Coherent and convex risk measures for bounded càdlàg processes

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Abstract

If the random future evolution of values is modelled in continuous time, then a risk measure can be viewed as a functional on a space of continuous-time stochastic processes. We extend the notions of coherent and convex risk measures to the space of bounded càdlàg processes that are adapted to a given filtration. Then, we prove representation results that generalize earlier results for one- and multi-period risk measures, and we discuss some examples.

Key words: Coherent risk measures, convex risk measures, coherent utility functionals, concave money based utility functionals, càdlàg processes, representation theorem.

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1 Introduction

The notion of a coherent risk measure was introduced in [ADEH1] and [ADEH2], where it was also shown that every coherent risk measure on the space of all random variables on a finite probability space can be represented as a supremum of linear functionals. In [De1] (see also [De2]), the concept of a coherent risk measure was extended to general probability spaces, and applications to risk measurement, premium calculation and capital allocation problems were presented. It turned out that the definitions and results of [ADEH1] and [ADEH2] have a direct analog in the setting of a general probability space if one restricts risk measures to the space of bounded random variables. On the space of all random variables, coherent risk measures can in general only exist if they are allowed to take the value ∞. In [ADEHK1] and [ADEHK2], the results of [ADEH1] and [ADEH2] were

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generalized to multi-period models. In [FS1] and [FS2] the more general concept of a convex risk measure was introduced and representation results of [De1] were generalized to convex risk measures on the space of all bounded variables on a general probability space.

The purpose of this paper is the study of risk measures that take into account the future evolution of values over a whole time-interval rather than at just finitely many times. Our main focus will be on representation results for such risk measures.

We model the future evolution of a discounted value by a stochastic process $(X_t)_{t\in[0,T]}$, and we call such a process X a discounted value process. The use of discounted values is common practice in finance. It just means that we use a numéraire (which may also be modelled as a stochastic process) and measure all values in multiples of the numéraire. A few of the many possible interpretations of a discounted valued process are:

- the evolution of the discounted market value of a firm's equity
- the evolution of the discounted accounting value of a firm's equity
- the evolution of the discounted market value of a portfolio of financial securities
- the evolution of the discounted surplus of an insurance company

(see also Subsection 2.1 of [ADEHK2]).

Since one might want to build new discounted value processes from old ones by adding and scaling them, it is natural to let the class of discounted value processes considered for risk measurement be a vector space. In this paper the class of discounted value processes is the space $\mathcal{R}^\infty$ of bounded càdlàg processes that are adapted to the filtration of a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, P)$ that satisfies the usual assumptions. Strictly speaking, by bounded, we mean essentially bounded, and we identify indistinguishable processes. In doing so we are referring to the probability measure P. However, the space $\mathcal{R}^\infty$ stays invariant if we change to an equivalent probability measure. Hence, by introducing P, we only specify the set of events with probability zero. For modelling purposes, the space $\mathcal{R}^0$ of all càdlàg processes that are adapted to the filtration $(\mathcal{F}_t)$ is more interesting than $\mathcal{R}^\infty$. Note that $\mathcal{R}^0$ is also invariant under change to an equivalent probability measure. The reason why in this paper we work with $\mathcal{R}^\infty$ is that in contrast to $\mathcal{R}^0$, it easily lends itself to the application of duality theory, which will be crucial in the proof of Theorem 3.3, the main result of this paper. In a forthcoming paper we will study risk measures on $\mathcal{R}^0$ and discuss conditions under which a convex risk measure on $\mathcal{R}^\infty$ can be extended to $\mathcal{R}^0$. Whereas we require coherent and convex risk measures on $\mathcal{R}^\infty$ to be real-valued, we will allow coherent and convex risk measures on $\mathcal{R}^0$ to take values in $(-\infty, \infty]$.

Although we consider continuous-time discounted value processes, the risk measures treated in this paper are static as we only measure the risk of a discounted value process at the beginning of the time period. In [ADEHK1], [ADEHK2] and [ES] one can find a discussion of dynamic risk measures for random variables in a discrete-time framework. Dynamic risk measures for stochastic processes in a continuous-time setup is the subject of ongoing research.

The structure of the paper is a follows: Section 2 contains notation and definitions. In Section 3, we state results on the representation of coherent and convex risk measures for bounded càdlàg processes and sketch those proofs that are simple generalizations of proofs in [De1] or [FS1]. We will also show how results of [De1] and [FS1] can be extended to representation results for real-valued coherent and convex risk measures on the spaces $\mathcal{R}^p := \left\{X \in \mathcal{R}^0 \mid \sup_{0 \le t \le T} |X_t| \in L^p\right\}$ for $p \in [1, \infty)$. In Section 4, we give that part of the proofs of the results of Section 3 which is not a straightforward extension of arguments from [De1] or [FS1]. Section 5 contains examples of coherent and convex risk measures for continuous-time stochastic processes.

2 Notation and Definitions

Let $T \in (0, \infty)$ and $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0 \le t \le T}, P)$ be a filtered probability space that satisfies the usual assumptions, that is, the probability space $(\Omega, \mathcal{F}, P)$ is complete, $(\mathcal{F}_t)$ is right-continuous and $\mathcal{F}_0$ contains all null-sets of $\mathcal{F}$. For all $p \in [1, \infty]$, we write L^p for the space $L^p(\Omega, \mathcal{F}, P)$.

As in [DM2] we set for an $(\mathcal{F}_t)$-adapted, càdlàg process $(X_t)_{t \in [0,T]}$ and $p \in [1, \infty]$,

$$||X||_{\mathcal{R}^p} := ||X^*||_p \,, \text{ where } X^* := \sup_{0 \le t \le T} |X_t| \,.$$

Obviously, equipped with this norm, the space

$$\mathcal{R}^p := \left\{ X \,:\, [0,T] \times \Omega \to \mathbb{R} \;\middle|\; \begin{array}{l} X \text{ càdlàg} \\ (\mathcal{F}_t)\text{-adapted} \\ ||X||_{\mathcal{R}^p} < \infty \end{array} \right\}$$

is a Banach space. For a stochastic process $b : [0,T] \times \Omega \to \mathbb{R}$ with right-continuous paths of finite variation, there exists a unique decomposition $b = b^+ - b^-$ such that b^+ and b^- are stochastic processes with right-continuous, non-decreasing paths, and almost surely, the nonnegative measures induced by b^+ and b^- on $[0,T]$ have disjoint support. The variation of such a process is given by the random variable $\mathrm{Var}\,(b) := b^+(T) + b^-(T)$. If b is optional (predictable), both processes b^+ and b^- are optional (predictable).

For $q \in [1, \infty]$, we set

$$\mathcal{A}^q := \left\{ a \,:\, [0,T] \times \Omega \to \mathbb{R}^2 \;\middle|\; \begin{array}{l} a = (a^{\mathrm{pr}}, a^{\mathrm{op}}) \\ a^{\mathrm{pr}}, a^{\mathrm{op}} \text{ right continuous, finite variation} \\ a^{\mathrm{pr}} \text{ predictable, } a_0^{\mathrm{pr}} = 0 \\ a^{\mathrm{op}} \text{ optional, purely discontinuous} \\ \mathrm{Var}\,(a^{\mathrm{pr}}) + \mathrm{Var}\,(a^{\mathrm{op}}) \in L^q \end{array} \right\} ,$$

It can be shown that $\mathcal{A}^q$ with the norm

$$||a||_{\mathcal{A}^q} := ||\mathrm{Var}\,(a^{\mathrm{pr}}) + \mathrm{Var}\,(a^{\mathrm{op}})\,||_q \,,\; a \in \mathcal{A}^q \,,$$

is also a Banach space. We set

$$\mathcal{A}^q_+ := \{a = (a^{\mathrm{pr}}, a^{\mathrm{op}}) \in \mathcal{A}^q \mid a^{\mathrm{pr}} \text{ and } a^{\mathrm{op}} \text{ are non-decreasing}\} \,.$$

It can easily be checked that for all $p, q \in [1, \infty]$ such that $p^{-1} + q^{-1} = 1$,

$$\langle X, a \rangle := \mathrm{E}\left[\int_{]0,T]} X_{t-} da_t^{\mathrm{pr}} + \int_{[0,T]} X_t da_t^{\mathrm{op}}\right] \tag{2.1}$$

is a well-defined bilinear form on $\mathcal{R}^p \times \mathcal{A}^q$, and

$$|\langle X, a \rangle| \le ||X||_{\mathcal{R}^p} ||a||_{\mathcal{A}^q} \quad \text{for all } X \in \mathcal{R}^p \text{ and } a \in \mathcal{A}^q \,. \tag{2.2}$$

