Asymptotic Support Theorem for Planar Isotropic Brownian Flows

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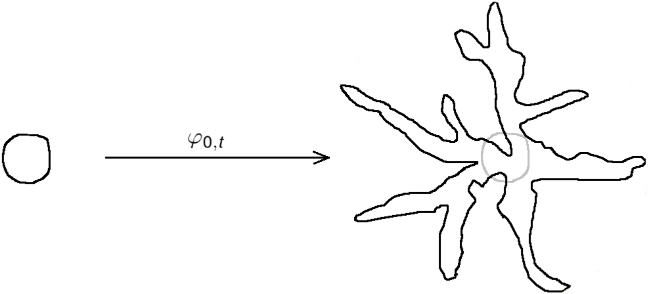
IRTG Summer School 2010 – Disentis/Mustér

Question of interest

We are interested in the evolution of a set $\mathcal{X} \subset \mathbf{R}^2$



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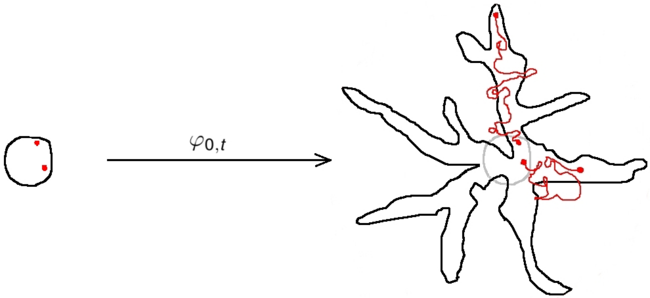


... or more precisely in the behaviour of the individual trajectories.

Stochastic Flows

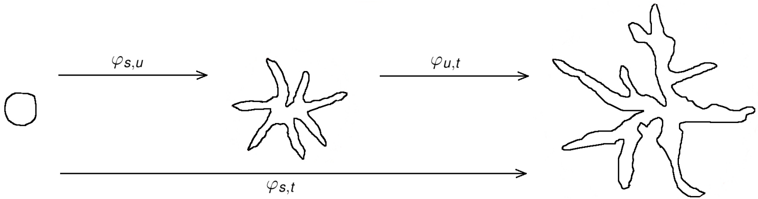
Stochastic Flow

A familiy $(\varphi_{s,t} : s, t \in \mathbf{R}_+)$ of random homeomorphims on $\mathbf{R}^d$ is called a *stochastic flow of homeomorphisms* if (almost surely)

i) $\varphi_{s,s} = \mathrm{id}|_{\mathbf{R}^d}$ for all $s \in \mathbf{R}_+$,

ii) $\varphi_{u,t} \circ \varphi_{s,u} = \varphi_{s,t}$ for all $s, t, u \in \mathbf{R}_+$ and

iii) $(x, s, t) \mapsto \varphi_{s,t}(x)$ is continuous.



Describe the evolution of passive tracers within a turbulent fluid, e.g. oil spill on the surface of an ocean.

Isotropic Brownian Flows (IBF)

Brownian Flow

A stochastic flow φ is called a *Brownian flow* if it

i) has independent increments, i.e. for $0 \leq t_1 < \ldots < t_n < \infty$ the random homeomorphisms $(\varphi_{t_i,t_{i+1}})_i$ are independent and

ii) is time homogeneous, i.e. the law of $(\varphi_{s+h,t+h})_{s,t\in\mathbf{R}_+}$ does not depend on $h \in \mathbf{R}_+$.

Isotropic Brownian Flow

A Brownian flow φ is called an *isotropic Brownian flow* if it is

i) space homogeneous, i.e. for all $\xi \in \mathbf{R}^d$ the laws of $(\varphi_{s,t}(\cdot + \xi))_{s,t\in\mathbf{R}_+}$ and $(\varphi_{s,t}(\cdot) + \xi)_{s,t\in\mathbf{R}_+}$ coincide and

ii) rotation and reflection invariant, i.e. the laws of $(\varphi_{s,t}(O\cdot))_{s,t\in\mathbf{R}_+}$ and $(O\varphi_{s,t}(\cdot))_{s,t\in\mathbf{R}_+}$ coincide for all orthogonal matrices O on $\mathbf{R}^d$.

Representation of IBF

Under suitable regularity conditions we have

$$\varphi_{s,t}(x) = x + \int_s^t M(du, \varphi_{s,u}(x)), \qquad \text{for } s \le t,$$

where $M : \mathbf{R}_+ \times \mathbf{R}^d \times \Omega \to \mathbf{R}^d$ is a mean-zero Gaussian martingale field – the *generating Brownian field* of φ.

The covariances of M are given by

$$\mathbf{E}\left[M(t,x)_i M(s,y)_j\right] = (s \wedge t) b_{i,j}(|x-y|) \qquad \text{for } i,j = 1, \ldots d,$$

where $b : \mathbf{R} \to \mathbf{R}^{d \times d}$ is the *covariance tensor*.

