

# Quantitative Models for Operational Risk: Extremes, Dependence and Aggregation

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# Introduction

## Context:

- Basel II (Banking) and Solvency 2 (Insurance)
- AMA approach to Operational Risk

## Our contribution:

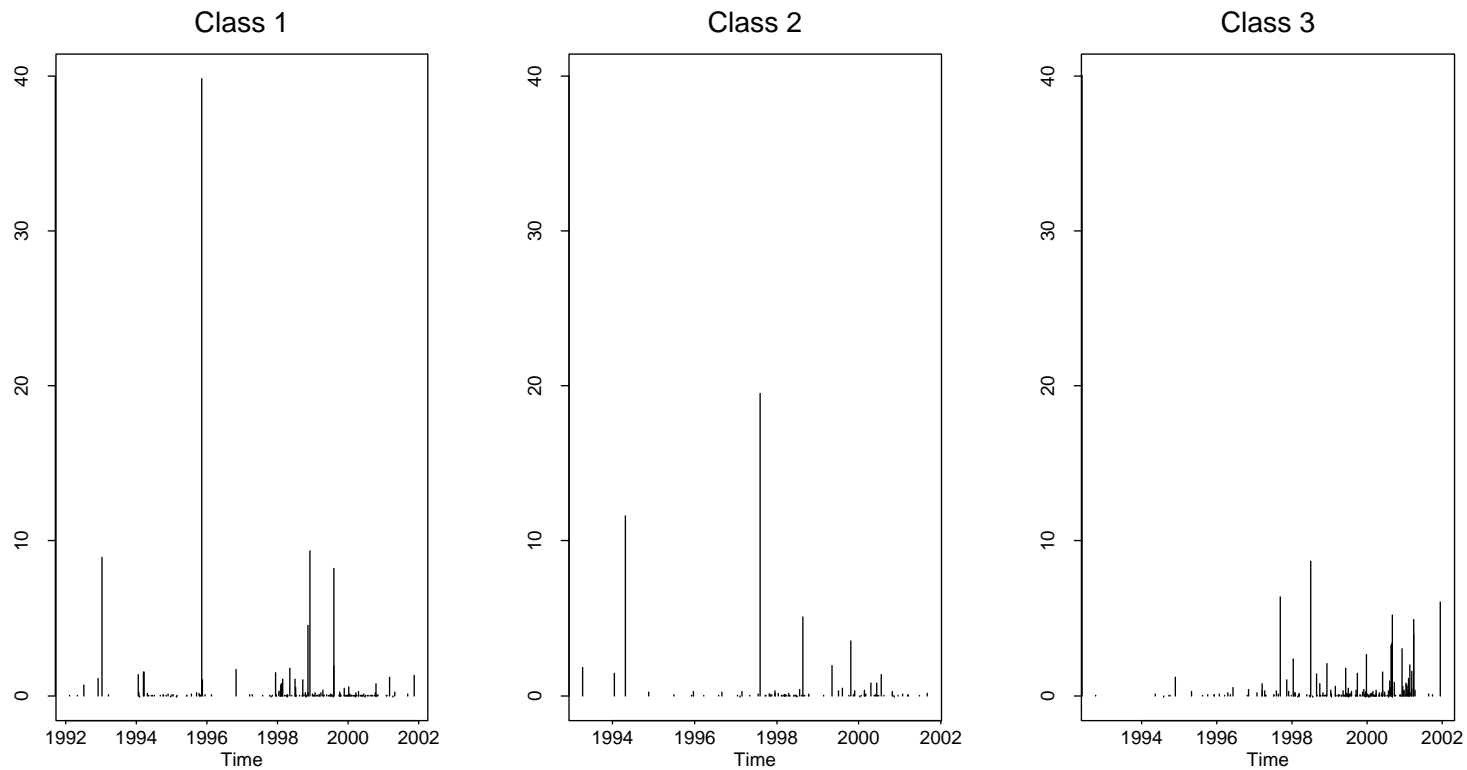
- Based on stylized facts of OpRisk data
- New stochastic methodology
- **No final** AMA solution, **but** contribution **towards** such a solution

# Outline of the talk

- Stylized facts of OpRisk data
- Methodological issues
- Non-stationary EVT models
- Dependence and Point Processes
- Aggregation
- Conclusion

# I. Stylized facts on OpRisk data

Some operational loss data:



# Stylized facts

- Different, non-homogeneous classes
- Extremes matter
- Non-repetitive/large claims versus repetitive/small claims
- Non-stationarity
- Interdependence
- Restricted data warehouses
- Thresholding

QIS-3: above facts are corroborated (Moscadelli)

# Regulatory issues

- An **appropriate (?)** risk measure:
  - Value-at-Risk
  - 1 year
  - $\alpha \geq 0.99$  (even 0.9997)
- Appropriate (?)
  - (non-) coherence ?
  - **beyond VaR**: expected shortfall ?
  - (very) **extreme quantiles**
  - yearly data: hardly any (**scaling ?**)

## II. Methodological issues

- EVT matters:
  - very high quantiles
  - non-stationarity
- Dependence matters:
  - “correlating” loss processes
  - copulas as a method for going beyond correlation
  - how to quantify dependence ?
- Aggregation matters:
  - global 1-year OpRisk measure
  - restricted/partial information
  - aggregate risk measures across risk classes

