

QUANTITATIVE RISK MANAGEMENT: CONCEPTS, TECHNIQUES AND TOOLS*

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OUTLINE OF THE TALK

- Introduction
- The Fundamental Theorems of Quantitative Risk Management
- PE's Desert-Island Copula
- Example 1: Credit Risk
- Example 2: A Copula Model for FX-data
- Some Further References
- Conclusion

INTRODUCTION

For copula applications to [financial risk management](#), it “all” started with the **RiskLab** report:

P. Embrechts, A.J. McNeil and **D. Straumann** (1997⁽¹⁾, 1999⁽²⁾, 2000⁽³⁾) Correlation and dependency in risk management: Properties and pitfalls.

and the **RiskMetrics** report:

D.X. Li (1998, 2000) On default correlation: A copula function approach. Working paper 99-07, RiskMetrics Group.

- (1) First version as RiskLab report
- (2) Abridged version published in RISK Magazine, May 1999, 69-71
- (3) Full length published in: Risk Management: Value at Risk and Beyond, ed. M.A.H. Dempster, Cambridge University Press, Cambridge (2000), 176-223

POTENTIAL COPULA APPLICATIONS

- Insurance:
 - Life (multi-life products)
 - Non-life (multi-line covers)
 - Integrated risk management (Solvency 2)
 - Dynamic financial analysis (ALM)
- Finance:
 - Stress testing (Credit)
 - Risk aggregation
 - Pricing/Hedging basket derivatives
 - Risk measure estimation under incomplete information
- Other fields:
 - Reliability, Survival analysis
 - Environmental science, Genetics
 - ...

FROM THE WORLD OF FINANCE

- “Extreme, synchronized rises and falls in financial markets occur infrequently but they do occur. The problem with the models is that they did not assign a high enough chance of occurrence to the scenario in which **many things go wrong at the same time** - the “perfect storm” scenario”

(Business Week, September 1998)

- Consulting for a large bank, topics to be discussed were:
 - general introduction to the topic of **EVT**
 - common pitfalls and its application to financial risk management
 - the application of EVT to the quantification of **operational risk**
 - general introduction to the topic of **copulae** and their possible use in **financial risk management**
 - **sources of information** to look at if we want to find out more

(London, March 2004)

THE REGULATORY ENVIRONMENT

- Basel Accord for Banking Supervision (1988)
 - Cooke ratio, “haircut” principle, too coarse
- Amendment to the Accord (1996)
 - VaR for Market Risk, Internal models, Derivatives, Netting
- **Basel II** (1998 – 2007)
 - Three Pillar approach
 - Increased granularity for Credit Risk
 - Operational Risk

THE REGULATORY ENVIRONMENT

- Solvency 1 (1997)
 - Solvency margin as % of premium (non-life), of technical provisions (life)
- **Solvency 2** (2000–2004)
 - Principle-based (not rule-based)
 - Mark-to-market (/model) for assets and liabilities (ALM)
 - Target capital versus solvency capital
 - Explicit modelling of dependencies and stress scenarios
- **Integrated Risk Management**

THE FUNDAMENTAL THEOREMS OF QUANTITATIVE RISK MANAGEMENT (QRM)

- **(FTQRM - 1)** For **elliptically distributed** risk vectors, classical Risk Management tools like VaR, Markowitz portfolio approach, ... work fine:

Recall:

- \mathbf{Y} in \mathbb{R}^d is **spherical** if $\mathbf{Y} \stackrel{d}{=} U\mathbf{Y}$ for all **orthogonal** matrices U
- $\mathbf{X} = A\mathbf{Y} + \mathbf{b}$, $A \in \mathbb{R}^{d \times d}$, $\mathbf{b} \in \mathbb{R}^d$ is called **elliptical**
- Let $\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$, $W \geq 0$, **independent** of \mathbf{Z} , then

$$\mathbf{X} = \boldsymbol{\mu} + W\mathbf{Z}$$

is **elliptical** (**multivariate Normal variance-mixtures**)

- If one takes

$W = \sqrt{\nu/V}$, $V \sim \chi_\nu^2$, then \mathbf{X} is **multivariate t_ν**

W normal inverse Gaussian, then \mathbf{X} is **generalized hyperbolic**

THE FUNDAMENTAL THEOREMS OF QUANTITATIVE RISK MANAGEMENT (QRM)

- **(FTQRM - 2)** Much more important!

