On EVT, aggregation and diversification in finance

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Consider two heavy-tailed risks X_1, X_2 with arbitrary dependence structure.

Questions:

- When does subadditivity of VaR hold:

$$\boxed{\mathrm{VaR}_\alpha(X_1 + X_2) \le \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) \ ?}$$

- Or equivalently, when is the measure of concentration ≤ 1:

$$\boxed{C_\alpha(X_1, X_2) := \frac{\mathrm{VaR}_\alpha(X_1 + X_2)}{\mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2)} \le 1 \ ?}$$

- Or, at least, when does asymptotic subadditivity hold:

$$\boxed{\lim_{\alpha \nearrow 1} C_\alpha(X_1, X_2) \le 1 \ ?}$$

MRV = Multivariate Regular Variation

Property (Daníelsson et al., 2005):
(Under regular variation: see later)
Finite mean $\Rightarrow$ asymptotic subadditivity

Fallacy:
Infinite mean $\Rightarrow$ asymptotic superadditivity

Example

(X_1, X_2) meta-t distributed with

- $t_{0.9}$ margins (infinite "mean"!),
- t_ν copula.

Model 1:
$t_{0.9}$ copula $\Rightarrow$ (X_1, X_2) elliptically distributed
$\Rightarrow$ subadditivity (well-known fact)

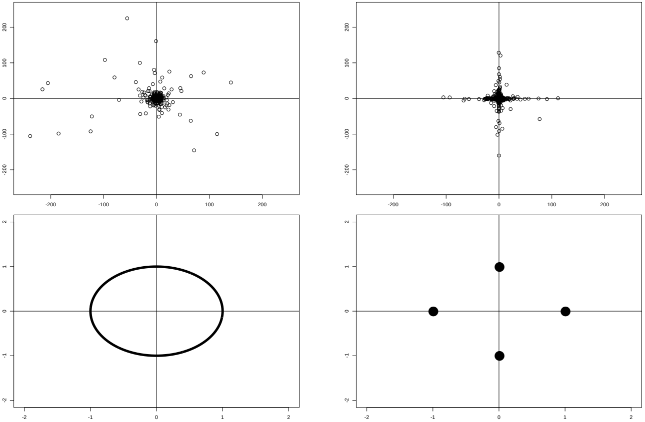
Model 2:
t_∞ copula $=$ Gauss copula $\Rightarrow$ (X_1, X_2) asymptotically independent
$\Rightarrow$ superadditivity (well-known fact)

$t_{0.9}$ **margins with:**

$t_{0.9}$ copula | t_∞/Gauss copula

simulated sample cloud

spectral measure



elliptical = subadditive | independent = superadditive

Intuition

Asymptotic subadditivity for infinite mean models

if and only if

(very) high values of one risk are sufficiently compensated
by (very) low values of the other risk.

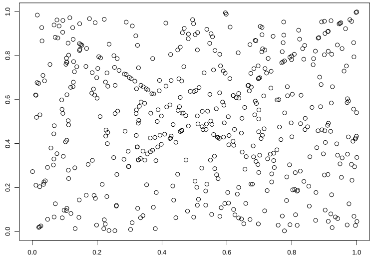
Remark: This intuition is indeed correct and can be made mathematically precise.
$\Longrightarrow$ Look at upper left and lower right corner!

Tail dependence and subadditivity

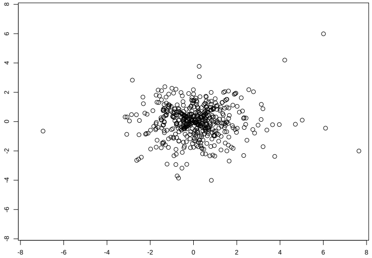
Fréchet family

$$C_{p_1,p_2}(u,v) = p_1(u \wedge v) + p_2(u+v-1)^+ + (1-p_1-p_2)uv.$$

Copula:



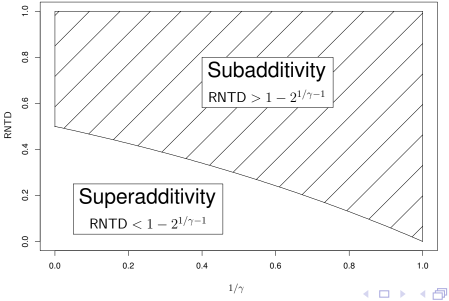
t_6-marginals:



Precise formulation (for two-sided risk, e.g., market risk)

$\lambda^+ = P(\text{risk 2 high}|\text{risk 1 high})$
$\lambda^- = P(\text{risk 2 low}|\text{risk 1 high})$
$\text{RNTD} = \lambda^-/(1-\lambda^+)$: relative negative tail dependence (RNTD)
ξ: heavy-tailedness (i.e. the bigger ξ, the heavier the tails)

Theorem (ELW, 2009) Under certain conditions (Fréchet family) the following holds asymptotically:



Elliptical vs Archimedean copulas

Negative tail dependence

- Elliptical distributions (under MRV) always have negative tail dependence.
- (Strict) Archimedean copulas never have negative tail dependence.