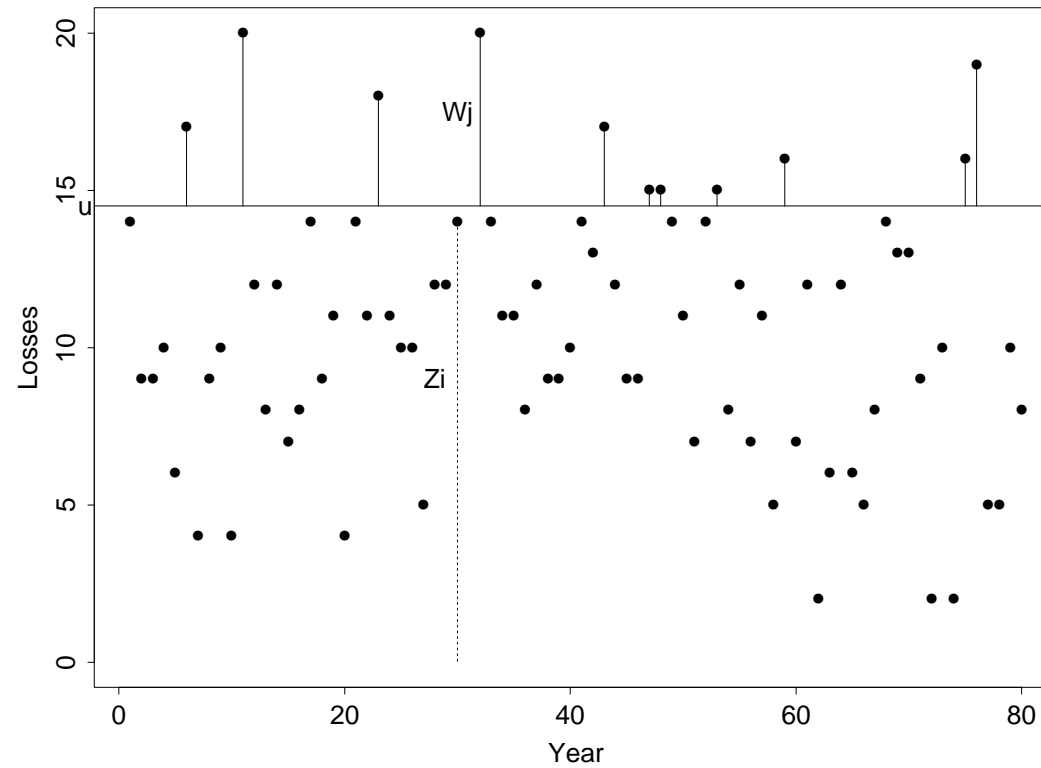
# III. Non-stationary EVT models

## Basic EVT methodology

- Ground-up losses are denoted by  $Z_1, Z_2, \dots, Z_q$
- $u$  is a typically high threshold
- $W_1, \dots, W_n$  are the excess losses from  $Z_1, \dots, Z_q$  above  $u$ , i.e.

$$W_j = Z_i - u \quad \text{for some } j = 1, \dots, n \text{ and } i = 1, \dots, q, \text{ where } Z_i > u$$

# The Point Process of Exceedances within the Peaks over Threshold (POT) Method



# The Peaks Over Threshold Method

For iid losses, and for  $u$  high enough:

- Conditional losses  $W_1, \dots, W_n$  follow a Generalized Pareto Distribution (GPD)

$$G_{\kappa, \sigma}(w) = \begin{cases} 1 - (1 + \kappa w / \sigma)_+^{-1/\kappa}, & \kappa \neq 0 \\ 1 - \exp(-w / \sigma), & \kappa = 0 \end{cases}$$

- The exceedance points of  $Z_1, \dots, Z_q$  of the threshold  $u$  follow (approximately) a homogeneous Poisson process with intensity  $\lambda > 0$
- The conditional losses  $W_1, \dots, W_n$  and the Poisson exceedance process are (approximately) independent, and
- An approximate log-likelihood function  $l(\lambda, \sigma, \kappa)$  can be derived

# An advanced POT model: non-stationarity

In general:

$$(\lambda, \kappa, \sigma) \rightsquigarrow (\lambda(t, \tau), \kappa(t, \tau), \sigma(t, \tau))$$

