(This is a slightly revised version of the original paper)

Another Proof of Pick's Area Theorem

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Let P be a *simple lattice polygon*, i.e., a polygonal Jordan domain in the plane whose vertices have integer coordinates. *Pick's theorem* says that the area $\mu(P)$ is given by

$$\mu(P) = i + \frac{b}{2} - 1\,,$$

where i and b denote the number of lattice points in the interior and on the boundary of P, respectively. Several proofs of this theorem can be found in the literature (see [1] for a recent list of references); most of them use a dissection argument combined with an analysis of certain triangles. Here I offer a proof of a more conceptional nature; it has the form of a *Gedankenexperiment* (thought-experiment).

Assume that at time 0 a unit of heat is concentrated at each lattice point. This heat will be distributed over the whole plane by heat conduction, and at time ∞ it is equally distributed on the plane with density 1. In particular, the amount of heat contained in P will be $\mu(P)$. Where does this amount of heat come from? Consider a segment e between two consecutive boundary lattice points. The midpoint m of e is a symmetry center of the lattice, so at each instant the heat flow is centrally symmetric with respect to m. This implies that the total heat flux across e is 0. As a consequence, the final amount of heat within P comes from the i interior lattice points and from the b boundary lattice points. To account for the latter, orient ∂P so that the interior is to the left of ∂P. The amount of heat going from a boundary lattice point into the interior of P is a half, minus the turning angle of ∂P at that point, measured in units of 2π. Since the sum of all turning angles for a simple polygon is known to be one full turn, we arrive at the stated formula.

Günter M. Ziegler has remarked that one can replace the units of heat, e.g., by thin circular cylinders of unit volume and let these cylinders collectively melt.

REFERENCE

1. B. Grünbaum and G.C. Shephard: Pick's theorem, *Amer. Math. Monthly* 100 (1993), 150–160.