_n} is uniformly integrable and therefore we have that for all t

$$E[X_t] = \lim_n E[X_{\tau_n}] = 1.$$

We therefore proved the following

Theorem 1.6 *Under the conditions of theorem 1.4 we have that the process X is a martingale, i.e. $E[X_t] = 1$ for all t, if and only if $\int_1^{+\infty} \frac{x}{\sigma^2(x)}\, dx = +\infty$.*

As an application we can state the following now facts for the equation ($\sigma > 0$ is a constant):

$$dX_t = \sigma X_t^\rho dW_t.$$

1 If $\rho = 1$ we get a geometric Brownian motion, it does not reach zero and it is a martingale. It cannot be closed at ∞.

2 If $\rho < 1$ the process X is a martingale but with certainty it reaches zero in finite time. It cannot be closed at ∞.

3 If $\rho > 1$ the process X does not reach zero but it is only a local martingale. The reader can find out the relation with Bessel processes.

2 General Model Analysis

Next we shall consider the existence of an equivalent local martingale measure for the following general Markovian stock price model

$$dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = 1$$

where W is a P-Brownian motion and where the process is stopped when it reaches zero. Of course if such an equivalent measure, Q, exists we have that under Q, the process follows the equation

$$dX_t = \sigma(X_t)dW'_t, \quad X_0 = 1$$

for some $Q-$ Brownian motion W' and where again the process is stopped when it reaches zero. The time horizon is supposed to be finite, i.e. we work with the time interval $[0, t_0]$, where $t_0 < +\infty$. All stopping times take values in the interval $[0,t_0]$, in particular we suppose that $\inf\emptyset = t_0$. We furthermore assume the following conditions.

(i) $X_0 = 1$ and X_t is stopped when 0 is reached.

(ii) $dX_t = b(X_t)dt + \sigma(X_t)dW_t, \quad X_0 = 1,$ has a unique weak solution, defining the law P on the space $C[0,t_0]$ of continuous functions.

(iii) $dX_t = \sigma(X_t)dW'_t, \quad X_0 = 1,$ has a unique weak solution, defining the law P' on the space $C[0,t_0]$ of continuous functions.

(iv) $\sigma(\cdot)$ is bounded and bounded away from zero on compact sets of $(0,\infty)$.

(v) $b(\cdot)$ is bounded on compact intervals of $(0,\infty)$.

(vi) b and σ are Borel measurable

The problem is to find necessary and sufficient conditions under which the two measures P and P' are equivalent. This is of course not always the case as the following heuristic reasoning explains. If 0 is reached under P' (i.e. $\int_0^1 \frac{u}{\sigma^2(u)}du < \infty$), and is not reached under P (i.e. b is very positive in a neighborhood of 0), then clearly P and P' cannot be equivalent.

Let us define

$$\begin{aligned} Z_t &= \exp\left\{-\int_0^t \frac{b(X_u)}{\sigma(X_u)}dW_u - \frac{1}{2}\int_0^t \frac{b^2(X_u)}{\sigma^2(X_u)}du\right\}, \\ Z'_t &= \exp\left\{\int_0^t \frac{b(X_u)}{\sigma(X_u)}dW'_u - \frac{1}{2}\int_0^t \frac{b^2(X_u)}{\sigma^2(X_u)}du\right\}. \end{aligned}$$

Clearly Z *should* describe $\frac{d\mathbf{P}'}{d\mathbf{P}}$ and Z' *should* describe $\frac{d\mathbf{P}}{d\mathbf{P}'}$. Here "should" comes from the fact the they are only local martingales and there is no need that $P \ll P'$ or $P' \ll P$.

Let us fix the time horizon t_0 to define

$$\begin{aligned} \nu &= \inf\{u \;;\; X_u = 0\}, \\ \tau &= \inf\{u \;;\; W_u = 0\} \; (W_0 = 1), \\ \tau_n &= \inf\{u \;;\; Z'_u \geq n \text{ or } Z'_u \leq \frac{1}{n}\}. \end{aligned}$$

Clearly on $\mathcal{F}_{\tau_n}$, we have

$$\begin{aligned} P|_{\mathcal{F}_{\tau_n}} &\sim P'|_{\mathcal{F}_{\tau_n}}, \\ \left.\frac{dP}{dP'}\right|_{\mathcal{F}_{\tau_n}} &= Z'_{\tau_n}, \\ \left.\frac{dP'}{dP}\right|_{\mathcal{F}_{\tau_n}} &= Z_{\tau_n}, \\ Z_{\tau_n}Z'_{\tau_n} &= 1. \end{aligned}$$

Under P', Z' is certainly defined and Z' can become 0 only if $\int_0^\cdot \frac{b^2(X_u)}{\sigma^2(X_u)}du$ becomes ∞.