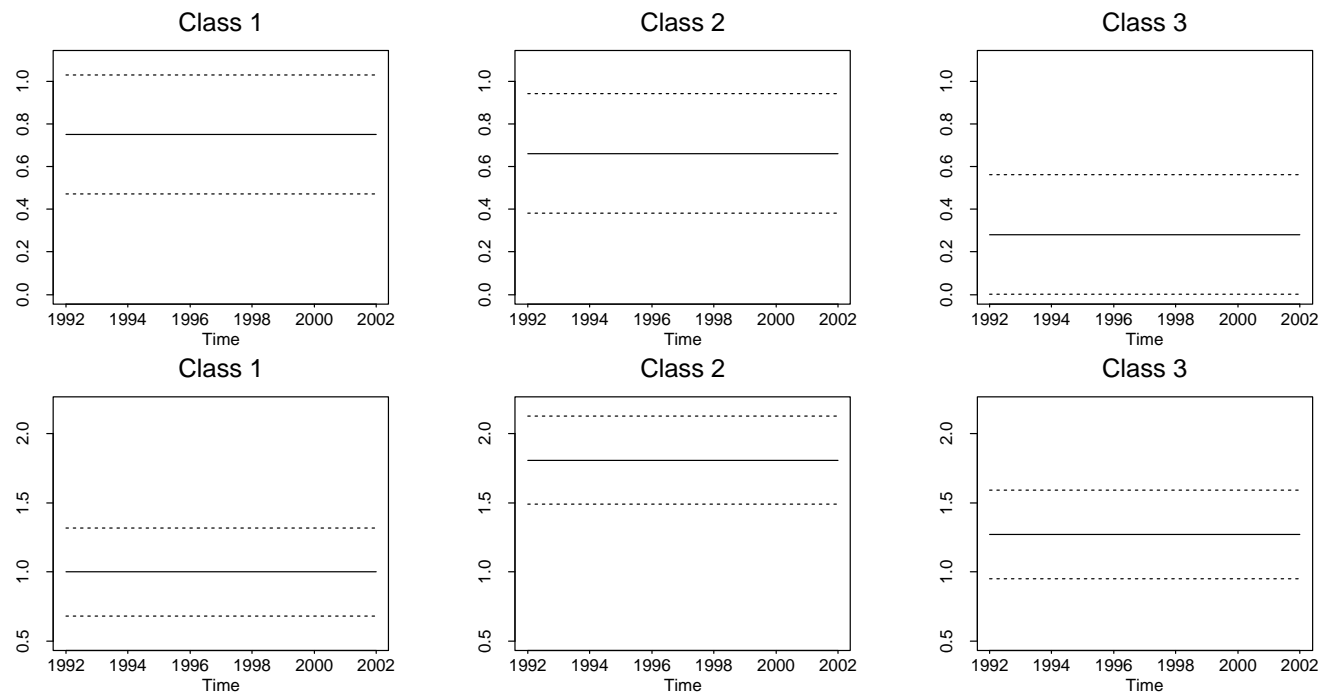
For **pooled** OpRisk:

$$\log \lambda(t, \tau) = \gamma_{\tau} I_{\tau} + \beta I_{(t > t_c)} + g(t)$$

- where
- $I_{\tau}$  class indicator
  - $I_{(t > t_c)}$  change point indicator
  - $g(t)$  general smooth function

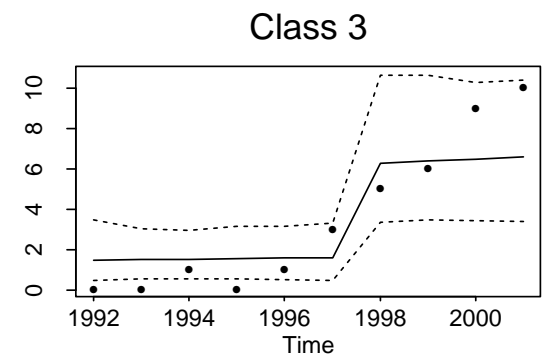
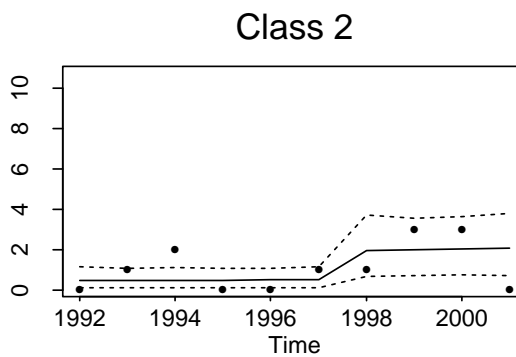
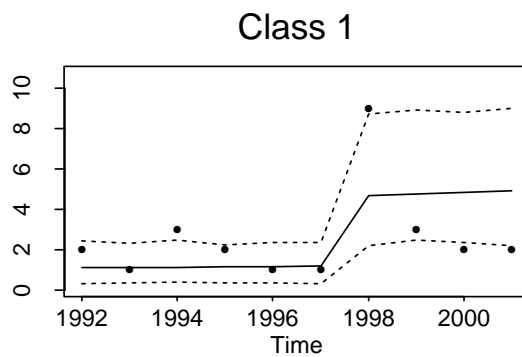
# The advanced POT model: results of fit to pooled OpRisk data

- $\hat{\kappa} = \hat{\kappa}(\tau)$  (upper panels)
- $\hat{\sigma} = \hat{\sigma}(\tau)$  (lower panels)