For **non**-elliptically distributed risk vectors, classical RM tools break down:

- VaR is typically non-subadditive
- risk capital allocation is non-consistent
- portfolio optimization is risk-measure dependent
- correlation based methods are insufficient

- **A(n early) stylized fact:**

In practice, portfolio risk factors typically are **non-elliptical**

Questions: - are these deviations relevant, important

- what are tractable, non-elliptical models

- how to go from static (one-period) to dynamic (multi-period) RM

SOME COMMON RISK MANAGEMENT FALLACIES

- **Fallacy 1:** **marginal** distributions and their correlation matrix uniquely determine the **joint** distribution

True for elliptical families, **wrong** in general

- **Fallacy 2:** given two one-period risks X_1, X_2 , **VaR**($X_1 + X_2$) is **maximal** for the case where the correlation $\rho(X_1, X_2)$ is maximal

True for elliptical families, **wrong** in general (non-coherence of VaR)

- **Fallacy 3:** **small** correlation $\rho(X_1, X_2)$ implies that X_1 and X_2 are **close** to being independent

AND THEIR SOLUTION

- **Fallacy 1:** standard copula construction (Sklar)

- **Fallacy 2:** many related (copula-) publications

Reference: P. Embrechts and G. Puccetti (2004). Bounds on Value-At-Risk, preprint ETH Zürich, www.math.ethz.ch/~embrechts

- **Fallacy 3:** many copula related examples

An economically relevant example: Two country risks X_1, X_2

- $Z \sim N(0, 1)$ independent of scenario generator $U \sim \text{UNIF}(\{-1, +1\})$
- $X_1 = Z, X_2 = UZ \sim N(0, 1)$
- $\rho(X_1, X_2) = 0$
- X_1, X_2 are strongly dependent
- $X_1 + X_2 = Z(1 + U)$ is **not** normally distributed

FOR THE QUANTITATIVE RISK MANAGER

WHY ARE COPULAE USEFUL

- pedagogical: “Thinking **beyond** linear correlation”
- **stress testing** dependence: joint extremes, spillover, contagion, ...
- **worst case analysis** under incomplete information:

given: $X_i \sim F_i, i = 1, \dots, d$, marginal 1-period risks

$\Psi(\mathbf{X})$: a financial position

Δ : a 1-period risk or pricing measure

task: find $\min \Delta(\Psi(\mathbf{X}))$ and $\max \Delta(\Psi(\mathbf{X}))$ under the above constraints

- eventually: finding better fitting dynamic models

THE BASIC MESSAGE FOR (STATIC) COPULA APPLICATIONS TO QRM

$\mathbf{X} = (X_1, \dots, X_d)'$ one-period risks

$$F_{\mathbf{X}}(\mathbf{x}) = P(X_1 \leq x_1, \dots, X_d \leq x_d)$$

$$F_i(x_i) = P(X_i \leq x_i), \quad i = 1, 2, \dots, d$$

$$F_{\mathbf{X}} \iff (F_1, \dots, F_d; C)$$

with copula C (via Sklar's Theorem)

“ \Rightarrow ” : finding useful copula models

“ \Leftarrow ” : stress testing

PE's DESERT-ISLAND COPULA

CLAIM: For applications in QRM, the most useful copula is the *t*-copula, $C_{\nu, \Sigma}^t$ in dimension $d \geq 2$ (McNeil, 1997)

Its derived copulae include:

- skewed or non-exchangeable *t*-copula
- grouped *t*-copula
- conditional excess or *t*-tail-copula

Reference: S. Demarta and A.J. McNeil (2004). The *t* copula and related copulas, preprint ETH Zürich, www.math.ethz.ch/~mcneil

THE MULTIVARIATE SKEWED t DISTRIBUTION

The random vector \mathbf{X} is said to have a multivariate skewed t distribution if

$$\mathbf{X} \stackrel{d}{=} \boldsymbol{\mu} + W\boldsymbol{\gamma} + \sqrt{W}\mathbf{Z}$$