Remark 2.1
It is easy to see that two different elements a and b of $\mathcal{A}^q$ induce different linear forms on $\mathcal{R}^p$. But there exist other pairs of processes in the more general space

$$\tilde{\mathcal{A}}^q := \left\{ a : [0,T] \times \Omega \to \mathbb{R}^2 \;\middle|\; \begin{array}{l} a = (a^{\mathrm{pr}}, a^{\mathrm{op}}) \\ a^{\mathrm{pr}}, a^{\mathrm{op}} \text{ right continuous, finite variation} \\ a^{\mathrm{pr}} \text{ predictable, } a_0^{\mathrm{pr}} = 0 \\ a^{\mathrm{op}} \text{ optional} \\ \mathrm{Var}\,(a^{\mathrm{pr}}) + \mathrm{Var}\,(a^{\mathrm{op}}) \in L^q \end{array} \right\} ,$$

that define the same linear form on $\mathcal{R}^p$ as a given $a \in \mathcal{A}^q$, for instance, $(a^{\mathrm{pr}} + c, a^{\mathrm{op}} - c)$ for any continuous, adapted, finite variation process c such that $c_0 = 0$ and $\mathrm{Var}\,(c) \in L^q$. However, if $(\tilde{a}^{\mathrm{pr}}, \tilde{a}^{\mathrm{op}}) \in \tilde{\mathcal{A}}^q$ induces the same linear form on $\mathcal{R}^p$ as $(a^{\mathrm{pr}}, a^{\mathrm{op}}) \in \mathcal{A}^q$, the process $\tilde{a}^{\mathrm{op}}$ can be split into a purely discontinuous, optional, finite variation process $\tilde{a}^{\mathrm{d}}$ and a continuous, finite variation process $\tilde{a}^{\mathrm{c}}$ such that $\tilde{a}_0^{\mathrm{c}} = 0$. Then, $(a^{\mathrm{pr}}, a^{\mathrm{op}}) = (\tilde{a}^{\mathrm{pr}} + \tilde{a}^{\mathrm{c}}, \tilde{a}^{\mathrm{d}})$ and $\mathrm{Var}\,(\tilde{a}^{\mathrm{op}}) = \mathrm{Var}\,(\tilde{a}^{\mathrm{c}}) + \mathrm{Var}\,\left(\tilde{a}^{\mathrm{d}}\right)$. Hence,

$$\begin{aligned} & \mathrm{Var}\,(a^{\mathrm{pr}}) + \mathrm{Var}\,(a^{\mathrm{op}}) = \mathrm{Var}\,(\tilde{a}^{\mathrm{pr}} + \tilde{a}^{c}) + \mathrm{Var}(\tilde{a}^{\mathrm{d}}) \\ \le\; & \mathrm{Var}\,(\tilde{a}^{\mathrm{pr}}) + \mathrm{Var}\,(\tilde{a}^{c}) + \mathrm{Var}\,(\tilde{a}^{\mathrm{op}}) - \mathrm{Var}\,(\tilde{a}^{c}) = \mathrm{Var}\,(\tilde{a}^{\mathrm{pr}}) + \mathrm{Var}\,(\tilde{a}^{\mathrm{op}}) \,, \end{aligned}$$

which shows that

$$||\mathrm{Var}\,(a^{\mathrm{pr}}) + \mathrm{Var}\,(a^{\mathrm{op}})\,||_q \le ||\mathrm{Var}\,(\tilde{a}^{\mathrm{pr}}) + \mathrm{Var}\,(\tilde{a}^{\mathrm{op}})\,||_q \,.$$

Definition 2.2 *A convex risk measure on $\mathcal{R}^\infty$ is a mapping $\rho : \mathcal{R}^\infty \to \mathbb{R}$ that satisfies the following properties:*

(1) $\rho\,(\lambda X + (1-\lambda) Y) \le \lambda \rho\,(X) + (1-\lambda)\rho\,(Y)$, *for all $X, Y \in \mathcal{R}^\infty$ and $\lambda \in [0,1]$*

(2) $\rho(X) \ge \rho(Y)$ *for all $X, Y \in \mathcal{R}^\infty$ such that $X \le Y$*

(3) $\rho\,(X + m) = \rho\,(X) - m$, *for all $X \in \mathcal{R}^\infty$ and $m \in \mathbb{R}$.*

A coherent risk measure on $\mathcal{R}^\infty$ is a convex risk measure ρ on $\mathcal{R}^\infty$ that satisfies the additional property:

(4) $\rho\,(\lambda X) = \lambda \rho\,(X)$, *for all $X \in \mathcal{R}^\infty$ and $\lambda \in \mathbb{R}_+$.*

We find it more convenient to work with the negative of a risk measure $\phi := -\rho$. We call ϕ the utility functional corresponding to the risk measure ρ. In terms of ϕ, Definition 2.2 can be rephrased as follows:

Definition 2.2' *A concave money based utility functional on $\mathcal{R}^\infty$ is a mapping $\phi : \mathcal{R}^\infty \to \mathbb{R}$ that satisfies the following properties:*

(1) $\phi(\lambda X + (1-\lambda)Y) \geq \lambda\phi(X) + (1-\lambda)\phi(Y)$, *for all* $X, Y \in \mathcal{R}^\infty$ *and* $\lambda \in [0,1]$

(2) $\phi(X) \leq \phi(Y)$ *for all* $X, Y \in \mathcal{R}^\infty$ *such that* $X \leq Y$

(3) $\phi(X+m) = \phi(X) + m$, *for all* $X \in \mathcal{R}^\infty$ *and* $m \in \mathbb{R}$.

A coherent utility functional on $\mathcal{R}^\infty$ is a concave money based utility functional ρ on $\mathcal{R}^\infty$ that satisfies the additional property:

(4) $\phi(\lambda X) = \lambda\phi(X)$, *for all* $X, Y \in \mathcal{R}^\infty$ *and* $\lambda \in \mathbb{R}_+$.

We say that a concave money based utility functional ϕ on $\mathcal{R}^\infty$ satisfies the Fatou property if

$$\limsup_{n\to\infty} \phi(X^n) \leq \phi(X)$$

for every bounded sequence $(X^n)_{n\geq 1} \subset \mathcal{R}^\infty$ and $X \in \mathcal{R}^\infty$ such that $(X^n - X)^* \xrightarrow{P} 0$.

We say that ϕ is continuous for bounded decreasing sequences if

$$\lim_{n\to\infty} \phi(X^n) = \phi(X)$$

for every decreasing sequence $(X^n)_{n\geq 1} \subset \mathcal{R}^\infty$ such that $(X^n - X)^* \xrightarrow{P} 0$ for some $X \in \mathcal{R}^\infty$.

Remark 2.3 It can be deduced from the properties (2) and (3) of Definition 2.2' that every concave money based utility functional ϕ on $\mathcal{R}^\infty$ is Lipschitz-continuous with respect to the $\mathcal{R}^\infty$-norm, that is,

$$|\phi(X) - \phi(Y)| \leq ||X - Y||_{\mathcal{R}^\infty}, \quad \text{for all } X, Y \in \mathcal{R}^\infty. \tag{2.3}$$

The acceptance set $\mathcal{C}$ corresponding to a concave money based utility functional ϕ on $\mathcal{R}^\infty$ is given by

$$\mathcal{C} := \{X \in \mathcal{R}^\infty \mid \phi(X) \geq 0\}.$$

Obviously, it is convex and has the following property: If $X, Y \in \mathcal{R}^\infty$, $X \in \mathcal{C}$ and $X \leq Y$, then $Y \in \mathcal{C}$ as well. If ϕ is coherent, then $\mathcal{C}$ is a convex cone.

3 Representation results

In this section we give results on the representation of coherent and concave money based utility functionals on $\mathcal{R}^\infty$ that extend Theorem 3.2 of [De1] and Theorem 6 of [FS1]. At the end of the section we also discuss representation results for real-valued concave money based utility functionals on $\mathcal{R}^p$ for $p \in [1,\infty)$.

In our framework the set

$$\mathcal{D}_\sigma := \{a \in \mathcal{A}^1_+ \,|\, ||a||_{\mathcal{A}^1} = 1\}$$

plays the role that is played by the set $\{f \in L1 \,|\, f \geq 0\,,\, \mathrm{E}[f] = 1\}$ in the papers [De1] and [FS1].

Definition 3.1 *We call a function $\gamma : \mathcal{D}_\sigma \to [-\infty,\infty)$ a penalty function if*

$$-\infty < \sup_{a\in\mathcal{D}_\sigma} \gamma(a) < \infty\,.$$

Definition 3.2 *For a given concave money based utility functional ϕ on $\mathcal{R}^\infty$ we define the conjugate function $\phi^* : \mathcal{A}^1 \to [-\infty,\infty)$ by*

$$\phi^*(a) := \inf_{X\in\mathcal{R}^\infty} \{\langle X,a\rangle - \phi(X)\}\ ,\ a \in \mathcal{A}^1\,.$$

As in [FS1], one can deduce from the translation invariance of ϕ that for all $a \in \mathcal{D}_\sigma$,

$$\phi^*(a) = \inf_{X\in\mathcal{C}} \langle X,a\rangle\ .$$

Theorem 3.3 *The following are equivalent:*

(1) *ϕ is a mapping defined on $\mathcal{R}^\infty$ that can be represented as*

$$\phi(X) = \inf_{a\in\mathcal{D}_\sigma} \{\langle X,a\rangle - \gamma(a)\}\ ,\ X \in \mathcal{R}^\infty\,, \tag{3.1}$$

for a penalty function $\gamma : \mathcal{D}_\sigma \to [-\infty,\infty)$.

(2) *ϕ is a concave money based utility functional on $\mathcal{R}^\infty$ whose acceptance set $\mathcal{C} := \{X \in \mathcal{R}^\infty \,|\, \phi(X) \geq 0\}$ is $\sigma(\mathcal{R}^\infty,\mathcal{A}^1)$-closed.*

(3) *ϕ is a concave money based utility functional on $\mathcal{R}^\infty$ that satisfies the Fatou property.*

(4) *ϕ is a concave money based utility functional on $\mathcal{R}^\infty$ that is continuous for bounded decreasing sequences.*

Moreover, if (1)-(4) are satisfied, then the restriction of ϕ^* to $\mathcal{D}_\sigma$ is a penalty function, $\phi^*(a) \geq \gamma(a)$ for all $a \in \mathcal{D}_\sigma$, and the representation (3.1) also holds if γ is replaced by ϕ^*.