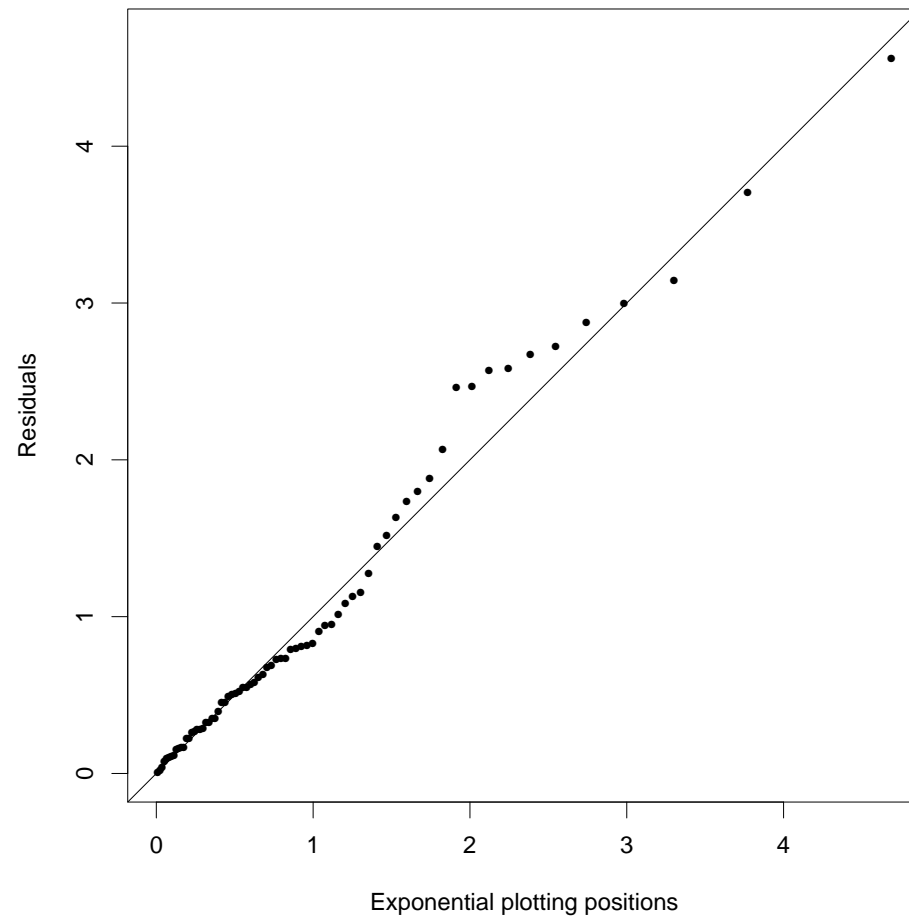


# The advanced POT model: intensity function

$$\log \hat{\lambda}(t, \tau) = \hat{\gamma}_\tau I_\tau + \hat{\beta} I_{(t > t_c)} + \hat{g}(t)$$



# The advanced POT model: Goodness-of-fit



# The advanced POT model: (dynamic) risk measures

Some results for  $\widehat{dVaR}_\alpha^\tau(2002)$  and  $\widehat{dES}_\alpha^\tau(2002)$

	$\widehat{dVaR}_{0.99}^\tau(2002)$	$\widehat{dES}_{0.99}^\tau(2002)$
$\tau = 1$	138.1	556.3
$\tau = 2$	90.6	271.6
$\tau = 3$	23.9	34.9

# IV. Dependence and Point Processes

Stylized situation:

$$\left. \begin{array}{l} \text{Loss class 1 : } L_{1,T} = \sum_{i=1}^{N_1(T)} X_{i,1} \\ \text{Loss class 2 : } L_{2,T} = \sum_{i=1}^{N_2(T)} X_{i,2} \\ \vdots \\ \text{Loss class } d : L_{d,T} = \sum_{i=1}^{N_d(T)} X_{i,d} \end{array} \right\} \begin{array}{l} \bullet \text{ interdependence} \\ \bullet \text{ aggregation} \end{array}$$

# Interdependence: available tools

- Correlation (linear, rank):
  - one-number summary:  $\rho, \tau, \rho_S \dots$
- Copula:  $L_{i,T} \sim F_i$ 
  - $P(L_{1,T} \leq l_1, \dots, L_{d,T} \leq l_d) = \mathcal{C}(F_1(l_1), \dots, F_d(l_d))$
  - determine the joint distribution function
- Joint **dynamic models** for the compound processes:
  - $\{(L_{1,t}, \dots, L_{d,t}) : t \geq 0\}$
  - marked **point processes**

# Interdependence in the joint dynamic models

Consider  $d = 2$ :  $L_{k,T} = \sum_{i=1}^{N_k(T)} X_{i,k}$ ,  $k = 1, 2$

- and

- (1) make  $(X_{i,1})$  and  $(X_{i,2})$  dependent,  $i \geq 1$

- (2) make  $\{N_1(t) : t \leq T\}$  and  $\{N_2(t) : t \leq T\}$  dependent

- (3) combination of both

- via

- (1) standard copula techniques

- (2) for **t fixed**, copulas of discrete distributions

- **however**: we want to couple the **dynamic** processes

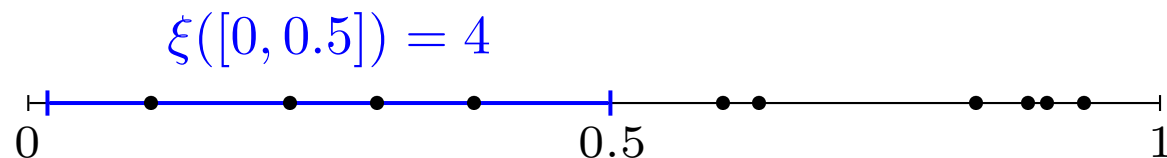
# An introduction of dynamic dependence models for point processes

## 1. Dependent loss frequencies (time dependence)

$$\xi = \sum_{i=1}^N I_{\mathbf{T}_i}, \quad \mathbf{T}_i = (T_{i,1}, \dots, T_{i,d}) \in \mathbb{R}_+^d$$