where $\boldsymbol{\mu}, \boldsymbol{\gamma} \in \mathbb{R}^d$

$$\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$$

W has an inverse gamma distribution depending on ν

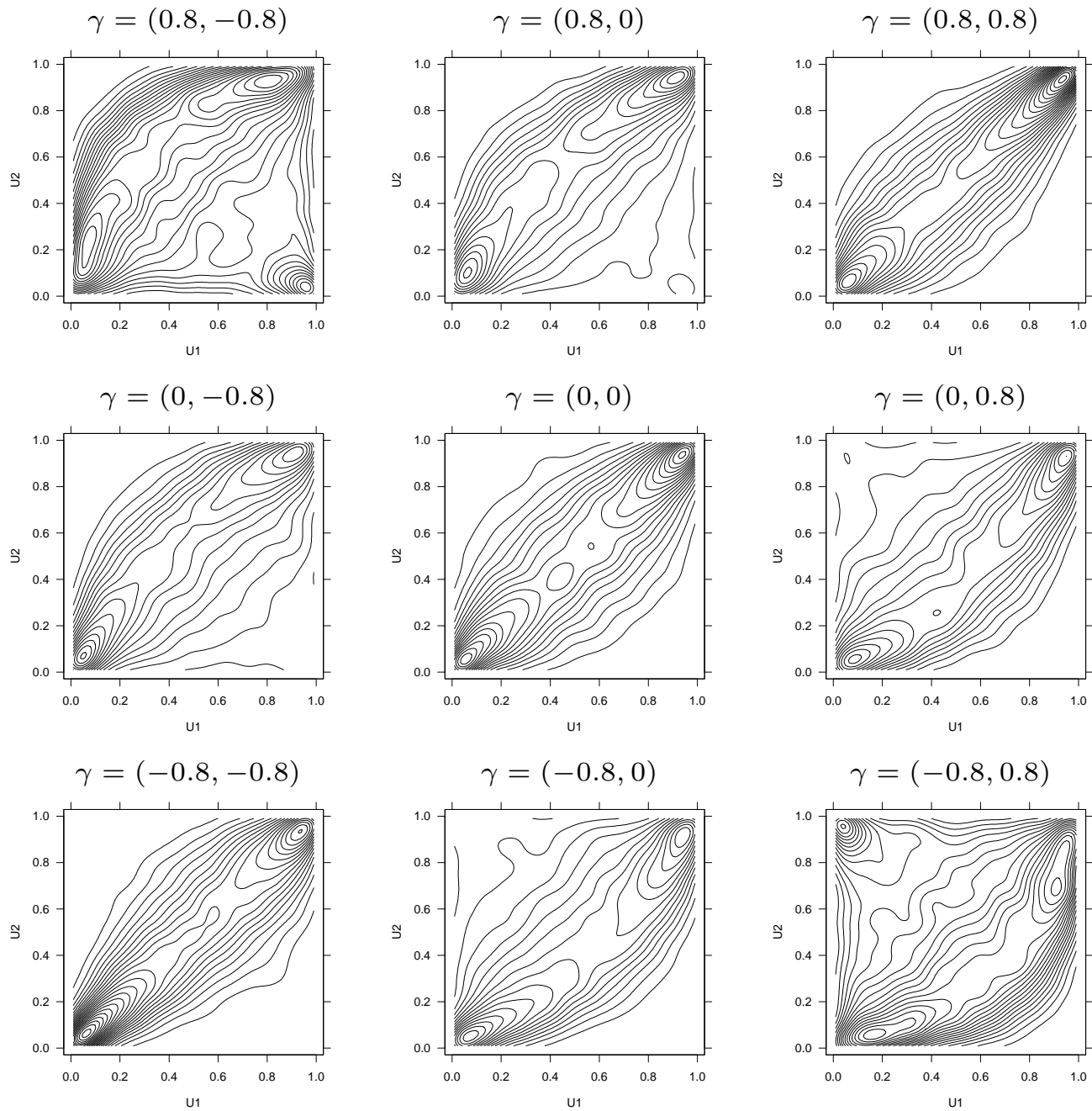
W and \mathbf{Z} are independent

Density:

$$f(\mathbf{x}) = c \frac{K_{\frac{\nu+d}{2}}\left(\sqrt{(\nu+Q(\mathbf{x}))\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma}}\right) \exp\left((\mathbf{x}-\boldsymbol{\mu})'\Sigma^{-1}\boldsymbol{\gamma}\right)}{\left(\sqrt{(\nu+Q(\mathbf{x}))\boldsymbol{\gamma}'\Sigma^{-1}\boldsymbol{\gamma}}\right)^{-\frac{\nu+d}{2}} \left(1+\frac{Q(\mathbf{x})}{\nu}\right)^{\frac{\nu+d}{2}}}$$

where $Q(\mathbf{x}) = (\mathbf{x} - \boldsymbol{\mu})'\Sigma^{-1}(\mathbf{x} - \boldsymbol{\mu})$, $c = \frac{2^{1-(\nu+d)/2}}{\Gamma(\frac{\nu}{2})(\pi\nu)^{d/2}|\Sigma|^{1/2}}$

and K_λ denotes a modified Bessel function of the third kind



THE GROUPED t-COPULA

The grouped t-copula is closely related to a t-copula where different subvectors of the vector \mathbf{X} have different levels of tail dependence