Consequences

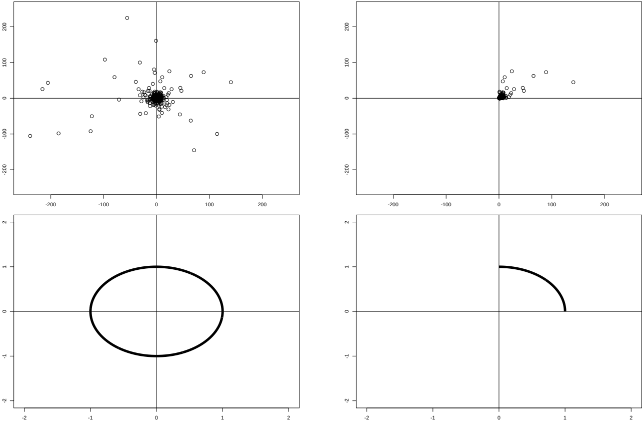
- for spectral measure
- for VaR properties

$t_{0.9}$ margins with:

$t_{0.9}$ copula | only positive values

simulated sample cloud

spectral measure



elliptical = subadditive | NOT subadditive!

Proposition (ELW, 2009)

Let $\boldsymbol{X} = (X_1, X_2)$ be elliptically distributed. Then (under MRV):

a) VaR asymptotically subadditive for $\boldsymbol{X}$ in finite and infinite mean models.

b) VaR asymptotically subadditive for $\boldsymbol{X}|\boldsymbol{X} > \mathbf{0}$ only in finite mean models.

Multivariate regular variation

Definition A random vector $\boldsymbol{X} = (X_1, \ldots, X_n)$ is multivariate regularly varying with index $-1/\xi < 0$, if there exists a probability measure μ, a measurable function $b : (0, \infty) \to (0, \infty)$ with $\lim_{t\to\infty} b(t) = \infty$ and a scalar $q = q(b) > 0$ such that for all $r > 0$,

$$\lim_{t\to\infty} t\, P\left(\|\boldsymbol{X}\| > rb(t),\ \frac{\boldsymbol{X}}{\|\boldsymbol{X}\|} \in G\right) = q r^{-1/\xi} \mu(G), \qquad (1)$$

for any Borel set $G \subset \aleph_{\|\cdot\|}^{n-1} = \{\boldsymbol{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n \mid \|\boldsymbol{x}\| = 1\}$. We write $\boldsymbol{X} \in \mathrm{MRV}_n(-1/\xi)$.

First observations

- For $\boldsymbol{X} \in \mathrm{MRV}_n(-1/\xi)$ with identically distributed marginals and Ψ measurable

$$q(\xi, \Psi) := \lim_{x \to \infty} \frac{P(\Psi(\boldsymbol{X}) > x)}{P(X_1 > x)} = \lim_{\alpha \nearrow 1} \left(\frac{\mathrm{VaR}_\alpha(\Psi(\boldsymbol{X}))}{\mathrm{VaR}_\alpha(X_1)} \right)^{1/\xi}.$$

- If $\Psi(\boldsymbol{X}) = \|\boldsymbol{X}\|$, then $q(\xi, \Psi)$ corresponds to the q in (1).
- (One) relevant case: $\Psi(\boldsymbol{X}) = \sum_{i=1}^n X_i$.
- Problem: $\sum_{i=1}^n X_i$ is only a norm if $\boldsymbol{X} \geq \mathbf{0}$.

Exploiting spectral measures (for positive rvs)

Theorem (Resnick) $\boldsymbol{X} \in \mathrm{MRV}_n(-1/\xi) \Leftrightarrow$
$\exists$ Radon measure ν_ξ: $\lim_{t\to\infty} \; t\,P(\boldsymbol{X}/b(t) \in B) = \nu_\xi(B)$, for all $B \subset [0,\infty]^n \setminus \{\mathbf{0}\}$ relatively compact with $\nu_\xi(\partial B) = 0$.

Spectral measure: $S_{\|\cdot\|} = \nu_1 \left\{ \boldsymbol{z} \in [0,\infty]^n \mid \|\boldsymbol{z}\| > 1, \boldsymbol{z}/\|\boldsymbol{z}\| \in \cdot \right\}$.