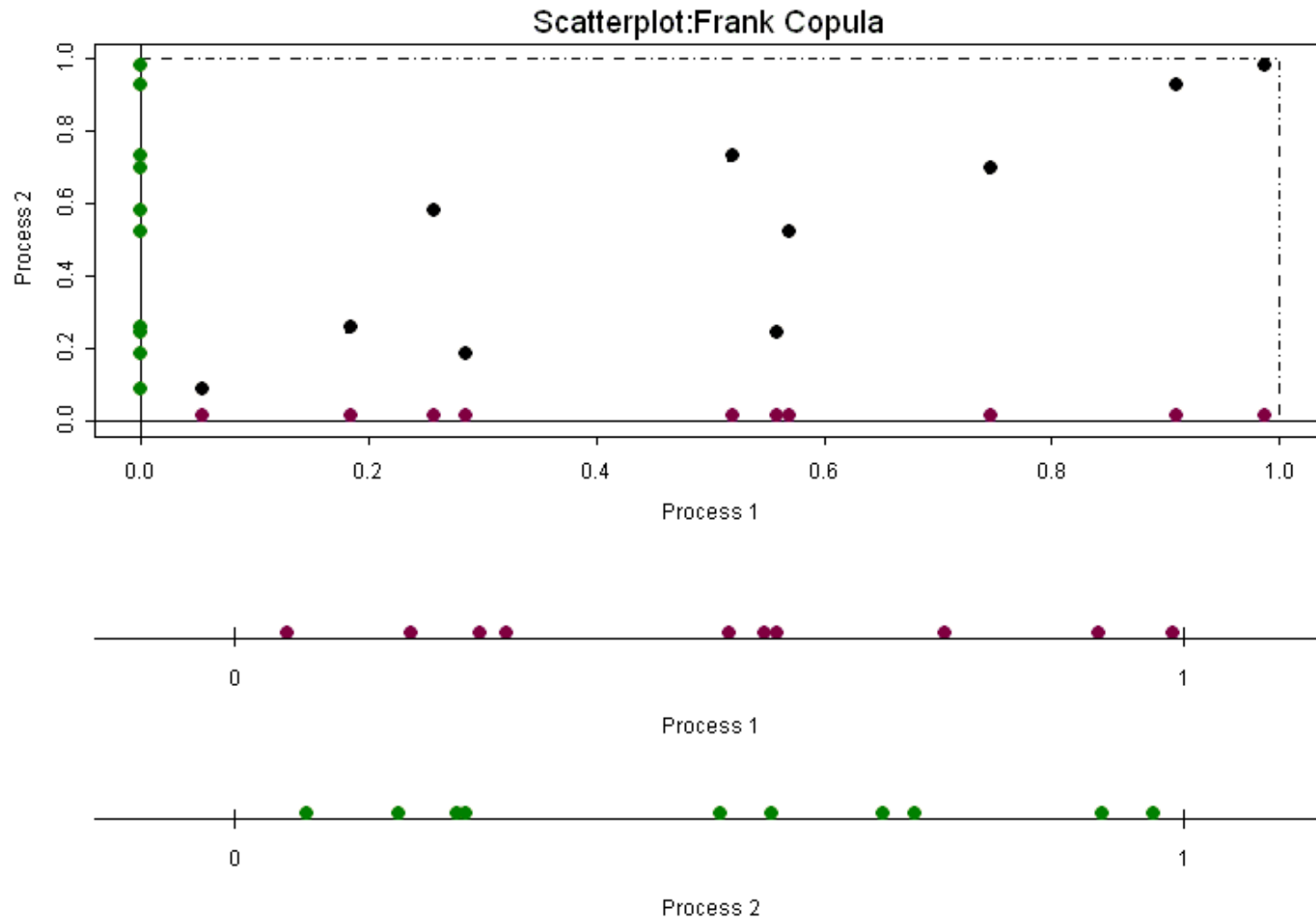
- $N$  gives the total (random) number of occurrences
- The  $\mathbf{T}_i$ 's are the exact locations
- In the **Poisson case**,  $N$  is Poisson,  $N$  and  $\{\mathbf{T}_i\}_i$  are independent and the  $\mathbf{T}_i$ 's are iid

An illustration for  $d = 1$ :



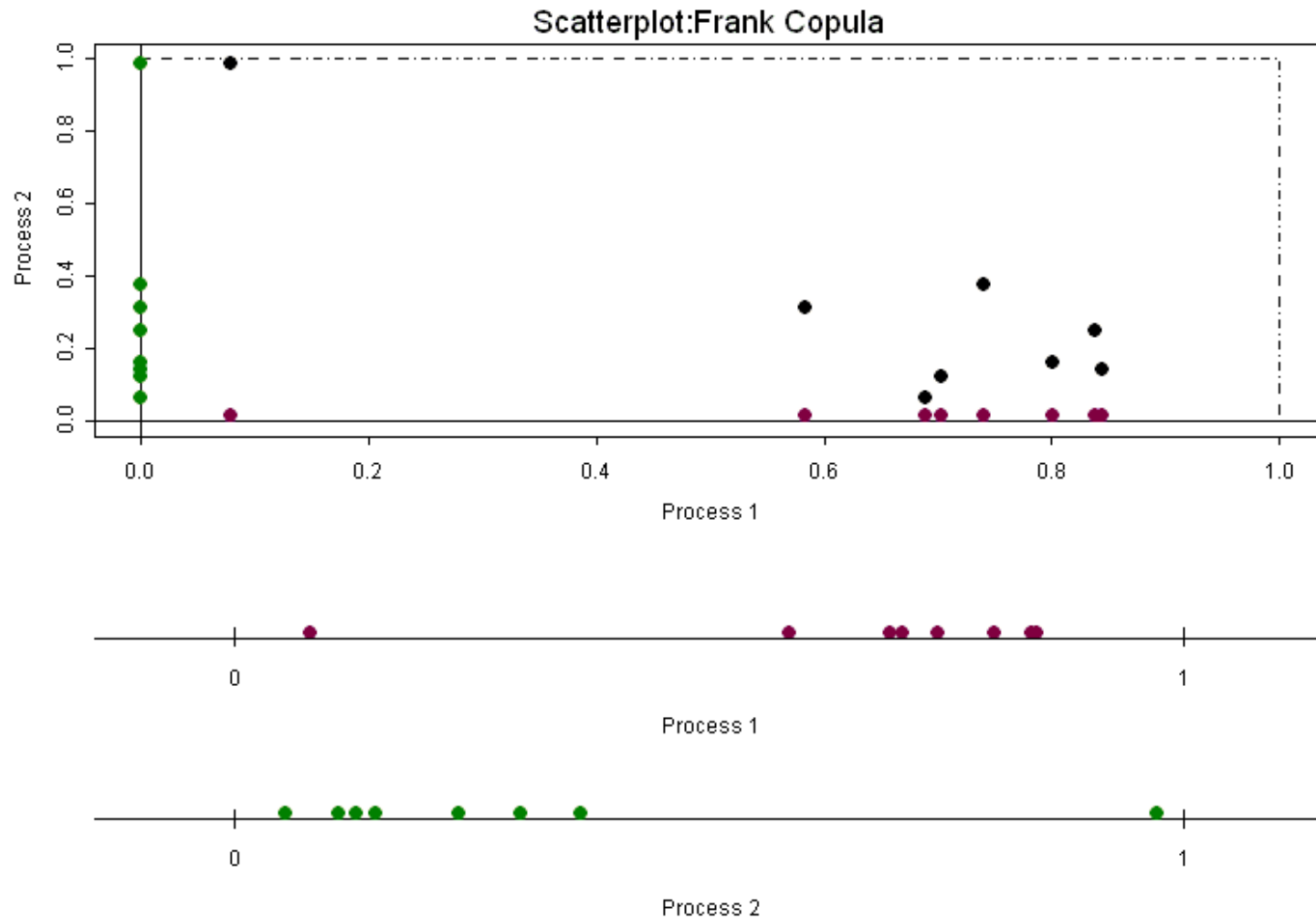
# Moving to $d \geq 2$ : Method I – Projections

