BSDEs With Rough Drivers

Joscha Diehl
(joint work with Peter Friz, TU Berlin)

29. July 2010
IRTG Summer School
Disentis

Outline

Classical BSDE theory concerns the solution of

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr - \int_t^T Z_r dW_r.$$

We are interested in giving meaning to

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \sum_{k=1}^{d} \int_t^T H_k(Y_r)d\zeta_r^k - \int_t^T Z_r dW_r.$$

When ζ is smooth, this is just a Lipschitz BSDE

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T H(Y_r)\dot{\zeta}_r dr - \int_t^T Z_r dW_r,$$

Classical BSDE theory concerns the solution of

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr - \int_t^T Z_r dW_r.$$

We are interested in giving meaning to

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T H(Y_r)d\zeta_r - \int_t^T Z_r dW_r.$$

When ζ is smooth, this is just a Lipschitz BSDE

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T H(Y_r)\dot{\zeta}_r dr - \int_t^T Z_r dW_r,$$

Classical BSDE theory concerns the solution of

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr - \int_t^T Z_r dW_r.$$

We are interested in giving meaning to

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T H(Y_r)d\zeta_r - \int_t^T Z_r dW_r.$$

When ζ is smooth, this is just a Lipschitz BSDE

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T H(Y_r)\dot{\zeta}_r dr - \int_t^T Z_r dW_r,$$

Short reminder on rough path theory

It is well known that the solution mapping $\zeta \mapsto x$ to the (deterministic) ODE

$$x_t = x_0 + \int_0^t H(x_r) d\zeta_r = x_0 + \int_0^t H(x_r) \dot{\zeta}_r dr,$$

defined for *smooth paths* ζ is in general *not* continuous in supremum norm.

For $p \geq 1$, rough path theory

- embeds smooth paths $\zeta \in C^\infty([0,T], \mathbb{R}^d) \subset C([0,T], G^{[p]}(\mathbb{R}^d))$,
- defines a metric d_{p-var} on this space (p-variation),
- can then handle the closure of all smooth paths with respect to this metric (*geometric p-rough paths*).

Short reminder on rough path theory

It is well known that the solution mapping $\zeta \mapsto x$ to the (deterministic) ODE

$$x_t = x_0 + \int_0^t H(x_r) d\zeta_r = x_0 + \int_0^t H(x_r)\dot{\zeta}_r dr,$$

defined for *smooth paths* ζ is in general *not* continuous in supremum norm.

For $p \geq 1$, rough path theory

- embeds smooth paths $\zeta \in C^\infty([0,T],\mathbb{R}^d) \subset C([0,T],G^{[p]}(\mathbb{R}^d))$,
- defines a metric d_{p-var} on this space (p-variation),
- can then handle the closure of all smooth paths with respect to this metric (*geometric p-rough paths*).

Back to BSDEs

We factorize the randomness of the vector field H through an Ito process, i.e. we consider

$$X_t = x + \int_0^t b_r dr + \int_0^t \sigma_r dW_r,$$

$$Y_t = \xi + \int_t^T f(\omega; r, Y_r, Z_r) dr + \int_t^T H(X_r, Y_r) d\zeta_r - \int_t^T Z_r dW_r.$$

Where

- $\xi \in L^\infty(\mathcal{F}_T)$,
- ζ is a p-rough path for a $p \geq 1$,
- b, σ are predictable and bounded,
- f is predictable and Lipschitz in (y, z),
- $H = (H_1, \cdots, H_d)$ has bounded derivatives up to order $2 + p$.

Back to BSDEs

We factorize the randomness of the vector field H through an Ito process, i.e. we consider

$$X_t = x + \int_0^t b_r dr + \int_0^t \sigma_r dW_r,$$
$$Y_t = \xi + \int_t^T f(\omega; r, Y_r, Z_r) dr + \int_t^T H(X_r, Y_r) d\zeta_r - \int_t^T Z_r dW_r.$$

Where

- $\xi \in L^\infty(\mathcal{F}_T)$,
- ζ is a p-rough path for a $p \geq 1$,
- b, σ are predictable and bounded,
- f is predictable and Lipschitz in (y, z),
- $H = (H_1, \cdots, H_d)$ has bounded derivatives up to order $2 + p$.