Proof. Here, we prove or sketch a proof of the implications (2) $\Rightarrow$ (1), (1) $\Rightarrow$ (3) and (3) $\Rightarrow$ (4), as well as the second part of the theorem, all of which can be shown by adapting the corresponding demonstrations in [De1] and [FS1]. For the implication (4) $\Rightarrow$ (2) we need a new proof. It is the main mathematical contribution of this paper and is given in Section 4.

(2) $\Rightarrow$ (1):
The function

$$-\phi^* : \mathcal{A}^1 \to (-\infty, \infty]$$

can be viewed as a generalized Fenchel transform of the convex function

$$X \mapsto -\phi(-X)\,.$$

Since ϕ is translation invariant, it follows from (2) that the set $\{X \in \mathcal{R}^\infty \mid \phi(X) \geq m\}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed for all $m \in \mathbb{R}$. Therefore, one can extend standard arguments from convex analysis (see, for instance, the proof of Theorem 3.1 in [Au]) to the locally convex topological vector space $(\mathcal{R}^\infty, \sigma(\mathcal{R}^\infty, \mathcal{A}^1))$ to conclude that

$$\phi(X) = \inf_{a \in \mathcal{A}^1} \{\langle X, a\rangle - \phi^*(a)\}\,.$$

Since ϕ is not only concave but a concave money based utility functional, it can also be proved that

$$\phi(X) = \inf_{a \in \mathcal{D}_\sigma} \{\langle X, a\rangle - \phi^*(a)\}\,. \tag{3.2}$$

For details we refer to the proof of Theorem 5 in [FS1]. Note that it follows from (3.2) that ϕ^* is a penalty function.

(1) $\Rightarrow$ (3):
It can easily be checked that (3.1) defines a concave money based utility functional ϕ on $\mathcal{R}^\infty$. To see that it satisfies the Fatou property, let $(X^n)_{n\in\mathbb{N}}$ be a bounded sequence in $\mathcal{R}^\infty$ with $\left(X^n - X\right)^* \xrightarrow{P} 0$ for some $X \in \mathcal{R}^\infty$. For every $\varepsilon > 0$ there exists an $a \in \mathcal{D}_\sigma$ such that

$$\begin{aligned}
\phi\left(X\right) + \varepsilon &\geq \mathrm{E}\left[\int_{]0,T]} X_{t-} da_t^{\mathrm{pr}} + \int_{[0,T]} X_t da_t^{\mathrm{op}}\right] - \gamma(a) \\
&= \lim_{n\to\infty} \mathrm{E}\left[\int_{]0,T]} X_{t-}^n da_t^{\mathrm{pr}} + \int_{[0,T]} X_t^n da_t^{\mathrm{op}}\right] - \gamma(a) \\
&\geq \limsup_{n\to\infty} \inf_{b\in\mathcal{D}_\sigma} \{\langle X^n, b\rangle - \gamma(b)\} \\
&= \limsup_{n\to\infty} \phi(X^n)\,.
\end{aligned}$$

Hence, we have $\limsup_{n\to\infty} \phi(X^n) \leq \phi(X)$, and (3) is proved.

(3) $\Rightarrow$ (4):
Let $(X^n)_{n\geq 1}$ be a decreasing sequence in $(\mathcal{R}^\infty)$ such that $(X^n - X)^* \xrightarrow{P} 0$ for some $X \in \mathcal{R}^\infty$.

If (3) is satisfied, then $\limsup_{n\to\infty}\phi(X^n) \le \phi(X)$, which, by monotonicity (property (2) of Definition 2.2') of ϕ, implies that $\lim_{n\to\infty}\phi(X^n) = \phi(X)$. Hence, (4) holds as well.

It remains to be shown that if ϕ is a concave money based utility functional on $\mathcal{R}^\infty$ that can be represented in the form (3.1) for a penalty function $\gamma : \mathcal{D}_\sigma \to [-\infty, \infty)$, then $\phi^*(a) \ge \gamma(a)$, for all $a \in \mathcal{D}_\sigma$. Note that (3.1) implies that

$$\gamma(a) \le \langle X, a\rangle - \phi(X)\,,$$

for all $X \in \mathcal{R}^\infty$ and $a \in \mathcal{D}_\sigma$. Hence, $\gamma(a) \le \phi^*(a)$, for all $a \in \mathcal{D}_\sigma$. $\square$

Corollary 3.4 *The following are equivalent:*

(1) *ϕ is a mapping defined on $\mathcal{R}^\infty$ that can be represented as*

$$\phi(X) = \inf_{a\in\mathcal{Q}_\sigma} \langle X, a\rangle \;,\; X \in \mathcal{R}^\infty\,, \tag{3.3}$$

for a non-empty set $\mathcal{Q}_\sigma \subset \mathcal{D}_\sigma$.

(2) *ϕ is a coherent utility functional on $\mathcal{R}^\infty$ whose acceptance set $\mathcal{C} := \{X \in \mathcal{R}^\infty \mid \phi(X) \ge 0\}$ is $\sigma\left(\mathcal{R}^\infty, \mathcal{A}^1\right)$-closed.*

(3) *ϕ is a coherent utility functional on $\mathcal{R}^\infty$ that satisfies the Fatou property.*

(4) *ϕ is a coherent utility functional on $\mathcal{R}^\infty$ that is continuous for bounded decreasing sequences.*

Proof. It can easily be verified that (1) implies (3). That (2), (3) and (4) are equivalent follows directly from Theorem 3.3. Finally, if one of the properties (2), (3) or (4) holds, then it follows from Theorem 3.3 that for all $X \in \mathcal{R}^\infty$,

$$\phi(X) = \inf_{a\in\mathcal{D}_\sigma} \{\langle X, a\rangle - \phi^*(a)\}\;. \tag{3.4}$$

Since ϕ is positively homogenous (property (4) of Definition 2.2'), the conjugate function ϕ^* takes only the values 0 and $-\infty$. Thus, the representation (3.4) can be written in the form (3.3) for $\mathcal{Q}_\sigma := \{a \in \mathcal{D}_\sigma \,|\, \phi^*(a) = 0\}$. $\square$

Remark 3.5 Let ϕ be a coherent utility functional on $\mathcal{R}^\infty$ that has a representation of the form (3.3) for a non-empty subset $\mathcal{Q}_\sigma \subset \mathcal{D}_\sigma$. Then, the $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-closure $\overline{\mathrm{conv}\mathcal{Q}_\sigma}$ of the convex hull of $\mathcal{Q}_\sigma$ induces the same coherent utility functional ϕ.

Remark 3.6 The set $\mathcal{D}_\sigma$ is contained in the set

$$\hat{\mathcal{D}}_\sigma := \left\{ a : [0,T]\times\Omega \to \mathbb{R}^2 \;\middle|\; \begin{array}{l} a = (a^{\mathrm{l}}, a^{\mathrm{r}})\,,\, a^{\mathrm{l}}_0 = 0 \\ a^{\mathrm{l}}, a^{\mathrm{r}} \text{ measurable, right-continuous, non-decreasing} \\ \mathrm{E}\left[a^{\mathrm{l}}_T + a^{\mathrm{r}}_T\right] = 1 \end{array} \right\}.$$

As in (2.1), we set for all $X \in \mathcal{R}^\infty$ and $a \in \hat{\mathcal{D}}_\sigma$,

$$\langle X, a \rangle := \mathrm{E}\left[\int_{]0,T]} X_{t-} da^{\mathrm{l}}_t + \int_{[0,T]} X_t da^{\mathrm{r}}_t\right] .$$

For every $a \in \hat{\mathcal{D}}_\sigma$, there exists a unique $\Pi^* a \in \mathcal{D}_\sigma$ such that for all $X \in \mathcal{R}^\infty$,

$$\langle X, a \rangle = \langle X, \Pi^* a \rangle . \tag{3.5}$$

Indeed, denote for each $a \in \hat{\mathcal{D}}_\sigma$ by $\tilde{a}^{\mathrm{l}}$ the dual predictable projection of a^{l} and by $\tilde{a}^{\mathrm{r}}$ the dual optional projection of a^{r}. Then, $\tilde{a} = (\tilde{a}^{\mathrm{l}}, \tilde{a}^{\mathrm{r}})$ is in the set $\tilde{\mathcal{A}}^1$ introduced in Remark 2.1.1 and for all $X \in \mathcal{R}^\infty$,

$$\langle X, a \rangle = \langle X, \tilde{a} \rangle .$$

Hence, it follows from what we have shown in Remark 2.1.1 that there exists a unique $\Pi^* a \in \mathcal{D}_\sigma$ that satisfies (3.5).

For a function $\hat{\gamma} : \hat{\mathcal{D}}_\sigma \to [-\infty, \infty)$ that satisfies

$$-\infty < \sup_{a \in \hat{\mathcal{D}}_\sigma} \hat{\gamma}(a) < \infty ,$$

the mapping $\gamma : \mathcal{D}_\sigma \to [-\infty, \infty)$ given by

$$\gamma(a) := \sup \left\{ \hat{\gamma}(\hat{a}) \mid \Pi^* \hat{a} = a \right\} ,$$

is a penalty function and for all $X \in \mathcal{R}^\infty$,

$$\phi(X) := \inf_{\hat{a} \in \hat{\mathcal{D}}_\sigma} \left\{ \langle X, \hat{a} \rangle - \hat{\gamma}(\hat{a}) \right\} = \inf_{a \in \mathcal{D}_\sigma} \left\{ \langle X, a \rangle - \gamma(a) \right\} ,$$

which shows that ϕ is a concave money based utility functional on $\mathcal{R}^\infty$ that satisfies the Fatou property.