**Example 1:** positive dependence (Frank copula with parameter  $\theta = 20$ )



# Moving to $d \geq 2$ : Method I – Projections

**Example 2:** **negative** dependence (Frank copula with parameter  $\theta = -20$ )

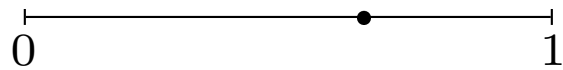


# Method I: Discussion

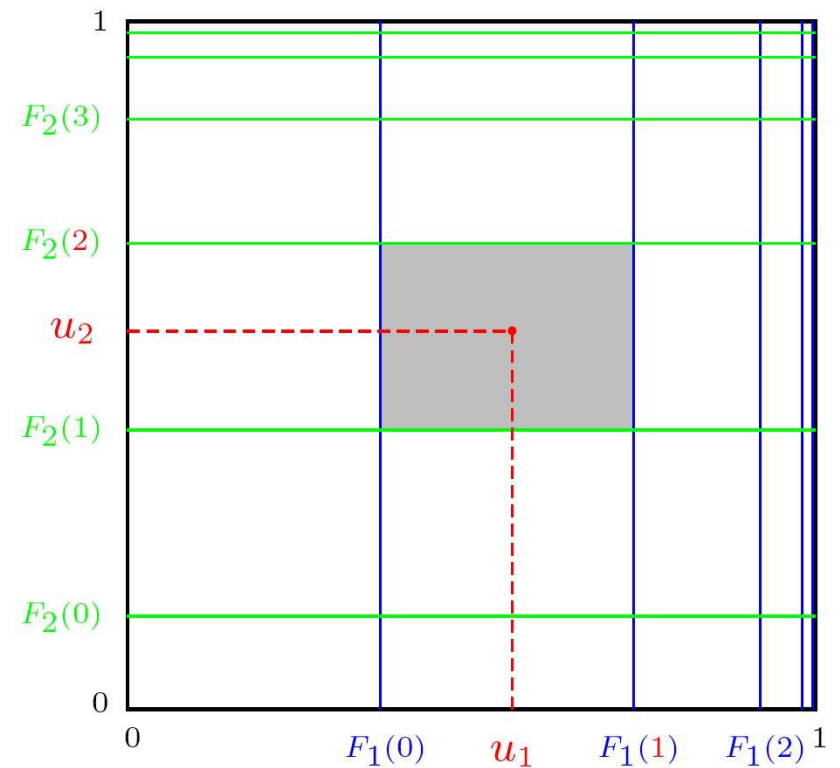
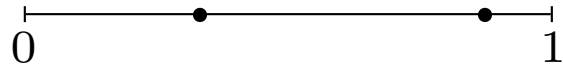
- Allows for construction of “dependence scenarios” like “dependence engineering”
- Construction holds for arbitrary  $d$
- Does **not** allow for different frequencies across  $d$  classes
- Common shock scenarios
- Resulting processes are **always positively dependent**, the same holds for aggregated loss processes

# Moving to different frequencies: Method II

- $N \rightsquigarrow (N_1, \dots, N_d)$
- $N_1, \dots, N_d$  dependent (copulas)