With our *realistic* hypothesis on $b(\cdot)$, $\sigma(\cdot)$, this means Z' remains > 0 and it can only become zero at time ν. That is, in the notation of section 1, where we introduced the solution by means of the method of time changes:

$$Z'_{t_0} > 0 \; P'\text{-a.s.} \iff \int_0^\nu \frac{b^2(X_u)}{\sigma^2(X_u)}du < \infty \; P'\text{-a.s.}$$

But as pointed out in section 1, we have that

$$\int_0^\nu \frac{b^2(X_u)}{\sigma^2(X_u)}du \overset{\text{(Law)}}{=} \int_0^{A_{T_0}} \frac{b^2(W_{C_u})}{\sigma^4(W_{C_u})}dC_u.$$

Hence, from the analysis in section 1 and because of the assumptions on b and σ, we get

$$Z'_{t_0} > 0 \; P'\text{-a.s.} \iff \int_0^{T_0} \frac{b^2(W_s)}{\sigma^4(W_s)}ds < \infty \iff \int_0^1 \frac{b^2(x)x}{\sigma^4(x)}dx < \infty.$$

Summarizing we have the following result :

(A) If $\displaystyle\int_0^1 \frac{u}{\sigma^2(u)}du = \infty$, then under P', X does not attain 0 and $Z'_{t_0} > 0$ P'-a.s..

(B) If $\displaystyle\int_0^1 \frac{u}{\sigma^2(u)}du < \infty$, then under P', X reaches 0 and $Z'_{t_0} > 0$ P'-a.s. $\iff$ $\displaystyle\int_0^1 \frac{b^2(u)}{\sigma^4(u)}u\,du < \infty$.

In both cases where we have $Z_{t_0} > 0$, $P'-$ a.s., we have $P' \ll P$ on $\mathcal{F}_{t_0}$

Indeed take $A \in \mathcal{F}_{t_0}$ such that $P[A] = 0$. Then

$$P[A \cap \{\tau_n > t_0\}] = 0 \;\forall\; n,$$

and hence

$$P'[A \cap \{\tau_n > t_0\}] = 0 \;\forall\; n.$$

Since $\cup_{n\geq 1}\{\tau_n > t_0\} = \Omega\ P'-a.s.$, we have $P'[A] = 0$.

In the same way we can show

Theorem 2.1 *If $Z_{t_0} > 0$ P-a.s., then $P \ll P'$ on $\mathcal{F}_{t_0}$.*

Proof. Let $\sigma_n = \inf\{t\ ;\ Z_t \geq n \text{ or } Z_t \leq \frac{1}{n}\}$. Take $A \in \mathcal{F}_{t_0}$ such that $P'[A] = 0$. Then

$$P'[A \cap \{\sigma_n > t_0\}] = 0 \;\forall\; n,$$

and hence

$$P[A \cap \{\sigma_n > t_0\}] = 0 \;\forall\; n.$$

Since $\cup_n\{\sigma_n > t_0\} = \Omega$ P-a.s., we have $P[A] = 0$. This concludes the proof. $\square$

The results can also be seen as follows.

$$Z_{t_0\wedge\tau_n} Z'_{t_0\wedge\tau_n} = 1,\ P \text{ and } P'\text{-a.s..}$$

So when $Z_{t_0\wedge\tau_n} \to 0$ somewhere P, then $Z'_{t_0\wedge\tau_n} \to \infty$ somewhere P. Since $Z'_{t_0\wedge\tau_n}$ remains bounded as for P', we cannot have $P \ll P'$. We can use the same argument for $Z'_{t_0\wedge\tau_n} \to 0$ somewhere P. Hence we get,

$$\begin{array}{rcll} Z_{t_0} > 0\ P\text{-a.s.} & \iff & P \ll P' & \text{and then}\quad E_{P'}[Z'_{t_0}] = 1, \\ Z'_{t_0} > 0\ P'\text{-a.s.} & \iff & P' \ll P & \text{and then}\quad E_P[Z_{t_0}] = 1. \end{array}$$

