Risk Measures or Measures that Describe Risk?

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The definition of risk is extremely complicated if not impossible. When dealing with situations involving risk, human beings act in many, sometimes contradictory ways. Modern economics have tried to give a theory of such behaviour, but when testing these theories, researchers found out that we do not always behave according to these basic rules. It is not surprising that a theory that tries to describe risk or tries to measure risk is the subject of criticism. When we presented our approach, we were told that we cannot represent the risk by just one number. Unfortunately we need such a theory as we have to make decisions that deal with the availability of money under uncertainty. In such a case we have to give a yes-no answer, meaning a one-bit representation that uses a lot less information than a number. The applications of risk measurement go from classical lotteries to the regulation of financial business by the national or international authorities. In this little note we will try to give a small overview on how we see the problem.

Perhaps the easiest example lies in the analysis of a trivial lottery. Let us say that we are proposed a deal that consists of winning 50 with probability 1/2 or winning 150 with probability 1/2. The average is 100. But probably very few people are willing to pay 100 for this game. It would result in a future wealth of -50 or +50, each with probability 1/2. The game looks neutral or fair but most people would prefer to have 0 for sure than having + or -50 with probability one-half. This attitude is called risk averseness. We deliberately forgot to give monetary units, since we realise that the answer could depend on the fact whether the units were Rappen, CHF, KiloCHF or MegaCHF. But for the applications we have in mind, we cannot make distinctions between this – difficult to explain – influence of psychology. So we will suppose that we are risk averse. As a result we will not be willing to pay the average value for this lottery. We probably will be willing to enter the game for less than 100, where the exact quantity we would be willing to pay could be related to the amount of risk-averseness. So let us say we are willing to pay 90. Although easy and simple, this example reflects already the basics of modern finance. The price we are willing to pay is not the average value! Usually translated by: "we want to be compensated for the risk we are taking" or "if there is no risk premium, we cannot enter the deal". There are at least two viewpoints that both merit to be mentioned. The first viewpoint is that when calculating prices, we change the probabilities. In our example this would mean that the probabilities we were using were not 50-50 but rather 40-60. By following this viewpoint prices become averages and this has the important consequence that prices are linear, meaning that the price of the sum of two positions is the sum of the prices. In modern terminology, we replaced the original measure by a so-called risk neutral measure. This practice of changing the measure to cope with risk averseness is not new. In life insurance, the mortality table used for pensions and for pure death risk are not the same. Also tha ge used to calculate the life insurance premium is not the age of the person but is lower or higher depending on whether a pension or a capital has to be insured. The second viewpoint is that prices are calculated by a different procedure not leading to a linear price mechanism. The former viewpoint is applicable to a very efficient market, where transaction costs are negligible and where products can be bought and sold without restrictions ("going short is allowed"). The latter viewpoint is applicable to more general situations and allows for a more detailed modelling of the diversification effect. By not assuming that prices are linear we can say that the price of the sum of two positions if greater than the sum of the prices since by buying two positions we might benefit from diversification or might benefit from the fact that we eliminate risk completely. This is for instance the case in a large company or bank where one unit takes a position in a foreign currency and another unit takes an opposite position, sometimes without knowing from each other that such positions exist.

In this example we supposed that the probability measure was known. This is in many cases a very optimistic assumption. In economics, where we talk of future outcomes for a financial deal or when we talk of the probability that the market will move in one or other direction, the word probability is not used in a physical sense. In is not the result of a statistical estimation based on many observations of experiments all done in more or less the same circumstances. The probability measure is rather the outcome of a personal analysis of the real world and is therefore rather subjective. This explains (but it is by far not the only explanation) that somebody wants to buy a stock and others would rather prefer to sell it. In case we deal with large sums, risk averseness may disappear. This is for instance the case when we buy car-insurance or life insurance. We are willing to pay a premium much higher than the average claim we will cause. Maybe that we attribue a higher probability to the event of causing an accident or maybe we are afraid of the consequences or maybe we just buy insurance because the law says we have to.

But how did we define risk in this example? We will not give a definition but rather use the word in a non-mathematical sense. When referring to measures of risk or risk-measures we do not claim that we have a precise mathematical definition of the concept "risk". It might therefore be better to speak of measures that describe risk, or measures that allow us to give the manager or decision maker a "quantitative" tool to compare – or better help to compare – different alternatives. We know that the theory has shortcomings but we also realise that if we want quantitative tools, we better use a theory instead of unfounded number juggling. To be useful such a theory should be free of contradictions and it should reflect the basic thinking on risk taking. Therefore we want to be very precise on the definitions of the *measures of risk* and allow ourselves to remain silent when it comes to the definition of *risk* itself. The former we consider as a mathematical object, to be defined in a precise way, the latter we consider as a concept from real life subject to different interpretations and strongly dependent on the mood of the persons.