Theorem (Definition/Existence/Uniqueness of solution)

Assume $\zeta^n \to \zeta$ in p-variation. Let (Y^n, Z^n) solve

$$Y_t^n = \xi + \int_t^T f(r, Y_r^n, Z_r^n)dr + \int_t^T H(X_r, Y_r^n)d\zeta_r^n - \int_t^T Z_r^n dW_r.$$

Then there exists a process (Y, Z) such that

$$Y^n \to Y, \quad Z^n \to Z.$$

It only depends on ζ and not on the approximating sequence.
We write (formally)

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T H(X_r, Y_r)d\zeta_r - \int_t^T Z_r dW_r. \tag{1}$$

Proof idea

- Define the (deterministic) flow

$$\phi^n(t,x,y) = y + \int_t^T H(x, \phi^n(r,x,y)) d\zeta^n_r.$$

- $\tilde{Y}^n_t := [\phi^n]^{-1}(t, X_t, Y^n_t)$, $\tilde{Z}^n_t := \dots$ then satisfies the BSDE

$$\tilde{Y}^n_t = \xi + \int_t^T \tilde{f}^n(r, X_r, \tilde{Y}^n_r, \tilde{Z}^n_r) dr - \int_t^T \tilde{Z}^n_r dW_r.$$

where $\tilde{f}^n$ is quadratic in z (e.g. if $f \equiv 0$ then $\tilde{f}^n(t,x,y,z) = \frac{\partial_{yy}\phi^n(t,x,y)}{\partial_y \phi^n(t,x,y)} |z|^2$).
- Use standard rough path results to get $\tilde{f}^n \to \tilde{f}^0$ and use comparison for BSDEs (Kobylanski '00) to show that $\tilde{Y}^n$ and $\tilde{Z}^n$ converge

Proof idea

- Define the (deterministic) flow

$$\phi^n(t,x,y) = y + \int_t^T H(x,\phi^n(r,x,y))d\zeta^n_r.$$

- $\tilde{Y}^n_t := [\phi^n]^{-1}(t,X_t,Y^n_t)$, $\tilde{Z}^n_t := \dots$ then satisfies the BSDE

$$\tilde{Y}^n_t = \xi + \int_t^T \tilde{f}^n(r,X_r,\tilde{Y}^n_r,\tilde{Z}^n_r)dr - \int_t^T \tilde{Z}^n_r dW_r.$$

where $\tilde{f}^n$ is quadratic in z (e.g. if $f \equiv 0$ then $\tilde{f}^n(t,x,y,z) = \frac{\partial_{yy}\phi^n(t,x,y)}{\partial_y \phi^n(t,x,y)}|z|^2$).

- Use standard rough path results to get $\tilde{f}^n \to \tilde{f}^0$ and use comparison for BSDEs (Kobylanski '00) to show that $\tilde{Y}^n$ and $\tilde{Z}^n$ converge

Proof idea

- Define the (deterministic) flow

$$\phi^n(t,x,y) = y + \int_t^T H(x, \phi^n(r,x,y)) d\zeta^n_r.$$

- $\tilde{Y}^n_t := [\phi^n]^{-1}(t, X_t, Y^n_t)$, $\tilde{Z}^n_t := \ldots$ then satisfies the BSDE

$$\tilde{Y}^n_t = \xi + \int_t^T \tilde{f}^n(r, X_r, \tilde{Y}^n_r, \tilde{Z}^n_r) dr - \int_t^T \tilde{Z}^n_r dW_r.$$

  where $\tilde{f}^n$ is quadratic in z (e.g. if $f \equiv 0$ then $\tilde{f}^n(t,x,y,z) = \frac{\partial_{yy}\phi^n(t,x,y)}{\partial_y \phi^n(t,x,y)} |z|^2$).
- Use standard rough path results to get $\tilde{f}^n \to \tilde{f}^0$ and use comparison for BSDEs (Kobylanski '00) to show that $\tilde{Y}^n$ and $\tilde{Z}^n$ converge