If

- $\mathbf{Z} \sim N_d(\mathbf{0}, \Sigma)$
- G_ν denotes the df of a univariate inverse gamma, $\text{Ig}(\frac{\nu}{2}, \frac{\nu}{2})$ distribution
- $U \sim \text{UNIF}(0, 1)$ is a uniform variate independent of \mathbf{Z}
- \mathcal{P} is a partition of $\{1, \dots, d\}$ into m sets of sizes $\{s_k : k = 1, \dots, m\}$
- ν_k is the degrees of freedom parameter associated with set of size s_k
- $W_k = G_{\nu_k}^{-1}(U)$

then

$$\mathbf{X} = \left(\sqrt{W_1} Z_1, \dots, \sqrt{W_1} Z_{s_1}, \dots, \dots, \sqrt{W_m} Z_{d-s_m+1}, \dots, \sqrt{W_m} Z_d \right)'$$

has a grouped t copula (similarly, grouped elliptical copula)

THE TAIL LIMIT COPULA

Lower Tail Limit Copula Convergence Theorem: Let C be an exchangeable copula such that $C(v, v) > 0$ for all $v > 0$. Assume that there is a strictly increasing continuous function $K : [0, \infty) \rightarrow [0, \infty)$ such that

$$\lim_{v \rightarrow 0} \frac{C(vx, v)}{C(v, v)} = K(x), \quad x \in [0, \infty).$$

Then there is $\eta > 0$ such that $K(x) = x^\eta K(1/x)$ for all $(0, \infty)$. Moreover, for all $(u_1, u_2) \in (0, 1]^2$

$$C_0^{lo}(u_1, u_2) = G(K^{-1}(u_1), K^{-1}(u_2)),$$

where $G(x_1, x_2) := x_2^\eta K(x_1/x_2)$ for $(x_1, x_2) \in (0, 1]^2$, $G := 0$ on $[0, 1]^2 \setminus (0, 1]^2$ and C_0^{lo} denotes the lower tail limit copula

Observation: The function $K(x)$ fully determines the tail limit copula

THE t LOWER TAIL LIMIT COPULA

For the bivariate t -copula $C_{\nu, \rho}^t$ with tail dependence coefficient λ we have for the t -LTL copula that

$$K(x) = \frac{xt_{\nu+1} \left(\frac{-(x^{1/\nu} - \rho)}{\sqrt{1-\rho^2}} \sqrt{\nu+1} \right) + t_{\nu+1} \left(\frac{-(x^{-1/\nu} - \rho)}{\sqrt{1-\rho^2}} \sqrt{\nu+1} \right)}{\lambda}$$

with $x \in [0, 1]$, whereas for the Clayton-LTL copula, with parameter θ ,

$$K(x) = ((x^{-\theta} + 1)/2)^{-1/\theta}$$

Important for practice: “for any pair of parameter values ν and ρ the K -function of the t -LTL copula may be very closely approximated by the K -function of the Clayton copula for some value of θ ” (S. Demarta and A.J. McNeil (2004))

EXAMPLE 1: the Merton model for corporate default (firm value model, latent variable model)

- portfolio $\{(X_i, k_i) : i = 1, \dots, d\}$ firms, obligors
- obligor i defaults by end of year if $X_i \leq k_i$
(firm value is less than value of debt, properly defined)
- modelling joint default: $P(X_1 \leq k_1, \dots, X_d \leq k_d)$
 - classical Merton model: $\mathbf{X} \sim N_d(\boldsymbol{\mu}, \Sigma)$
 - KMV: calibrate k_i via “distance to default” data
 - CreditMetrics: calibrate k_i using average default probabilities for different rating classes
 - Li model: X_i 's as survival times are assumed exponential and use Gaussian copula
- hence standard industry models use Gaussian copula
- improvement using t-copula