If $\hat{\mathcal{Q}}_\sigma$ is a non-empty subset of $\hat{\mathcal{D}}_\sigma$, then for all $X \in \mathcal{R}^\infty$,

$$\phi(X) := \inf_{\hat{a} \in \hat{\mathcal{Q}}_\sigma} \langle X, \hat{a} \rangle = \inf_{a \in \Pi^* \hat{\mathcal{Q}}_\sigma} \langle X, a \rangle ,$$

and therefore, ϕ is a coherent utility functional on $\mathcal{R}^\infty$ that satisfies the Fatou property.

We finish this section by discussing a variation of Theorem 3.3 and giving some reasons why we stated Theorem 3.3 as we did.

First note that it follows from (2.2) that for all $p, q \in [0, \infty]$ such that $p^{-1} + q^{-1} = 1$, every element $a \in \mathcal{A}^q$ induces a continuous linear functional on the space $\mathcal{R}^p$. By Theorem 65 on page 254 of [DM2], for $1 < p < \infty$, the Banach space $\mathcal{A}^q$ can be identified with the dual $(\mathcal{R}^p)'$ of $\mathcal{R}^p$. On the other hand, the fact that $(L^\infty)'$ can be embedded into $(\mathcal{R}^\infty)'$, shows that $(\mathcal{R}^\infty)'$ contains functionals that are not sigma-additive. Hence, $\mathcal{A}^1$ corresponds to a strict subspace of $(\mathcal{R}^\infty)'$. It follows from Theorem 67 on page 255 of [DM2] that if the filtration $(\mathcal{F}_t)_{t \in [0,T]}$ is constant over time, then $\mathcal{A}^\infty = (\mathcal{R}^1)'$. However, if the filtration is not constant, then $\mathcal{A}^\infty$ is smaller than $(\mathcal{R}^1)'$.

Proposition 3.7 *Consider a $p \in [1,\infty]$ and let ϕ be a mapping from $\mathcal{R}^p$ to $\mathbb{R}$ that satisfies the properties (1), (2) and (3) of Definition 2.2' for all $X, Y \in \mathcal{R}^p$, $\lambda \in [0,1]$ and $m \in \mathbb{R}$. Then ϕ can be represented as*

$$\phi(X) = \inf_{a \in \mathcal{D}^p} \{\langle X, a\rangle - \gamma(a)\} \,, \quad X \in \mathcal{R}^p \,, \tag{3.6}$$

where

$$\mathcal{D}^p := \left\{ a \in (\mathcal{R}^p)'_+ \mid \langle 1, a\rangle = 1 \right\}$$

and γ is a function from $\mathcal{D}^p$ to $[-\infty,\infty)$ such that

$$-\infty < \sup_{a \in \mathcal{D}^p} \gamma(a) < \infty \,.$$

Remark 3.8 Note that for $p \in (1,\infty)$, the set $\mathcal{D}^p$ can be identified with $\mathcal{D}_\sigma \cap \mathcal{A}^q$, where $\frac{1}{p} + \frac{1}{q} = 1$.

Before we prove Proposition 3.7, we shortly explain why our main result, Theorem 3.3, is about concave money based utility functionals on $\mathcal{R}^\infty$ that satisfy the Fatou property or are continuous for bounded decreasing sequences. There are two reasons why of all the spaces $\mathcal{R}^p$, $p \in [1,\infty]$, Theorem 3.3 deals only with $\mathcal{R}^\infty$. First, in real risk management situations it is often not clear what probability to assign to possible future events, and a whole set of different probability measures is taken into consideration. The space $\mathcal{R}^\infty$ (as well as $\mathcal{R}^0$) is the same for all these measures as long as they are equivalent. This is not true for the spaces $\mathcal{R}^p$ for $p \in [1,\infty)$. Secondly, a representation result of the form (3.6) for $p \in [1,\infty)$ is of minor importance because of the following: Since a coherent or convex risk measure is meant to give the amount of cash that has to be added to a financial position to make it acceptable (see [ADEH2], [De1] or [FS1]), it is natural to require such a measure on $\mathcal{R}^\infty$ to be real-valued. On the other hand, it will be shown in a forthcoming paper that for an interesting theory of coherent and convex risk measures on $\mathcal{R}^0$, one has to allow them to take values in $(-\infty,\infty]$ (equivalently, coherent and concave money based utility functionals should be allowed to take values in $[-\infty,\infty)$). So if one restricts oneself to real-valued coherent or convex risk measures on $\mathcal{R}^p$ for some $p \in [1,\infty)$, one might miss interesting examples of coherent and convex risk measures on $\mathcal{R}^\infty$ that can be extended to $\mathcal{R}^0$ but are equal to ∞ for certain $X \in \mathcal{R}^p$.

Proposition 3.7 also includes that every concave money based utility functional on $\mathcal{R}^\infty$ has a representation in terms of $\mathcal{D}^\infty \subset (\mathcal{R}^\infty)'$. However, not all elements of the dual $(\mathcal{R}^\infty)'$ of $\mathcal{R}^\infty$ have a nice representation. Theorem 3.3 characterizes those concave money based utility functionals on $\mathcal{R}^\infty$ that can be represented with elements of the "nicer" space $\mathcal{A}^1$. Note that it is in general not easy to check whether the acceptance set of a concave money based utility functional on $\mathcal{R}^\infty$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed, nor has this property a direct economic interpretation. The Fatou property and continuity for bounded decreasing sequences on the other hand, are very intuitive conditions and can in many cases easily be verified.

In the following proof we associate to a concave money based utility functional on $\mathcal{R}^p$ a coherent utility functional that is defined on $\mathcal{R}^p \times \mathbb{R}$. This allows us to use properties of coherent utility functionals to conclude something for a concave money based utility functional.

Proof of Proposition 3.7. It is enough to show that the acceptance set

$$\mathcal{C} := \{X \in \mathcal{R}^p \mid \phi(X) \geq 0\} \quad \text{is norm-closed in} \quad \mathcal{R}^p \,. \tag{3.7}$$

Then the proposition follows from the arguments in the proof of Theorem 5 of [FS1].

For $p = \infty$, (3.7) follows directly from (2.3). For $p \in [1, \infty)$, we show (3.7) by a homogenization argument. Since ϕ is translation invariant, $\mathcal{C}$ is norm-closed in $\mathcal{R}^p$ if and only if

$$\tilde{\mathcal{C}} := \left\{ X \in \mathcal{R}^p \mid \tilde{\phi}(X) \geq 0 \right\} ,$$

is, where $\tilde{\phi}(X) := \phi(X) - \phi(0)$, for all $X \in \mathcal{R}^p$. Consider the convex cone

$$\hat{\mathcal{C}} := \bigcup_{\lambda > 0} \{(\lambda X, \lambda) \mid X \in \tilde{\mathcal{C}}\} \cup \{(X, 0) \mid X \in 0^+ \tilde{\mathcal{C}}\} ,$$

where

$$0^+ \tilde{\mathcal{C}} := \left\{ X \in \tilde{\mathcal{C}} \mid \lambda X \in \tilde{\mathcal{C}} \text{ for all } \lambda \geq 0 \right\} .$$

It can easily be checked that the function

$$\hat{\phi}(X, x) := \sup \left\{ \alpha \in \mathbb{R} \mid (X, x) - (\alpha, \alpha) \in \hat{\mathcal{C}} \right\} , \quad X \in \mathcal{R}^p \,, \, x \in \mathbb{R} \,,$$

is real-valued and satisfies the following properties:

(1) $\hat{\phi}(X + Y, x + y) \geq \hat{\phi}(X, x) + \hat{\phi}(Y, y)$ for all $(X, x), (Y, y) \in \mathcal{R}^p \times \mathbb{R}$
(2) $(X, x) \geq 0$ implies $\hat{\phi}(X, x) \geq 0$
(3) $\hat{\phi}(X + m, x + m) = \hat{\phi}(X, x) + m$ for all $(X, x) \in \mathcal{R}^p \times \mathbb{R}$ and $m \in \mathbb{R}$
(4) $\hat{\phi}(\lambda(X, x)) = \lambda \hat{\phi}(X, x)$ for all $(X, x) \in \mathcal{R}^p \times \mathbb{R}$ and $\lambda \geq 0$.