Outline

If ζ is a smooth path, it is well known that the FBSDE

$$X_t^{s,x} = x + \int_s^t \sigma(r, X_r^{s,x}) dW_r + \int_s^t b(r, X_r^{s,x}) dr,$$

$$Y_t^{s,x} = g(X_T^{s,x}) + \int_t^T f(r, X_r^{s,x}, Y_r^{s,x}, Z_r^{s,x}) dr + \int_t^T H(X_r^{s,x}, Y_r^{s,x}) d\zeta_r - \int_t^T Z_r^{s,x} dW_r.$$

is connected (via $u(t,x) := Y_t^{t,x}$) to the PDE

$$\partial_t u + Lu + f(t, x, u, Du \cdot \sigma(t,x)) + H(x,u)\dot{\zeta}_t = 0,$$
$$u(T,x) = g(x).$$

where

$$Lu(t,x) := \frac{1}{2} \mathrm{Tr}[\sigma(t,x)\sigma(t,x)^T D^2 u(t,x)] + \langle b(t,x), Du(t,x) \rangle$$

If $\zeta^n \to \zeta$, in p-variation, then by the previous section

$$u^n(t,x) \to u(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

- could hence define a solution to the *rough PDE* (Caruana/Friz/Oberhauser '09)
- but: not straightforward, via this approach, to show uniform convergence on compacta and continuity of the solution map
- hence work directly on PDE

If $\zeta^n \to \zeta$, in p-variation, then by the previous section

$$u^n(t,x) \to u(t,x), \qquad (t,x) \in [0,T] \times \mathbb{R}^n.$$

- could hence define a solution to the *rough PDE* (Caruana/Friz/Oberhauser '09)
- but: not straightforward, via this approach, to show uniform convergence on compacta and continuity of the solution map
- hence work directly on PDE

Theorem

Assume $\zeta^n \to \zeta$ in p-variation. Let $u^n \in BUC([0,T] \times \mathbb{R}^n)$ be the solution to

$$\partial_t u^n + Lu^n + f(t,x,u^n,Du^n \cdot \sigma(t,x)) + H(x,u^n)\dot{\zeta}_t^n = 0,$$
$$u^n(T,x) = g(x).$$

Then (under apropriate assumptions on g, f and H) there exists a $u \in BUC([0,T] \times \mathbb{R}^n)$, only dependent on ζ but not on the approximating sequence ζ^n, such that

$$u^n \to u \qquad \text{locally uniformly.}$$

We write (formally)

$$du + [Lu + f(t,x,u,Du \cdot \sigma(t,x))]\, dt + H(x,u)d\zeta_t = 0,$$
$$u(T,x) = g(x).$$

Outline

BSDEs With Rough Drivers

Markovian Setting - Connection To Rough PDEs

Connection To BDSDEs

Let

$$\Omega^B = C([0,T],\mathbb{R}^d), \Omega^W = C([0,T],\mathbb{R}^m),$$

with Wiener measures $\mathbb{P}^B$, $\mathbb{P}^W$. Let

$$\Omega = \Omega^B \times \Omega^W, \mathbb{P} := \mathbb{P}^B \otimes \mathbb{P}^W.$$

Define $B(\omega^1,\omega^2) := \omega^1$, $W(\omega^1,\omega^2) = \omega^2$; so B is a d-dimensional Brownian motion, W is an independent m-dimensional Brownian motion.

In this setting Pardoux/Peng '92 introduced the notion of *BDSDEs*

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r) dr + \int_t^T H(X_r, Y_r) \circ dB_r - \int_t^T Z_r dW_r.$$

We note that it is $\mathcal{F}_t := \mathcal{F}^B_{t,T} \vee \mathcal{F}^W_{0,t}$ adapted, where $\mathcal{F}^B_{t,T} := \sigma(B_r : r \in [t,T]), \mathcal{F}^W_{0,t} := \sigma(W_r : r \in [0,t])$.