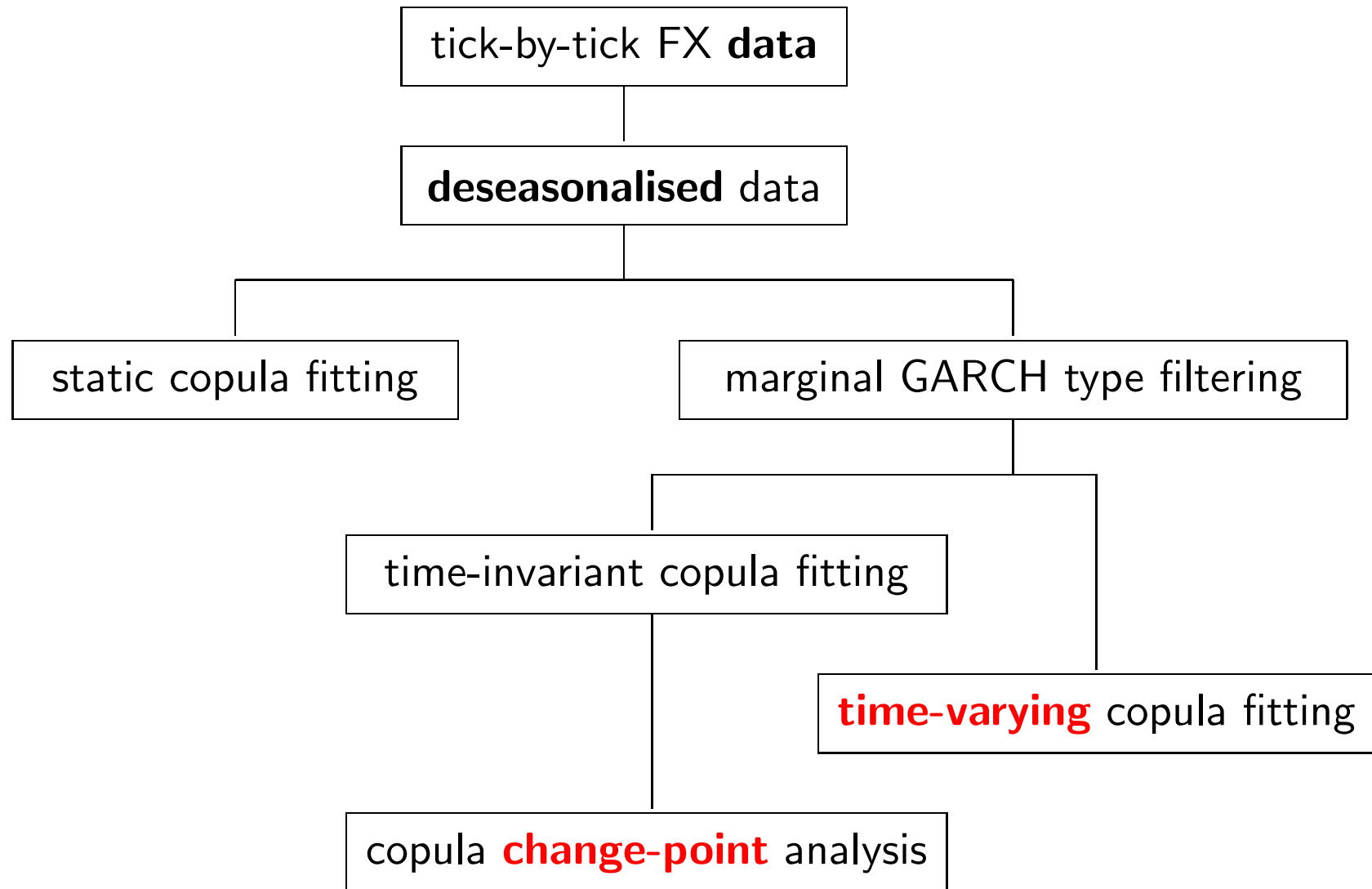
- standardised equicorrelation ($\rho_i = \rho = 0.038$) matrix Σ calibrated so that for $i = 1, \dots, d$, $P(X_i \leq k_i) = 0.005$ (medium credit quality in KMV/CreditMetrics)
- set $\nu = 10$ in t-model and perform 100 000 simulations on $d = 10\,000$ companies to find the loss distribution
- use VaR concept to compare risks