One of the first applications of measures of risk – although not called that way – was probably the calculation of premiums in insurance. Long before the theory was invented, insurers knew that the premium for an insurance had to be higher than the average pay-out. (That the premium is higher than the average and not lower is not in contradiction with our example where we advocated a number below the average. It is simply the fact that the income is taken with a positive sign and the pay-out is taken with a negative sign.) The reasoning is simple: we know that if a large group is insured, the final realisation of the claims will result in a sum that is not equal to the expected value but will be around this value, sometimes bigger, sometimes smaller. When only the mean value were used, the company would lose each time the deviation was positive. Nowadays we know, thanks to probability theory, that such a situation will eventually lead to bankruptcy, no matter how high the starting capital was. The first idea is then to use the deviation from the mean or average value to measure the risk taken and to calculate the loading or extra part of the premium. This suggest that we could use the standard deviation as a good measure. In modern finance this idea was used by Markowitz in his portfolio theory and it was also at the basis of the Capital Asset Pricing Model (CAPM). However the standard deviation has a big disadvantage. It treats the negative and the positive deviations from the mean in the same way. In our lottery example this means that the "50" was given the same influence as the "150". Already Markowitz realised this and in the footnotes of his book, he said that other measures can be used, e.g. the semi-variance. The CAPM itself can be obtained without referring to the standard deviation and it follows already from simple arbitrage arguments or even better it already follows from what is usually called the "law of one price", an extremely weak form of consistency between market prices. In insurance the use of the standard deviation is doubtful when we have to deal with probability distributions with fat tails. In such cases the standard deviation might not even exist. This is the case for reinsurance problems. The distributions used there, usually have fat tails and reflect the overshoot over a high level. These so-called extreme value distributions are the topic of recent scientific research and their use in insurance and finance is well accepted. It is clear that we need more sophisticated tools than just standard deviation.

Another example that got a lot of attention and became important was the value at risk or VaR. Given the probability distribution of the future wealth of a financial institution, the value at risk at the level α, is the quantity that is defined in such a way that in less than α% of the cases the wealth will drop below this level. In practice the time horizon used is the ten day period and the level is quite low, typically 0.5%, 1% or 5%. When this risk measure is used, we will accept positions as safe when in say, less than 1% of the cases, we get in trouble. In earlier days this definition as a quantile was not used, it was rather replaced by a multiple of the standard deviation. The author remembers a discussion where the subject was whether the multiple would have to be 3 or 5. This practice of is course not very intelligent. It uses the fact that the quantile is defined through the standard deviation and the mean, a fact that is only valid for a small class of probability distributions and is not satisfied by the extreme value distributions already mentioned. Also the positions built by the traders can have strange form (there is a mathematical theorem that says that any distribution can be arbitrarily approximated by a position in one stock and the options written on it). The standard deviation technique works quite well for the normal distribution but it does not for others. Here the word "normal distribution" is misleading. It suggest that the other distributions are abnormal and that in normal situations we indeed get the normal distribution. This remark is not only meant as a joke, some people even believe it. It would be better to adapt the French terminology and speak of "Gaussian distributions", the big advantage being that it suggests that it is a technical term.

The value at risk, even when seen as a quantile, has some dangers. When used we will accept positions that with high probability are good and that with a very small probability lead to bankruptcy. For the stockholder of a company this seems good, since in any case he/she is not liable for the amount of the bankruptcy. The situation changes if viewed from the side of the regulator, who is supposed to protect "Society". From his/her viewpoint the amount of the bankruptcy might be more important than just the fact that there is a bankruptcy.

Inside a financial institution, the use of value at risk is even more problematic. When applied to individual business units, it could lead to a position that is good in 99% of the cases but in one percent of the cases leaves the company with a big loss, a loss that has then to be covered by the gains of the other business units. The use of value at risk seems to lead to more risk taking, it favourises "free riders" and so called "doubling strategies", where in 99% of the cases the trader gets a gain (and consequently a bonus) and in one percent the trader makes a tremendous loss, gets fired or if really severe causes the company to file for bankruptcy. It favours the practice "take the money and run".

We proposed a more structural approach to risk measures and we started from a set of basic properties a risk measure would need to satisfy. These properties were discussed with practitioners and were tested against their opinion on how to deal with risky situations. The interested reader can look up our papers on risk measures (e.g. Artzner-Delbaen-Eber-Heath "Thinking Coherently" in Risk Magazine November 1997 or the Pisa lecture notes of the author, see http://www.math.ethz.ch/~delbaen) to get an idea and to see a discussion of the different properties we set forward. Since then, the theory has evolved a lot and we are presently working on a dynamic version of this theory. It is impossible to even summarize the main points in this theory. So let us mention just a couple of remarks. The risk measures are defined on the set of random variables so that it is possible to use correlation models when dealing with different positions. This reflects the fact that the acceptance of a new deal, say a new insurance contract or a new kind of option, is dependent on the other elements in the portfolio. The risk measures we propose are coherent, meaning that diversification has a positive impact. A diversified portfolio needs less economic capital than a non-diversified portfolio. (in a credit portfolio the use of value at risk on the individual loan level would rather give the opposite as was illustrated by Albanese). Since we know that in practice and especially in economics the exact distribution of the future outcomes is difficult to get, we propose the use of different probability measures. This allows for stress-testing to be built in.

The fact that diversification is rewarded and not punished also allows to treat the capital allocation problem in a consistent way. The capital allocation problem is the following. Once the economic capital for the firm is calculated, how can you allocate this capital to the different business units? This is not an investment problem. Phrased in another way, we could say that each business unit requires its own economic capital but that because they are in a bigger entity, they could benefit from the diversification effect. The problem is how? Such problems are important since more and more, business units are evaluated on their returns. But return on what? The use of coherent risk measures allows to define the risk adjusted capital for each business unit in a coherent way, meaning that the outcome is fair in a game theoretic sense.

It is difficult to predict whether the coherent risk measures will eventually replace the use of VaR. We already observed that they are used more and more.