In the next step we show that

$$\hat{\mathcal{C}} = \{(X, x) \in \mathcal{R}^p \times \mathbb{R} \mid \hat{\phi}(X, x) \geq 0 \} .$$

The inclusion "$\subset$" follows directly from the definition of $\hat{\phi}$. To show the other inclusion, it is enough to prove that

$$\hat{\phi}(X, x) < 0 \quad \text{for all} \quad (X, x) \notin \hat{\mathcal{C}} \,. \tag{3.8}$$

If $x < 0$, this is obvious. If $(X, x) \notin \hat{\mathcal{C}}$ and $x = 0$, then $X \notin 0^+ \tilde{\mathcal{C}}$, and we show (3.8) by contradiction. So assume that $\hat{\phi}(X, 0) \geq 0$. Then,

$$\frac{1}{\beta}(X + \beta) \in \tilde{\mathcal{C}}$$

for all $\beta > 0$, and therefore also

$$\frac{\lambda}{\beta} X + \lambda \in \tilde{\mathcal{C}}$$

for all $\beta > 0$ and $\lambda \in [0,1]$. This implies that $X \in 0^+\tilde{\mathcal{C}}$, which is a contradiction. Hence, $\hat{\phi}(X,0) < 0$. To prove (3.8) for a $(X,x) \notin \hat{\mathcal{C}}$ with $x > 0$, note that $\frac{X}{x} \notin \tilde{\mathcal{C}}$ and for all $Y, Z \in \mathcal{R}^p$, the function $\lambda \mapsto \tilde{\phi}(\lambda Y + (1-\lambda)Z)$ is concave and therefore also continuous. Hence, the set

$$\left\{\lambda \in [0,1] \mid \lambda Y + (1-\lambda)Z \in \tilde{\mathcal{C}}\right\} = \left\{\lambda \in [0,1] \mid \tilde{\phi}\left(\lambda Y + (1-\lambda)Z\right) \geq 0\right\}$$

is closed in $[0,1]$. This implies that also the set

$$\left\{\lambda \in [0,1] \mid \lambda(1,1) + (1-\lambda)(X,x) \in \hat{\mathcal{C}}\right\} = \left\{\lambda \in [0,1] \mid \frac{\lambda + (1-\lambda)x\frac{X}{x}}{\lambda + (1-\lambda)x} \in \tilde{\mathcal{C}}\right\}$$

is closed in $[0,1]$. Therefore, there exists a $\lambda > 0$ such that $\lambda(1,1) + (1-\lambda)(X,x) \notin \hat{\mathcal{C}}$. Hence, $\hat{\phi}\left(\lambda(1,1) + (1-\lambda)(X,x)\right) \leq 0$, and by the properties (3) and (4),

$$\hat{\phi}(X,x) \leq \frac{-\lambda}{1-\lambda} < 0\,.$$

Now, let $(f_i)_{i \in I}$ be the family of all linear functions $f_i : \mathcal{R}^p \times \mathbb{R} \to \mathbb{R}$ such that

$$\hat{\phi}(X,x) \leq f_i(X,x) \quad \text{for all} \quad (X,x) \in \mathcal{R}^p \times \mathbb{R}\,.$$

It follows from the Hahn-Banach theorem in its standard form that

$$\hat{\phi}(X,x) = \inf_{i \in I} f_i(X,x)\ . \tag{3.9}$$

In particular, all f_i, $i \in I$, are positive. It then follows from a classical result of Namioka, see [Na], that the f_i, $i \in I$, are continuous with respect to the norm $||\cdot||_{\mathcal{R}^p} + |\cdot|$. This implies that

$$\hat{\mathcal{C}} = \{(X,x) \in \mathcal{R}^p \times \mathbb{R} \,|\, \hat{\phi}(X,x) \geq 0\} = \bigcap_{i \in I} \{(X,x) \in \mathcal{R}^p \times \mathbb{R} \,|\, f_i(X,x) \geq 0\}$$

is $(||\cdot||_{\mathcal{R}^p} + |\cdot|)$-closed. Therefore,

$$\tilde{\mathcal{C}} \times \{1\} = \hat{\mathcal{C}} \cap (\mathcal{R}^p \times \{1\})$$

is $(||\cdot||_{\mathcal{R}^p} + |\cdot|)$-closed, which shows that $\tilde{\mathcal{C}}$ is $||\cdot||_{\mathcal{R}^p}$-closed. $\square$

4 Proof of (4) $\Rightarrow$ (2) in Theorem 3.3

A coherent or concave money based utility functional on L^∞ is a map $\tilde{\phi} : L^\infty \to \mathbb{R}$ that satisfies the corresponding properties of Definition 2.2'. It satisfies the Fatou property if

$$\limsup_{n\to\infty} \tilde{\phi}(X^n) \le \tilde{\phi}(X)\,,$$

for all bounded sequences $(X^n)_{n\ge1} \subset L^\infty$ and $X \in L^\infty$ such that $X^n \xrightarrow{P} X$. We say that $\tilde{\phi}$ is continuous for bounded decreasing sequences if

$$\lim_{n\to\infty} \tilde{\phi}(X^n) = \tilde{\phi}(X)\ ,$$

for every decreasing sequence $(X^n)_{n\ge1} \subset \mathcal{R}^\infty$ such that $X^n \xrightarrow{P} X$ for some $X \in \mathcal{R}^\infty$.

We need the following two lemmas for concave money based utility functionals on L^∞. They are part of the statement of Theorem 3.2 in [De1]. To make clear why these lemmas cannot immediately be generalized to the framework of continuous-time stochastic processes, we provide detailed proofs.

Lemma 4.1 *Let $\tilde{\phi}$ be a concave money based utility functional on L^∞ that is continuous for bounded decreasing sequences. Then it also satisfies the Fatou property.*

Proof. Assume that $\tilde{\phi}$ is continuous for bounded decreasing sequences, but there exists a bounded sequence $(X^n)_{n\ge1}$ in L^∞ and an $X \in L^\infty$ such that $X^n \xrightarrow{P} X$ and

$$\limsup_{n\to\infty} \tilde{\phi}(X^n) > \tilde{\phi}(X)\,.$$

Then, there exists a subsequence $(X^{n_j})_{j\ge1}$ such that $X^{n_j} \to X$ almost surely, and

$$\limsup_{j\to\infty} \tilde{\phi}(X^{n_j}) > \tilde{\phi}(X)\,. \tag{4.1}$$

The sequence

$$Y^j := \sup_{m\ge j}(X^{n_m} \vee X)\,,\; j \ge 1\,,$$

is non-increasing and converges to X almost surely. Therefore, $\lim_{j\to\infty} \tilde{\phi}(Y^j) = \tilde{\phi}(X)$. On the other hand, $\lim_{j\to\infty} \tilde{\phi}(Y^j) \ge \limsup_{j\to\infty} \tilde{\phi}(X^{n_j})$. This contradicts (4.1), and therefore $\limsup_{n\ge1} \tilde{\phi}(X^n) \le \tilde{\phi}(X)$, which proves the lemma. □

Lemma 4.2 *Let $\tilde{\phi}$ be a concave money based utility functional on L^∞ that satisfies the Fatou property. Then the acceptance set*

$$\tilde{\mathcal{C}} := \left\{X \in L^\infty \,\middle|\, \tilde{\phi}(X) \ge 0\right\}$$

is $\sigma(L^\infty, L^1)$-closed.

Proof. The key ingredient of this proof is the following result, which follows immediately from the Krein-Šmulian theorem (Theorem 5 in [KS]):

A convex set G in the dual E' of a Banach space E is $\sigma(E', E)$-closed if and only if for all $m > 0$, the set
$$G \cap \{x \in E' \,|\, ||x||_{E'} \leq m\} \text{ is } \sigma(E', E)\text{-closed.} \tag{4.2}$$

Clearly , the set $\tilde{\mathcal{C}}$ is convex. Hence, by (4.2), it is enough to show that for every $m > 0$, the convex set

$$\tilde{\mathcal{C}}_m := \tilde{\mathcal{C}} \cap \{X \in L^\infty \,|\, ||X||_\infty \leq m\} \quad \text{is } \sigma(L^\infty, L^1)\text{-closed}\,. \tag{4.3}$$

Let $(X^n)_{n\geq 1}$ be a sequence in $\tilde{\mathcal{C}}_m$ and $X \in L^1$ such that $||X^n - X||_{L^1} \to 0$. It can easily be checked that $||X||_\infty \leq m$. Since $\tilde{\phi}$ satisfies the Fatou property,

$$\tilde{\phi}(X) \geq \limsup_{n\to\infty} \tilde{\phi}(X^n) \geq 0\,,$$

which shows that $X \in \tilde{\mathcal{C}}_m$. Hence, $\tilde{\mathcal{C}}_m$ is a norm-closed subset of L^1. It follows from Theorem 3.12 of [Ru] that it is also a $\sigma(L^1, L^\infty)$-closed subset of L^1. This implies that it is a $\sigma(L^\infty, L^1)$-closed subset of L^∞, and the proof is complete. □

Remark 4.3 Together, Lemma 4.1 and Lemma 4.2 imply that for a concave money based utility functional $\tilde{\phi}$ on L^∞ that is continuous for bounded decreasing sequences, the acceptance set
$$\tilde{\mathcal{C}} := \left\{X \in L^\infty \,\middle|\, \tilde{\phi}(X) \geq 0\right\}$$
is $\sigma(L^\infty, L^1)$-closed. The proof of Lemma 4.1 cannot be generalized to concave money based utility functionals on $\mathcal{R}^\infty$ because for a sequence $(X^n)_{n\geq 1}$ of bounded càdlàg processes the supremum process $Y := \sup_{n\geq 1} X^n$ does not need to be càdlàg. The reason why the proof of Lemma 4.2 cannot be generalized to the setting of Theorem 3.3 is that $\mathcal{R}^\infty$ is not the dual of a metrizable locally convex topological vector space, and therefore, nor the Krein-Šmulian theorem nor the slightly more general Banach-Dieudonné theorem (see Theorem 2 on page 159 of [Gr]) can be applied. In fact, it can be shown with standard arguments from functional analysis that $\mathcal{R}^\infty$ cannot even be isomorphic to a complemented subspace of the dual of a metrizable locally convex topological vector space. But this is beyond the scope of this paper.