Results:

	min	25%	med	mean	75%	90%	95%	max
Gaussian	1	28	43	49.8	64	90	109	131
t	0	1	9	49.9	42	132	235	3 238

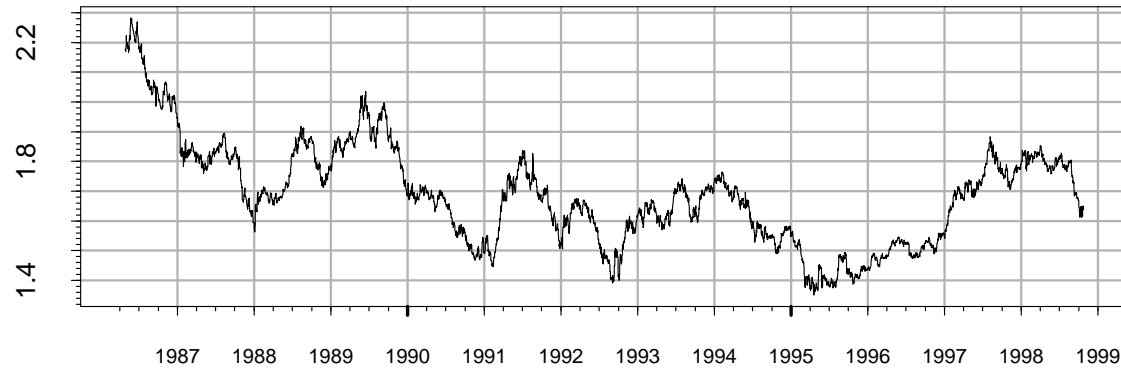
- more realistic t-model: **block-t-copula** (Lindskog, McNeil)
- has been used for banking and (re)insurance portfolios

Example 2: High-Frequency FX data



FX DATA SERIES

USD/DEM



USD/JPY



Olsen data set: Bivariate 5 minutes **logarithmic middle prices**

$$\bar{p}_t = \frac{1}{2} (\log p_{t,bid} + \log p_{t,ask})$$

ASYMPTOTIC CLUSTERING OF BIVARIATE EXCESSES

- **Extreme tail** dependence copula relative to a threshold t :

$$C_t(u, v) = P(U \leq F_t^{-1}(u), V \leq F_t^{-1}(v) | U \leq t, V \leq t)$$

with conditional distribution function

$$F_t(u) := P(U \leq u | U \leq t, V \leq t), \quad 0 \leq u \leq 1$$

- **Archimedean** copulae: there exists a continuous, strictly decreasing function $\psi : [0, 1] \mapsto [0, \infty]$ with $\psi(1) = 0$, such that