We assume (wlog): $b(0) = \mathrm{Id}|_{\mathbf{R}^d} \quad \Rightarrow \quad (M(t,x))_{t \ge 0}$ is BM

Known Results

IBFs have Lyapunov exponents $\mu_i, i = 1, \ldots, d$, which are determined by the covariance tensor b.

Linear Growth (Cranston, Scheutzow, Steinsaltz)

A non-negative top-Lyapunov exponent $\mu_1 \geq 0$ implies that the diameter of any non-trivial bounded connected set in $\mathbf{R}^d$ grows linearly in time.

Linear Growth Rate for Planar IBF (van Bargen)

If $\mu_1 > 0$ then the linear growth rate K for a planar IBF is a deterministic constant and can be described via the *stable norm*.

Question of Interest

Let $F_T(\mathcal{X})$ be the set of time-scaled trajectories starting in $\mathcal{X} \subseteq \mathbf{R}^2$, i.e.

$$F_T(\mathcal{X}) := \bigcup_{x \in \mathcal{X}} \left\{ [0,1] \ni t \mapsto \frac{1}{T} \varphi_{0,tT}(x) \right\}.$$

Main Theorem (2010+)

If $\mu_1 > 0$ then there exists a constant K such that for any $\varepsilon > 0$ and non-trivial compact connected $\mathcal{X} \subseteq \mathbf{R}^2$ we have

$$\lim_{T \to \infty} \mathbf{P} \left(d_H(F_T(\mathcal{X}), \mathrm{Lip}_0(K)) \le \varepsilon \right) = 1.$$

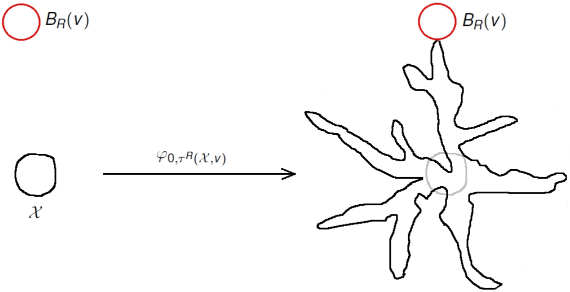
Where d_H is the Hausdorff-distance, i.e.

$$d_H(F_T(\mathcal{X}), \mathrm{Lip}_0(K)) := \max \left\{ \sup_{g \in F_T(\mathcal{X})} d(g, \mathrm{Lip}_0(K)); \sup_{f \in \mathrm{Lip}_0(K)} d(f, F_T(\mathcal{X})) \right\}.$$

Stable Norm

For $R \geq 1$, $\mathcal{X} \subseteq \mathbf{R}^2$ and $v \in \mathbf{R}^2$ define the hitting-time

$$\tau^R(\mathcal{X}, v) := \inf\left\{t \geq 0 : \varphi_{0,t}(\mathcal{X}) \cap B_R(v) \neq \emptyset; \operatorname{diam}(\varphi_{0,t}(\mathcal{X})) \geq 1\right\}$$



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Stable Norm

For $R \geq 1$ and $v \in \mathbf{R}^2$ let

$$\|v\|^R := \lim_{T \to \infty} \frac{1}{T} \sup_{\substack{\mathcal{X} \subseteq B_{2R}(0) \\ \operatorname{diam}(\mathcal{X}) \geq 1}} \mathbf{E}\left[\tau^R(\mathcal{X}, vT)\right].$$

- Stable norm does not depend on R.
- Lipschitz constant K is (Euclidean) radius of the unit ball in $\|\cdot\|^R$

Idea of the Proof

$$\lim_{T\to\infty} \mathbf{P}\left(d_H(F_T(\mathcal{X}), \mathrm{Lip}_0(K)) \leq \varepsilon\right) = 1.$$

Upper Bound

$$\mathbf{P}\left(\forall_{x\in\mathcal{X}}\ \exists_{f\in\mathrm{Lip}_0(K)}\ : \left\|\frac{1}{T}\varphi_{0,\cdot T}(x) - f\right\|_\infty \leq \varepsilon\right) \to 1,$$

in other words: $\mathcal{X}$ expands **not too fast** under the action of the flow $\rightsquigarrow \tau^R(\mathcal{X}, vT)/T$ is **not too small.**

Lower Bound

$$\mathbf{P}\left(\forall_{f\in\mathrm{Lip}_0(K)}\ \exists_{x\in\mathcal{X}}\ : \left\|\frac{1}{T}\varphi_{0,\cdot T}(x) - f\right\|_\infty \leq \varepsilon\right) \to 1,$$

in other words: $\mathcal{X}$ expands **fast enough** under the action of the flow $\rightsquigarrow \tau^R(\mathcal{X}, vT)/T$ is **not too big.**

Take-home Message and Future Prospects

Main Theorem

Characterization of the asymptotic support of the time-scaled trajectories starting in a non-trivial compact connected set.

Future Prospects

- ▷ Achieve almost sure convergence.
- ▷ Extend this result to higher dimensions.

Thank you for your attention.