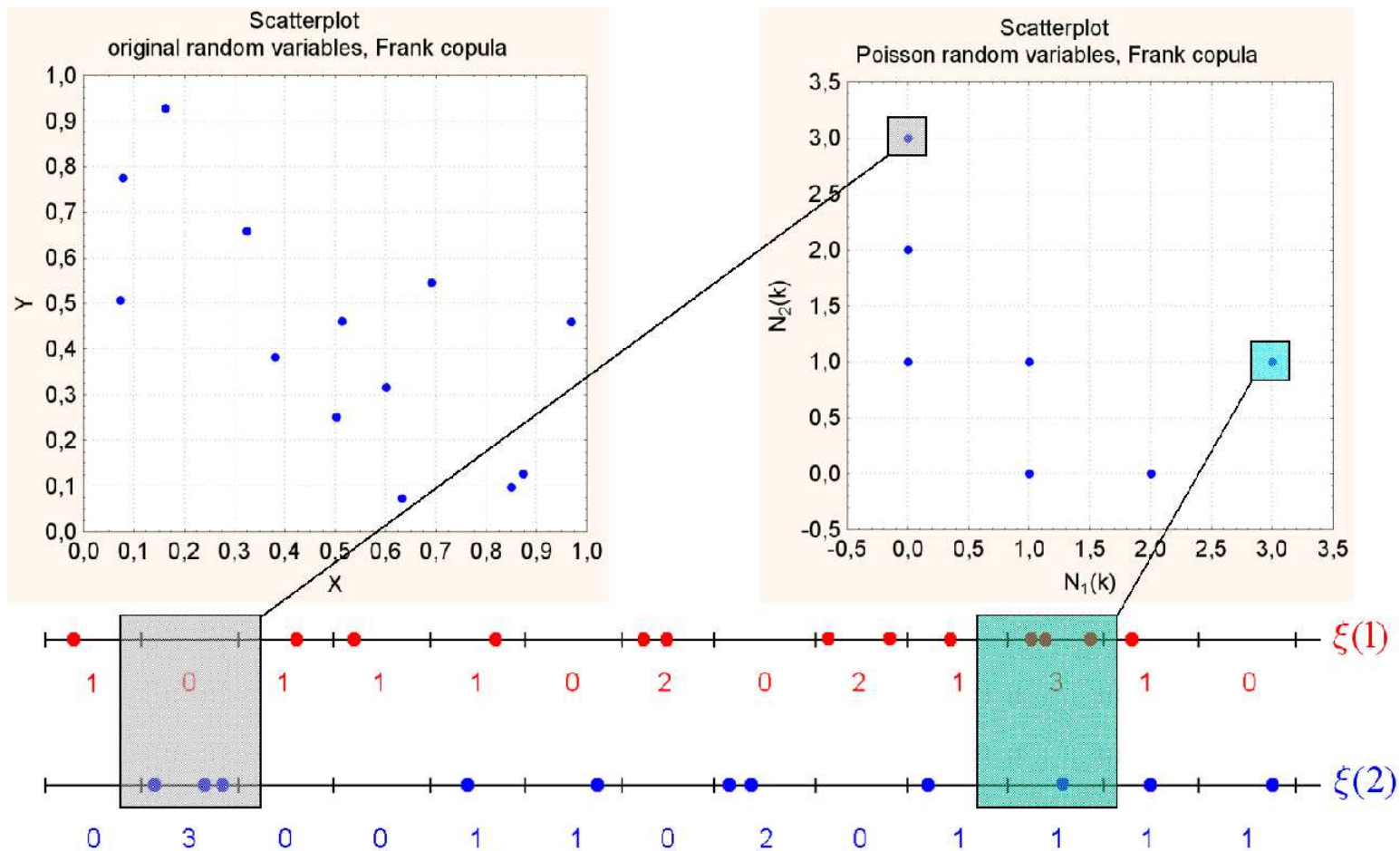


where?



# Moving to different frequencies: Method II cont'd

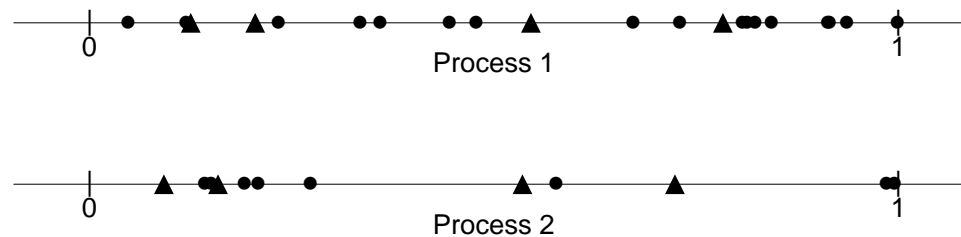
- Be careful:**
- dependence between number of events (above construction)
  - locations
    - independence: standard
    - dependence: possible but **tricky**



## Method II: Discussion

As in previous discussion, however

- Method II **does** allow for different frequencies across  $d$  classes
- Resulting processes can be **negatively** dependent, even if the locations are independent
- Method I and Method II can be **combined** (**superposition of processes**)





# Dependent loss processes

- Method I  $\Rightarrow \rho(L_{1,T}, L_{2,T}) = \frac{E(X_{1,1}X_{1,2})}{\sqrt{E(X_{1,1})^2 E(X_{1,2})^2}}$
- Method II  $\Rightarrow \rho(L_{1,T}, L_{2,T}) = \rho(N_1(T), N_2(T)) \frac{E(X_{1,1}X_{1,2})}{\sqrt{E(X_{1,1})^2 E(X_{1,2})^2}}$

Frachot, A., Roncalli, T. and Salomon, E. (2004) *The correlation problem in operational risk*. Crédit Lyonnais, Working paper.