Besides the Lemmas 4.1 and 4.2 we also need

Lemma 4.4 *Assume that $\mathcal{F}_t = \mathcal{F}$, for all $t \in [0, T]$, and let $Z \in L^1$. Then the set*

$$\left\{a \in \mathcal{A}^1 \,|\, \mathrm{Var}\,(a) \leq |Z|\right\} \quad \textit{is } \sigma(\mathcal{A}^1, \mathcal{R}^\infty)\textit{-compact}\,. \tag{4.4}$$

Proof. Since the filtration is constant over time, it follows from Theorem 67 of [DM2] page 255 that $\mathcal{A}^\infty$ is the dual of $\mathcal{R}^1$. Therefore, Alaoglu's theorem implies that the set

$$\{a \in \mathcal{A}^\infty \mid \mathrm{Var}\,(a) \le 1\}$$

is $\sigma(\mathcal{A}^\infty, \mathcal{R}^1)$-compact. It can easily be checked that the map

$$\mathcal{A}^\infty \to \mathcal{A}^1 \,, \quad a \mapsto |Z|\, a$$

is $\sigma(\mathcal{A}^\infty, \mathcal{R}^1)/\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-continuous. Hence, the lemma follows because

$$\left\{a \in \mathcal{A}^1 \mid \mathrm{Var}\,(a) \le |Z|\right\} \,,$$

is the image of the set

$$\{a \in \mathcal{A}^\infty \mid \mathrm{Var}\,(a) \le 1\}$$

under this map. □

We are now ready to prove the implication (4) $\Rightarrow$ (2) of Theorem 3.3:

a) We first assume that $\mathcal{F}_t = \mathcal{F}$ for all $t \in [0,T]$:

Obviously, $\mathcal{C}$ is convex. It follows from Mackey's theorem (see for instance, Corollary to Theorem 9, Section 13, Chapter 2 in [Gr]) that $\mathcal{A}^1$ is the dual of $\mathcal{R}^\infty$ equipped with the Mackey topology $\tau(\mathcal{R}^\infty, \mathcal{A}^1)$. Hence, it follows from Theorem 3.12 of [Ru] that $\mathcal{C}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed if we can show that it is $\tau(\mathcal{R}^\infty, \mathcal{A}^1)$-closed. So let $(X^\lambda)_{\lambda \in \Lambda}$ be a net in $\mathcal{C}$ and $X \in \mathcal{R}^\infty$ such that $X^\lambda \to X$ in $\tau(\mathcal{R}^\infty, \mathcal{A}^1)$. In order to show that $X \in \mathcal{C}$, we first construct a refining sequence $(S^n)_{n \ge 1}$ of increasing stopping times sequences. Since for all $t \in [0,T]$, $\mathcal{F}_t = \mathcal{F}$, stopping times are just random times. Define $S^n_0 := 0$ and recursively,

$$S^n_k := \inf \left\{ t > S^n_{k-1} \,\middle|\, t \in \left\{S^{n-1}_j \,,\, j \ge 1\right\} \text{ or } |X_t - X_{S^n_{k-1}}| \ge \frac{1}{n} \right\} \wedge T \,.$$

By construction, we clearly have $\{S^{n-1}_k, k \ge 1\} \subset \{S^n_k, k \ge 1\}$ almost surely. Moreover, it can be deduced from the fact that X has càdlàg paths that for every $n \ge 1$,

$$P\left[\bigcup_{k=1}^\infty \{S^n_k = T\}\right] = 1 \,.$$

Let $\mathcal{N}$ be the set of all subsets of $\mathbb{N}$ and define the measure ν on $(\mathbb{N}, \mathcal{N})$ by $\nu(k) := 2^{-k}$, $k \in \mathbb{N}$. An element

$$Y \in L^\infty(\Omega \times \mathbb{N}) := L^\infty(\Omega \times \mathbb{N}, \mathcal{F}_T \otimes \mathcal{N}, P \otimes \nu)$$

can be viewed as a bounded sequence $(Y_k)_{k \ge 1}$ in $L^\infty(\Omega)$.

We define the maps

$$\xi_n : \mathcal{R}^\infty \to L^\infty(\Omega \times \mathbb{N})$$

and

$$\xi^n : L^\infty(\Omega \times \mathbb{N}) \to \mathcal{R}^\infty$$

as follows: For $Y \in \mathcal{R}^\infty$, we set

$$(\xi_n Y)_k := \begin{cases} \sup_{S^n_{k-1} \le t < S^n_k} Y_t & \text{if } k \le K^n \\ Y_T & \text{if } k > K^n \end{cases},$$

where the random variable K^n is given by

$$K^n := \inf \{k \ge 1 \mid S^n_k = T\} \, .$$

For $Y \in L^\infty(\Omega \times \mathbb{N})$, we set

$$(\xi^n Y)_t := \begin{cases} Y_k & \text{if } S^n_{k-1} \le t < S^n_k \\ Y_{K^n+1} & \text{if } t = T \end{cases} .$$

Note that for all $Y \in \mathcal{R}^\infty$ and $n \ge 1$,

$$Y \le \xi^n \xi_n Y \, . \tag{4.5}$$

Furthermore, it can easily be checked that the mapping

$$\phi^n := \phi \circ \xi^n : L^\infty(\Omega \times \mathbb{N}) \to \mathbb{R}$$

is a concave money based utility functional on $L^\infty(\Omega \times \mathbb{N})$. Moreover, if $(Z^m)_{m \ge 1}$ is a decreasing sequence in $L^\infty(\Omega \times \mathbb{N})$ such that $Z^m \stackrel{P \otimes \nu}{\longrightarrow} Z$ for some $Z \in L^\infty(\Omega \times \mathbb{N})$, then the sequence $(\xi^n Z^m)_{m \ge 1}$ is decreasing and $(\xi^n Z^m - \xi^n Z)^* \stackrel{P}{\to} 0$ because K^n is almost surely finite. Hence,

$$\lim_{m \to \infty} \phi^n(Z^m) = \lim_{m \to \infty} \phi \circ \xi^n Z^m = \phi \circ \xi^n Z = \phi^n Z \, ,$$

which shows that ϕ^n is continuous for bounded decreasing sequences.

Let $Z \in L^1(\Omega \times \mathbb{N})$, that is, Z is a sequence $(Z_k)_{k \ge 1}$ of random variables such that $\mathrm{E}\left[\sum_{k \ge 1} 2^{-k} |Z_k|\right] < \infty$. By Lemma 4.4, the set

$$\mathcal{K} := \left\{ a \in \mathcal{A}^1 \,\middle|\, \mathrm{Var}\,(a) \le \sum_{k \ge 1} 2^{-k} |Z_k| \right\}$$

is $\sigma(\mathcal{A}^1, \mathcal{R}^\infty)$-compact, and obviously, it is absolutely convex. For the natural bilinear form on

$$\left(L^\infty(\Omega \times \mathbb{N}) \, , \, L^1(\Omega \times \mathbb{N}) \right) \, ,$$

we use the same notation $\langle ., . \rangle$ as for the bilinear form on $(\mathcal{R}^\infty, \mathcal{A}^1)$ defined in (2.1). Furthermore, for all $k \ge 1$, we denote by Θ_k the set of random variables θ such that

$S^n_{k-1} \leq \theta \leq S^n_k$. Then, we can write

$$
\begin{aligned}
& \left\langle \xi_n X^\lambda - \xi_n X, Z \right\rangle = \mathrm{E}\left[\sum_{k\geq 1}\Big((\xi_n X^\lambda)_k - (\xi_n X)_k\Big)\, 2^{-k} Z_k\right] \\
\leq \;& \sum_{k=1}^{K^n} \mathrm{E}\left[\sup_{S^n_{k-1}\leq t < S^n_k} \left|X^\lambda_t - X_t\right| 2^{-k}\,|Z_k|\right] + \sum_{k\geq K^n+1} \mathrm{E}\left[\left|X^\lambda_T - X_T\right| 2^{-k}\,|Z_k|\right] \\
\leq \;& \sum_{k\geq 1} \sup_{\theta_k\in\Theta_k} \mathrm{E}\left[\left|X^\lambda_{\theta_k} - X^\lambda_{\theta_k}\right| 2^{-k}\,|Z_k|\right] \\
\leq \;& \sup_{a\in\mathcal{K}} \mathrm{E}\left[\int_{]0,T]} (X^\lambda_{t_-} - X^\lambda_{t_-})\,da^{\mathrm{pr}}_t + \int_{[0,T]} (X^\lambda_t - X^\lambda_t)\,da^{\mathrm{op}}_t\right] \\
= \;& \sup_{a\in\mathcal{K}} \left\langle X^\lambda - X, a \right\rangle .
\end{aligned}
$$

Since $(X^\lambda)_{\lambda\in\Lambda}$ converges to X in $\tau(\mathcal{R}^\infty, \mathcal{A}^1)$, we have $\sup_{a\in\mathcal{K}}\left\langle X^\lambda - X, a\right\rangle \to 0$, and therefore also,

$$
\xi_n X^\lambda \to \xi_n X \quad \text{in} \quad \sigma(L^\infty(\Omega\times\mathbb{N}), L^1(\Omega\times\mathbb{N}))\,. \tag{4.6}
$$

The fact that all X^λ are in $\mathcal{C}$ and (4.5) imply that all $\xi_n X^\lambda$ are in

$$
\mathcal{C}_n := \{Y \in L^\infty(\Omega\times\mathbb{N}) \,|\, \phi^n(Y) = \phi\circ\xi^n(Y) \geq 0\}\,.
$$