Let

$$\Omega^B = C([0,T],\mathbb{R}^d), \Omega^W = C([0,T],\mathbb{R}^m),$$

with Wiener measures $\mathbb{P}^B$, $\mathbb{P}^W$. Let

$$\Omega = \Omega^B \times \Omega^W, \mathbb{P} := \mathbb{P}^B \otimes \mathbb{P}^W.$$

Define $B(\omega^1,\omega^2) := \omega^1$, $W(\omega^1,\omega^2) = \omega^2$; so B is a d-dimensional Brownian motion, W is an independent m-dimensional Brownian motion.

In this setting Pardoux/Peng '92 introduced the notion of *BDSDEs*

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T H(X_r, Y_r) \circ dB_r - \int_t^T Z_r dW_r.$$

We note that it is $\mathcal{F}_t := \mathcal{F}^B_{t,T} \vee \mathcal{F}^W_{0,t}$ adapted, where $\mathcal{F}^B_{t,T} := \sigma(B_r : r \in [t,T]), \mathcal{F}^W_{0,t} := \sigma(W_r : r \in [0,t])$.

Let $p \in (2,3)$. Let $\mathbf{B}_t = \exp(B_t + A_t)$ be the Enhanced Brownian motion (over B), i.e.

$$\mathbf{B}_t = (B_t, \begin{pmatrix} \int_0^t B_r^1 \circ dB_r^1 & \cdots & \int_0^t B_r^1 \circ dB_r^d \\ \cdots & \cdots & \cdots \\ \int_0^t B_r^d \circ dB_r^1 & \cdots & \int_0^t B_r^d \circ dB_r^d \end{pmatrix})$$

Especially $\mathbf{B} \in C_0^{p-\mathrm{var}}([0,T], G^2(\mathbb{R}^d))$, $\mathbb{P}^1 - a.s.$.
For almost every $\omega^1 \in \Omega^B$ we can construct the solution to the BSDE with rough driver

$$Y_t^{rp} = \xi + \int_t^T f(r, Y_r^{rp}, Z_r^{rp}) dr + \int_t^T H(X_r, Y^{rp}) d\mathbf{B}_r(\omega^1) - \int_t^T Z^{rp} dW_r, \quad t \in [0,T].$$

Theorem

Let (Y, Z) be the solution to the BDSDE

$$Y_t = \xi + \int_t^T f(r, Y_r, Z_r)dr + \int_t^T H(X_r, Y_r) \circ dB_r - \int_t^T Z_r dW_r.$$

For $\omega^1 \in \Omega^B$ let (Y^{rp}, Z^{rp}) be the solution to the BSDE with rough driver

$$Y_t^{rp} = \xi + \int_t^T f(r, Y_r^{rp}, Z_r^{rp})dr + \int_t^T H(X_r, Y^{rp})d\mathbf{B}_r(\omega^1) - \int_t^T Z^{rp} dW_r, \quad t \in [0, T].$$

We have for $\mathbb{P}^B$ a.e. ω^1 that $\mathbb{P}^W$-a.s.

$$Y_t(\omega^1, \cdot) = Y_t^{rp}(\omega^1, \cdot), \quad t \leq T.$$

Future Work

- connection to BDSDEs driven by fractional Brownian motion (Jing/Leon '10)
- numerical scheme (problems: either exploding Lipschitz constant *or* driver not Lipschitz in y, not $\frac{1}{2}$-Hölder in t)

Thank you!

M. Caruana, P. K. Friz, and H. Oberhauser.
A (rough) pathwise approach to a class of non-linear stochastic partial differential equations.
Arxiv preprint arXiv:0902.3352, 2009.

P. K. Friz and H. Oberhauser.
Rough path stability of SPDEs arising in non-linear filtering.
Arxiv preprint arXiv:1005.1781, 2010.

M. Kobylanski.
BSDEs and PDEs with quadratic growth.
Ann. Probab., 28(2):558–602, 2000.

É. Pardoux and S. Peng.
Backward stochastic differential equations and quasilinear parabolic partial differential equations.
In Stochastic partial differential equations and their applications (Charlotte, NC, 1991). Springer, 1992.