- Combination of Method I and Method II  $\Rightarrow$  common shock model as a special case

Powojowski, M., Reynolds, D. and Tuenter, H. (2002) Dependent events and operational risk. *Algo research quarterly*, **5**(2), 68-73

## V. Aggregation

- In practice:  $d \in \{7, 8, 56\}$
- Loss rvs:  $L_{1,T}, \dots, L_{d,T}$  **dependent**
- With given risk measures:  $\text{VaR}_{1,\alpha}^T, \dots, \text{VaR}_{d,\alpha}^T$
- Issue:

$$\text{VaR}_{\alpha}^T \left( \sum_{k=1}^d L_{k,T} \right)$$

Under **some** (or **no**) **idea of** the **interdependence**

# Question 1

$$\text{VaR}_\alpha^T \left( \sum_{k=1}^d L_{k,T} \right) \leq \sum_{k=1}^d \text{VaR}_{k,\alpha}^T \quad ?$$

No in general:

- highly skewed loss dfs
- (very) heavy-tailed loss dfs
- special dependence structures

This is an issue in OpRisk!

## Question 2

Find optimal bounds for

$$\text{VaR}_{l,\alpha}^T \leq \text{VaR}_{\alpha}^T \left( \sum_{k=1}^d L_{k,T} \right) \leq \text{VaR}_{u,\alpha}^T$$

given marginal VaR's and dependence information

**Solution:**

- Fréchet Problem
- Mass Transportation Problem

## Example 1: Comonotonic risks

If  $L_{1,T}, \dots, L_{d,T}$  are **comonotonic**, i.e. there exists a rv  $Z$  and increasing functions  $f_{1,T}, \dots, f_{d,T}$ , so that

$$L_{i,T} = f_{i,T}(Z) \quad i = 1, \dots, d,$$

then the VaR is **additive**, i.e.

$$\text{VaR}_\alpha^T \left( \sum_{k=1}^d L_{k,T} \right) = \sum_{k=1}^d \text{VaR}_{k,\alpha}^T$$

## Example 2: No dependence information

(Unconstrained optimization problem)

- Take  $L_{i,T} = L_i$ ,  $i = 1, \dots, d = 8$ ,
  - Marginal OpRisk loss dfs:

$$P(L_i \leq x) = 1 - (x + 1)^{-1.5}, \quad x \geq 0 \quad (\text{Pareto}(1, 1.5))$$

- $E(L_i) = 2 < \infty$ ,  $\text{Var}(L_i) = \infty$
- In the comonotonic case:

$$\text{VaR}_{0.99} \left( \sum_{i=1}^8 L_i \right) = \sum_{i=1}^8 \text{VaR}_{0.99}(L_i) = 0.16$$

$$\text{VaR}_{0.999} \left( \sum_{i=1}^8 L_i \right) = \sum_{i=1}^8 \text{VaR}_{0.999}(L_i) = 0.79$$

## Example 2: No dependence information cont'd

- The **unconstrained** bounds are (in thousand):

$$\text{VaR}_{0.99} \left( \sum_{i=1}^8 L_i \right) \leq 0.41$$

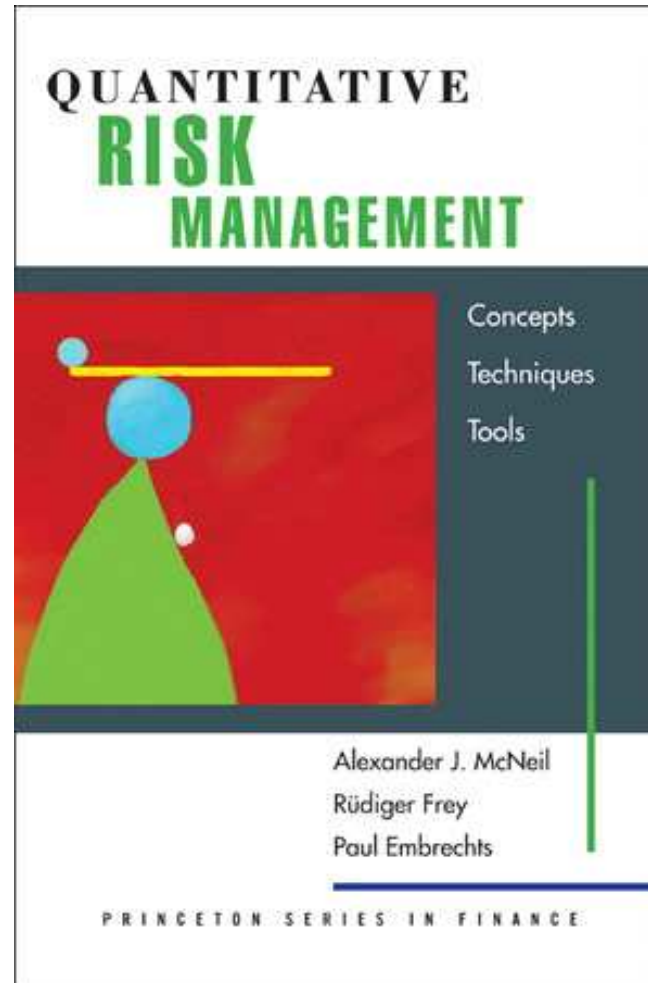
$$\text{VaR}_{0.999} \left( \sum_{i=1}^8 L_i \right) \leq 1.93$$

- The “worst dependence case” can be calculated

## VI. Conclusion

- OpRisk data are intricate
- Regulators ask for extreme risk measures
- At the moment we see **no satisfactory full AMA model** at the horizon
- There are interesting **methodological building blocks** working well for **specific** OpRisk data
- There are interesting **more advanced techniques**
- **More work** is needed
- **DATA!**

# Quantitative Risk Management: Concepts, Techniques and Tools



## References

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