Besides Feller's condition and its proof, is not easy to give a condition under which 0 is reached for the measure P . Since we do not have a nicer method we will summarize how it works. Contrary to the presentation in section 1, this is standard material. We will not give all details, since the convergence or divergence of the integrals is done exactly inthe same way as in section 1.

First we introduce the function h which is the solution of the equation

$$\frac{1}{2}\sigma^2(x)h''(x) + h'(x)b(x).$$

The solution of this equation (under our boundedness assumptions) is given by (x_0 still to be chosen in R_+):

$$h(x) = \int_{x_0}^x \exp\left(\int_{x_0}^u \frac{-2b(s)}{\sigma^2(s)}\,ds\right)\,du.$$

Since the function h is stricly increasing it has an inverse funtion h^{-1}. Under the measure P the process $Y = h(X)$ is a local martingale that satisfies

$$dY_t = h'(X_t)\sigma(X_t)\,dW_t \quad\text{ or }\quad dY_t = h'(h^{-1}(Y_t))\sigma(h^{-1}(Y_t))\,dW_t.$$

The function h is bounded above on compact intervals of R_+. The function h can however tend to $-\infty$ at zero, in which case $x_0 = 0$ cannot be used as a normalisation.

Depending on the convergence of certain integrals there are two normalisations possible

The first case: $h(0) = -\infty$. In this case we could take $x_0 = 1$. We can continue the analysis a little bit further. If one stops the process X when it reaches an arbitrary level, say n, we get a process Y^n that is a local martingale bounded above. This is necessarily a submartingale and therefore also bounded below a.s.. This means that the process X cannot reach zero in finite time. So we get that under the condition $h(0) = -\infty$, the process X cannot reach zero.

The second case: $h(0) = 0$. In this case we take $x_0 = 0$ and we can further multiply the function h with a constant in such a way that we get $h(1) = 1$. Whether the point 0 can be reached or not is not obvious. Applying the criterion of section 1 to the process Y gives us that the process X (or equivalently the process $h(X) = Y$) will reach zero if and only if

$$\int_0^1 \frac{x}{\left(h'(h^{-1}(x))\sigma(h^{-1}(x))\right)^2}\, dx < +\infty.$$

Making the obvious substitution $x = h(y)$ gives us that this condition is equivalent to the condition

$$\int_0^1 \frac{h(y)}{h'(y)\sigma^2(y)}\, dx < +\infty.$$

The case $b = 0$ and $\sigma(x) = x$ provides a case where $h(0) = 0$ and where the previous integral diverges. So the condition is not superfluous.

We can now summarize the results. We only give the necessary and sufficient conditions for the measures P and P' to be equivalent. The characterisation of the case $P \ll P'$ is left as an exercise.

1 $\int_0^1 \frac{x}{\sigma^2(x)}\, dx = +\infty$. Then the measures are equivalent if and only if either $h(0) = -\infty$ or $h(0) = 0$ and $\int_0^1 \frac{h(y)}{h'(y)\sigma^2(x)}\, dx = +\infty$.

2 $\int_0^1 \frac{x}{\sigma^2(x)}\, dx < +\infty$. Then the two measures are equivalent if and only if $\int_0^1 \frac{b^2(u)}{\sigma^4(u)} u\, du < \infty$ and $h(0) = 0$ and $\int_0^1 \frac{h(y)}{h'(y)\sigma^2(x)}\, dy < +\infty$ and $\int_0^1 \frac{b^2(y)h(y)}{h'(y)\sigma^4(y)}\, dy < +\infty$.

We will not further analyse the dependece between the convergence of all these integrals. It depends on the behaviour of b and σ around the origin. As the reader can see this is a matter of delicate real analysis and it goes beyond the scope of this paper. For practical applications we suggest that the reader should follow the discussion above and should not simply check the convergence/divergence of the integrals.

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