Therefore, it follows from (4.6), Lemma 4.1 and Lemma 4.2 that all $\xi_n X$ are in $\mathcal{C}_n$ as well, and hence, all $\xi^n\xi_n X$ are in $\mathcal{C}$. Since the sequence $(S^n)_{n\geq 1}$ of sequences of increasing stopping times is refining, the sequence $(\xi^n\xi_n X)_{n\geq 0}$ of stochastic processes is non-increasing, and by construction of $(S^n)_{n\geq 1}$, for all $n\geq 1$,

$$
X \leq \xi^n\xi_n X \leq X + \frac{1}{n}\,.
$$

In particular,

$$
(\xi^n\xi_n X - X)^* \xrightarrow{P} 0\,.
$$

Hence, since ϕ is continuous for bounded decreasing sequences,

$$
\phi(X) = \lim_{n\to\infty}\phi(\xi^n\xi_n X) \geq 0\,,
$$

which shows that $X\in\mathcal{C}$.

b) Now, let $(\mathcal{F}_t)_{t\in[0,T]}$ be a general filtration that satisfies the usual assumptions. We define the filtration $(\hat{\mathcal{F}}_t)_{t\in[0,T]}$ by

$$
\hat{\mathcal{F}}_t := \mathcal{F} \text{ for all } t\in[0,T]\,,
$$

and denote the corresponding dual pair by $(\hat{\mathcal{R}}^\infty, \hat{\mathcal{A}}^1)$. By Π_{op} we denote the optional projection from $\hat{\mathcal{R}}^\infty$ to $\mathcal{R}^\infty$. Theorem 47 on page 108 in [DM2] guarantees that the optional projection of a measurable, bounded, càdlàg process is again càdlàg. It can easily be checked that the mapping $\hat{\phi} := \phi \circ \Pi_{\text{op}} : \hat{\mathcal{R}}^\infty \to \mathbb{R}$ is a concave money based utility functional on $\hat{\mathcal{R}}^\infty$. To see that $\hat{\phi}$ is continuous for bounded decreasing sequences, let $(X^n)_{n\geq 1}$ be a decreasing sequence in $\hat{\mathcal{R}}^\infty$ such that $(X^n - X)^* \xrightarrow{P} 0$ for some $X \in \hat{\mathcal{R}}^\infty$. Then, also the sequence $(\Pi_{\text{op}} X^n)_{n\geq 1}$ is decreasing. Furthermore, $||(X^n - X)^*||_{L^2} \to 0$. Let the martingale $(M_t^n)_{t\in[0,T]}$ be given by

$$M_t^n := \mathrm{E}\left[(X^n - X)^* \,|\, \mathcal{F}_t\right], \; t \in [0,T] .$$

It follows from Doob's L^2-inequality that

$$\begin{aligned} & ||(\Pi_{\text{op}} X^n - \Pi_{\text{op}} X)^*||_2 = || \sup_{t\in[0,T]} \mathrm{E}\left[X_t^n - X_t \,|\, \mathcal{F}_t\right] ||_2 \leq ||(M^n)^*||_2 \\ \leq \; & 2||M_T^n||_2 \leq 2||(X^n - X)^*||_2 , \end{aligned}$$

which shows that $(\Pi_{\text{op}} X^n - \Pi_{\text{op}} X)^* \to 0$ in probability. Since ϕ is continuous for bounded decreasing sequences, we have

$$\lim_{n\to\infty} \hat{\phi}(X^n) = \lim_{n\to\infty} \phi \circ \Pi_{\text{op}}(X^n) = \phi \circ \Pi_{\text{op}}(X) = \hat{\phi}(X) ,$$

that is, $\hat{\phi}$ is also continuous for bounded decreasing sequences. It follows from part a) of the proof that the set

$$\hat{\mathcal{C}} := \left\{ X \in \hat{\mathcal{R}}^\infty \,\middle|\, \phi \circ \Pi_{\text{op}}(X) \geq 0 \right\}$$

is $\sigma(\hat{\mathcal{R}}^\infty, \hat{\mathcal{A}}^1)$-closed. Let $(X^\lambda)_{\lambda\in\Lambda}$ be a net in $\mathcal{C} \subset \hat{\mathcal{C}}$ and $X \in \mathcal{R}^\infty$ such that $X^\lambda \to X$ in $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$. The projection Π^* of Remark 3.6 can be extended to $\hat{\mathcal{A}}^1$, and for all $\hat{a} \in \hat{\mathcal{A}}^1$,

$$\left\langle X^\lambda - X, \hat{a} \right\rangle = \left\langle X^\lambda - X, \Pi^* \hat{a} \right\rangle \to 0 ,$$

that is, $X^\lambda \to X$ in $\sigma(\hat{\mathcal{R}}^\infty, \hat{\mathcal{A}}^1)$. Therefore, $X \in \hat{\mathcal{C}} \cap \mathcal{R}^\infty = \mathcal{C}$. This shows that $\mathcal{C}$ is $\sigma(\mathcal{R}^\infty, \mathcal{A}^1)$-closed, and the proof is complete. □

5 Examples

In this section we will repeatedly make use of the set

$$\tilde{\mathcal{D}}_\sigma := \left\{ f \in L^1 \,|\, f \geq 0 , \, \mathrm{E}\left[f\right] = 1 \right\}$$

and the set

$$\hat{\mathcal{D}}_\sigma := \left\{ a \, : [0,T] \times \Omega \to \mathbb{R}^2 \;\middle|\; \begin{array}{l} a = (a^{\mathrm{l}}, a^{\mathrm{r}}) \, , \, a_0^{\mathrm{l}} = 0 \\ a^{\mathrm{l}}, a^{\mathrm{r}} \text{ measurable, right-continuous, non-decreasing} \\ \mathrm{E}\left[a_T^{\mathrm{l}} + a_T^{\mathrm{r}}\right] = 1 \end{array} \right\}$$

introduced in Remark 3.6.

Before we start with concrete examples, note that different coherent or concave money based utility functionals on $\mathcal{R}^\infty$ can easily be combined to form new ones. Indeed, if $(\phi_j)_{j\geq 1}$ is a sequence of coherent utility functionals on $\mathcal{R}^\infty$ and $(\lambda_j)_{j\geq 1}$ a sequence of numbers in $[0,1]$ such that $\sum_{j\geq 1}\lambda_j = 1$, then $\sum_{j\geq 1}\lambda_j\phi_j$ is again a coherent utility functional on $\mathcal{R}^\infty$, and it satisfies the Fatou property if ϕ_j does for every j such that $\lambda_j > 0$. The same is true for a sequence $(\phi_j)_{j\geq 1}$ of concave money based utility functionals on $\mathcal{R}^\infty$ if the series $\sum_{j\geq 1}\lambda_j\phi(0)$ is convergent.

The obvious interpretation of the following examples in the context of risk management is left to the reader.

Example 5.1 For every subset $\mathcal{P}_\sigma$ of $\tilde{\mathcal{D}}_\sigma$, the function

$$\tilde{\phi}(X) := \inf_{f\in\mathcal{P}_\sigma} \mathrm{E}\left[Xf\right] \,,\; X \in L^\infty \,,$$

is a coherent utility functional on L^∞ that satisfies the Fatou property. If θ is a random variable taking values in $[0,T]$, then

$$\hat{\mathcal{Q}}_\sigma := \left\{ \left(0, f 1_{\{\theta\leq t\}}\right) \mid f \in \mathcal{P}_\sigma \right\} \,,$$

is a subset of $\hat{\mathcal{D}}_\sigma$, and the corresponding coherent utility functional on $\mathcal{R}^\infty$ is given by

$$\phi(X) := \inf_{\hat{a}\in\hat{\mathcal{Q}}_\sigma} \langle X, \hat{a}\rangle = \inf_{f\in\mathcal{P}_\sigma} \mathrm{E}\left[X_\theta f\right] = \tilde{\phi}(X_\theta) \,.$$

In terms of elements of $\mathcal{D}_\sigma$, ϕ can be represented as

$$\phi(X) = \inf_{a\in\Pi^*\hat{\mathcal{Q}}_\sigma} \langle X, a\rangle \,,$$

where Π^* is the projection from $\hat{\mathcal{D}}_\sigma$ to $\mathcal{D}_\sigma$ explained in Remark 3.6. It can easily be checked that if θ is a stopping time, then

$$\Pi^*\hat{\mathcal{Q}}_\sigma = \left\{ \left(0, \mathrm{E}\left[f \mid \mathcal{F}_\theta\right] 1_{\{\theta\leq t\}}\right) \,\middle|\, f \in \mathcal{P}_\sigma \right\} \,.$$

Example 5.2 As in Example 5.1, consider a subset $\mathcal{P}_\sigma$ of $\tilde{\mathcal{D}}_\sigma$ and let

$$\tilde{\phi}(X) := \inf_{f\in\mathcal{P}_\sigma} \mathrm{E}\left[Xf\right] \,,\; X \in L^\infty \,.$$

be the associated coherent utility functional on L^∞. The set

$$\hat{\mathcal{Q}}_\sigma := \left\{ \left(0, (f\frac{t}{T})_{t\in[0,T]}\right) \mid f \in \mathcal{P}_\sigma \right\} \,,$$

is a subset of $\hat{\mathcal{D}}_\sigma$, and the corresponding coherent utility functional is given by

$$\phi(X) := \inf_{\hat{a}\in\hat{\mathcal{Q}}_\sigma} \langle X, \hat{a}\rangle = \inf_{f\in\mathcal{P}_\sigma} \mathrm{E}\left[\frac{1}{T}\int_0^T X_t\, dt\, f\right] = \tilde{\phi}\left(\frac{1}{T}\int_0^T X_t\, dt\right) \,.$$

In terms of elements of $\mathcal{D}_\sigma$, ϕ can be represented as

$$\phi(X) = \inf_{a \in \Pi^* \hat{\mathcal{Q}}_\sigma} \langle X, a \rangle \ ,$$

where Π^* is the projection from $\hat{\mathcal{D}}_\sigma$ to $\mathcal{D}_\sigma$ explained in Remark 3.6. Note that the projection Π^* maps an element $\left(0, (f \frac{t}{T})_{t \in [0,T]}\right) \in \hat{\mathcal{Q}}_\sigma$ to $a = (a^{\mathrm{pr}}, a^{\mathrm{op}})$, where a^{pr} is the continuous part of the dual optional projection $\left(\mathrm{E}\left[f \mid \mathcal{F}_t\right] \frac{t}{T}\right)_{t \in [0,T]}$ of $\left(f \frac{t}{T}\right)_{t \in [0,T]}$ and a^{op} is the purely discontinuous part of $\left(\mathrm{E}\left[f \mid \mathcal{F}_t\right] \frac{t}{T}\right)_{t \in [0,T]}$.