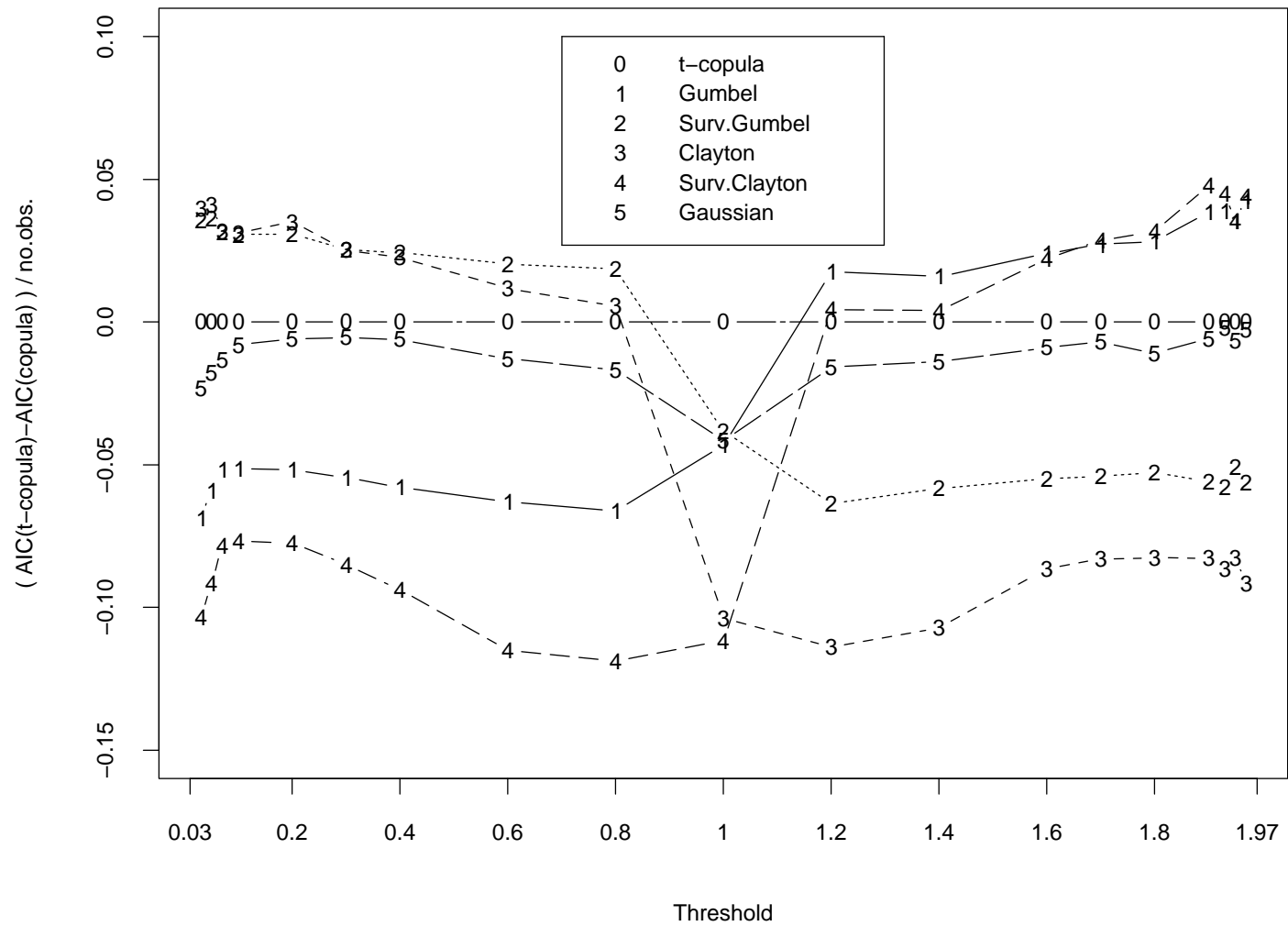
$$C(u, v) = \psi^{[-1]}(\psi(u) + \psi(v))$$

- For “sufficiently regular” Archimedean copulae (Juri and Wüthrich (2002)):