Example 5.3 Let $\mathcal{P}_\sigma$ be a subset of $\tilde{\mathcal{D}}_\sigma$. Then

$$\tilde{\phi}(X) := \inf_{f \in \mathcal{P}_\sigma} \mathrm{E}\left[Xf\right] \ , \ X \in L^\infty \ ,$$

is a coherent utility functional on L^∞ that satisfies the Fatou property. The set

$$\hat{\mathcal{Q}}_\sigma := \left\{ \left(0, f 1_{\{\theta \leq t\}}\right) \ \middle| \ f \in \mathcal{P}_\sigma \ , \ \theta \text{ a } [0,T]\text{-valued random variable} \right\} \ ,$$

is again a subset of $\hat{\mathcal{D}}_\sigma$, and for all $X \in \mathcal{R}^\infty$,

$$\phi(X) := \inf_{a \in \Pi^* \hat{\mathcal{Q}}_\sigma} \langle X, a \rangle = \inf_{\hat{a} \in \hat{\mathcal{Q}}_\sigma} \langle X, \hat{a} \rangle = \tilde{\phi}\left(\inf_{t \in [0,T]} X_t \right) . \tag{5.1}$$

The last equality in (5.1) can been shown as follows: Obviously, for all $X \in \mathcal{R}^\infty$,

$$\tilde{\phi}\left(\inf_{t \in [0,T]} X_t \right) \leq \mathrm{E}\left[X_\theta f\right] \ ,$$

for all $f \in \mathcal{P}_\sigma$ and all $[0,T]$-valued random variables θ. Therefore,

$$\tilde{\phi}\left(\inf_{t \in [0,T]} X_t \right) \leq \phi(X) \, .$$

On the other hand, for all $\varepsilon > 0$, there exists an $f \in \mathcal{P}_\sigma$ such that

$$\mathrm{E}\left[\inf_{t \in [0,T]} X_t \, f \right] \leq \tilde{\phi}\left(\inf_{t \in [0,T]} X_t \right) + \frac{\varepsilon}{2} \, .$$

Furthermore, it can be deduced from the cross section theorem (see for example, Theorem 44 of Chapter III in [DM1]) that there exists a $[0,T]$-valued random variable θ such that

$$X_\theta \leq \inf_{t \in [0,T]} X_t + \frac{\varepsilon}{2} \, .$$

Hence,

$$\mathrm{E}\left[X_\theta f\right] \leq \mathrm{E}\left[\inf_{t \in [0,T]} X_t \, f \right] + \frac{\varepsilon}{2} \leq \tilde{\phi}(\inf_{t \in [0,T]} X_t) + \varepsilon \, ,$$

and it follows that

$$\phi(X) \leq \tilde{\phi}\left(\inf_{t \in [0,T]} X_t \right) \, .$$

Example 5.4 Let $\tilde{\gamma} : \tilde{\mathcal{D}}_\sigma \to [-\infty, \infty)$ be a penalty function and

$$\tilde{\phi}(X) = \inf_{f \in \tilde{\mathcal{D}}_\sigma} \{\langle X, f \rangle - \tilde{\gamma}(f)\} \, , \; X \in L^\infty \, ,$$

the corresponding concave money based utility functional on L^∞. For a $[0,T]$-valued random variable θ we define the function $\hat{\gamma} : \hat{\mathcal{D}}_\sigma \to [-\infty, \infty)$ by

$$\hat{\gamma}(\hat{a}) := \begin{cases} \tilde{\gamma}(f) \, , & \text{if } \hat{a} \text{ is of the form } \left(0, f 1_{\{\theta \le t\}}\right) \text{ for some } f \in \tilde{\mathcal{D}}_\sigma \\ -\infty \, , & \text{otherwise} \end{cases} .$$

As in Remark 3.6, one can construct from $\hat{\gamma}$ a penalty function $\gamma : \mathcal{D}_\sigma \to [-\infty, \infty)$ such that

$$\inf_{a \in \mathcal{D}_\sigma} \{\langle X, a \rangle - \gamma(a)\} = \inf_{\hat{a} \in \hat{\mathcal{D}}_\sigma} \{\langle X, \hat{a} \rangle - \hat{\gamma}(\hat{a})\} = \tilde{\phi}(X_\theta) \, .$$

Example 5.5 Consider a penalty function $\tilde{\gamma} : \tilde{\mathcal{D}}_\sigma \to [-\infty, \infty)$ and the associated concave money based utility functional

$$\tilde{\phi}(X) = \inf_{f \in \tilde{\mathcal{D}}_\sigma} \{\langle X, f \rangle - \tilde{\gamma}(f)\} \, , \; X \in L^\infty \, .$$

Define the function $\hat{\gamma} : \hat{\mathcal{D}}_\sigma \to [-\infty, \infty)$ as follows

$$\hat{\gamma}(\hat{a}) := \begin{cases} \tilde{\gamma}(f) \, , & \text{if } \hat{a} \text{ is of the form } \left(0, \left(f \frac{t}{T}\right)_{t \in [0,T]}\right) \text{ for some } f \in \tilde{\mathcal{D}}_\sigma \\ -\infty \, , & \text{otherwise} \end{cases} .$$

As in Remark 3.6, one can construct a penalty function $\gamma : \mathcal{D}_\sigma \to [-\infty, \infty)$ such that

$$\inf_{a \in \mathcal{D}_\sigma} \{\langle X, a \rangle - \gamma(a)\} = \inf_{\hat{a} \in \hat{\mathcal{D}}_\sigma} \{\langle X, \hat{a} \rangle - \hat{\gamma}(\hat{a})\} = \tilde{\phi}\left(\frac{1}{T} \int_0^T X_t \, dt\right) .$$

Example 5.6 Let $\tilde{\phi} : L^\infty \to \mathbb{R}$ be a concave money based utility functional on L^∞ that can be represented as

$$\tilde{\phi}(X) = \inf_{f \in \tilde{\mathcal{D}}_\sigma} \{\langle X, f \rangle - \tilde{\gamma}(f)\} \, , \; X \in L^\infty \, ,$$

for some penalty function $\tilde{\gamma} : \tilde{\mathcal{D}}_\sigma \to [-\infty, \infty)$. Define the function $\hat{\gamma} : \hat{\mathcal{D}}_\sigma \to [-\infty, \infty)$ as follows:

$$\hat{\gamma}(\hat{a}) := \begin{cases} \tilde{\gamma}(f) \, , & \text{if } \hat{a} \text{ is of the form } \left(0, f 1_{\{\theta \le t\}}\right) \text{ for some } f \in \tilde{\mathcal{D}}_\sigma \\ & \text{and a } [0,T]\text{-valued random variable } \theta \\ -\infty \, , & \text{otherwise} \end{cases} .$$

By Remark 3.6, there exists a penalty function $\gamma : \mathcal{D}_\sigma \to [-\infty, \infty)$ such that

$$\inf_{a \in \mathcal{D}_\sigma} \{\langle X, a \rangle - \gamma(a)\} = \inf_{\hat{a} \in \hat{\mathcal{D}}_\sigma} \{\langle X, \hat{a} \rangle - \hat{\gamma}(\hat{a})\} \, ,$$

and, as in Example 5.3, it can be deduced from the cross section theorem that

$$\inf_{\hat{a} \in \hat{\mathcal{D}}_\sigma} \{\langle X, \hat{a} \rangle - \hat{\gamma}(\hat{a})\} = \tilde{\phi}\left(\inf_{t \in [0,T]} X_t\right) .$$

Example 5.7 The set

$$\mathcal{Q}_\sigma := \left\{ \left(0, 1_{\{\tau \leq t\}}\right) \mid \tau \text{ a } [0,T]\text{-valued stopping time} \right\} \subset \mathcal{D}_\sigma$$

induces the coherent utility functional

$$\phi(X) := \inf \left\{ \mathrm{E}\left[X_\tau\right] \mid \tau \text{ a } [0,T]\text{-valued stopping time} \right\} , \; X \in \mathcal{R}^\infty .$$

It follows from the theory of optimal stopping (see for instance, page 417 of [DM2]) that for all $X \in \mathcal{R}^\infty$,

$$\phi(X) = S_0(X) ,$$

where $(S_t(X))_{t \in [0,T]}$ is the largest submartingale that is dominated by the process X.

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