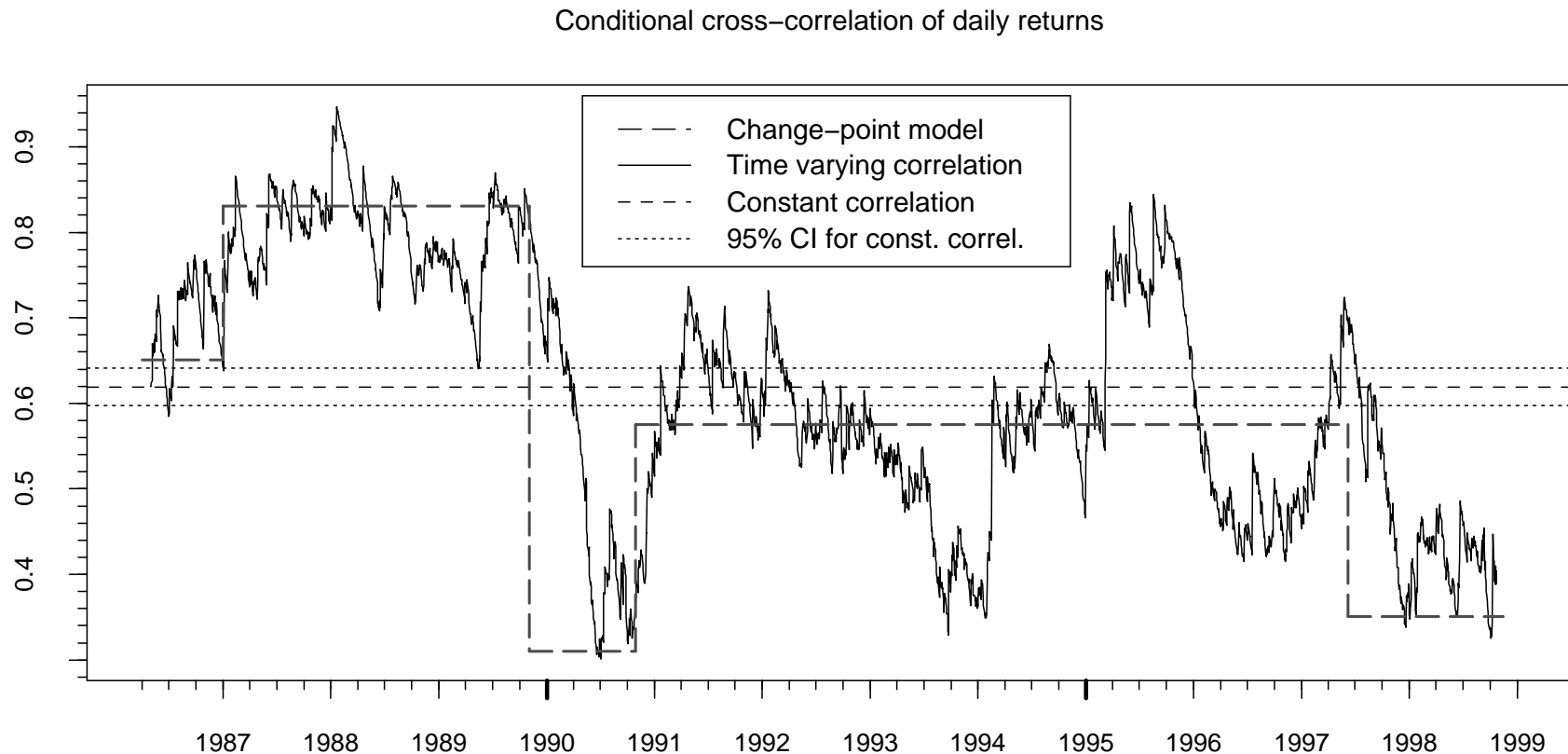
$$\lim_{t \rightarrow 0^+} C_t(u, v) = C_{\alpha}^{\text{Clayton}}(u, v)$$

Juri, A. and M. Wüthrich (2002). Copula convergence theorems for tail events. Insurance: Math. & Econom., 30: 405–420

ASYMPTOTIC CLUSTERING OF BIVARIATE EXCESSES



Estimated t-copula conditional correlation of daily returns on the FX USD/DEM and USD/JPY spot rates



A. Dias and P. Embrechts (2004). Dynamic copula models and change-point analysis for multivariate high-frequency data in finance, preprint ETH Zürich

HINTS* FOR FURTHER READING

(* There exists an already long and fast growing literature)

Books combining copula modelling with applications to finance:

- Bluhm, C., Overbeck, L. and Wagner, C. *An Introduction to Credit Risk Modeling*. Chapman & Hall/CRC, New York, 2002
- Cherubini, U., E. Luciano and W. Vecchiato (2004). *Copula Methods in Finance*, Wiley, To appear
- McNeil, A.J., R. Frey and P. Embrechts (2004). *Quantitative Risk Management: Concepts, Techniques and Tools*. Book manuscript, To appear
- Schönbucher, P.J. *Credit Derivatives Pricing Models*, Wiley Finance, 2003

(and there are more)

HINTS FOR FURTHER READING

Some papers for further reading:

- Cherubini, U. and E. Luciano, Bivariate option pricing with copulas, *Applied Mathematical Finance* **9**, 69–85 (2002)
- Dias, A. and P. Embrechts (2003). Dynamic copula models for multivariate high-frequency data in finance. Preprint, ETH Zürich
- Fortin, I. and C. Kuzmics (2002). Tail-dependence in stock-return pairs: Towards testing ellipticity. Working paper, IAS Vienna
- Patton, A.J. (2002). Modelling time-varying exchange rate dependence using the conditional copula. Working paper, UCSD
- Rosenberg, J., Nonparametric pricing of multivariate contingent claims, NYU, Stern School of Business (2001), working paper
- van den Goorbergh, C. Genest and B.J.M. Werker, Multivariate option pricing using dynamic copula models, Tilburg University (2003), discussion paper No. 2003-122

(and there are many, many more)

CONCLUSION

- Copulae are here to stay as a risk management tool
- Dynamic models
- Calibration / fitting
- High dimensions ($d \geq 100$, say)
- Most likely application: credit risk
- Limit theorems
- Link to multivariate extreme value theory