Theorem (Barbe, Fougères, Genest) $\boldsymbol{X} \in \mathrm{MRV}_n(-1/\xi)$, $\mathbb{R}^n_+$-valued with identically distributed marginals, then

$$q(\xi, \|\cdot\|) = \lim_{x\to\infty} \frac{P(\|\boldsymbol{X}\| > x)}{P(X_1 > x)} = \int_{\aleph^{n-1}_{+,\|\cdot\|}} \|\boldsymbol{z}^\xi\|^{1/\xi} \, S_{\|\cdot\|}(d\boldsymbol{z}),$$

where $\aleph^{n-1}_{+,\|\cdot\|} = \aleph^{n-1}_{\|\cdot\|} \cap \mathbb{R}^n_+$.

Direct consequences

Corollary Let $\Psi = \|\cdot\|_1$ be the l_1-norm in $\mathbb{R}_+^n$. Then:

$$n \le q(\xi, \|\cdot\|_1) \le n^{1/\xi} \quad \text{for} \quad \xi \le 1,$$
$$n \ge q(\xi, \|\cdot\|_1) \ge n^{1/\xi} \quad \text{for} \quad \xi \ge 1.$$

Proof: (Inverse) Triangular inequality for $L_{1/\xi}$-"norm".

Risk management interpretation: $\boldsymbol{X} \in \mathrm{MRV}_n(-1/\xi)$, $\mathbb{R}_+^n$-valued with identically distributed marginals, then VaR_α is asymptotically subadditive for $\boldsymbol{X}$ if and only if $\xi \le 1$.

Theorem for positive rvs (e.g., Operational Risk) (ELW, 2009)

Under MRV:

a) $X_1, \dots, X_n \geq 0$ with finite mean $(\xi \leq 1)$. Then:

$$n^{\xi-1} \leq \lim_{\alpha \nearrow 1} \frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right)}{\sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)} \leq 1.$$

b) $X_1, \dots, X_n \geq 0$ with infinite mean $(\xi \geq 1)$. Then:

$$1 \leq \lim_{\alpha \nearrow 1} \frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right)}{\sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)} \leq n^{\xi-1}.$$

Example:

(Asymptotic) best/worst case diversification benefit:

$$1 - \lim_{\alpha \nearrow 1} \frac{\mathrm{VaR}_\alpha\left(\sum_{i=1}^n X_i\right)}{\sum_{i=1}^n \mathrm{VaR}_\alpha(X_i)}$$

	$n = 2$	$n = 7$	$n = 8$	$n = 100$
$\xi = 0.1$	46%/0%	83%/0%	85%/0%	98%/0%
$\xi = 0.5$	29%/0%	62%/0%	65%/0%	90%/0%
$\xi = 0.9$	7%/0%	17%/0%	19%/0%	37%/0%
$\xi = 1$	0%/0%	0%/0%	0%/0%	0%/0%
$\xi = 1.1$	0%/ − 7%	0%/ − 21%	0%/ − 23%	0%/ − 58%

Immediate Corollary (ELW, 2009)

Under MRV:

a) $X_1, \ldots, X_n \geq 0$ with finite mean
$\Longrightarrow \text{VaR}_\alpha$ is asymptotically subadditive.

b) $X_1, \ldots, X_n \geq 0$ with infinite mean
$\Longrightarrow \text{VaR}_\alpha$ is asymptotically superadditive.

Does this corollary fully solve the subadditivity problem for positive risks?

Does this corollary fully solve the subadditivity problem for positive risks?

NO!

It is only an asymptotic statement, i.e., for $\alpha \to 1$.

And: asymptotically (for positive risks) only the finite/infinite mean issue matters (as seen before).

But: subadditivity of VaR also typically fails (although asymptotic subadditivity may hold!) for

- very skew distributions
- very special dependence (usually not a problem in practice)

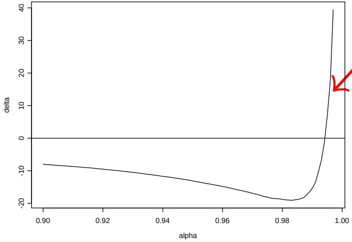
Counter-Example (Degen, E., L., 2007)

Suppose $X_1, X_2 \overset{\text{iid}}{\sim} g\text{-and-}h$ $(\overset{d}{=} \frac{e^{gZ}-1}{g} e^{hZ^2/2}$, $Z \sim \mathcal{N}(0,1))$, with skewness parameter $g = 2.4$, and heavy-tailedness parameter $h = 0.2$ (typical for Operational Risk).

(Absolute) diversification benefit:

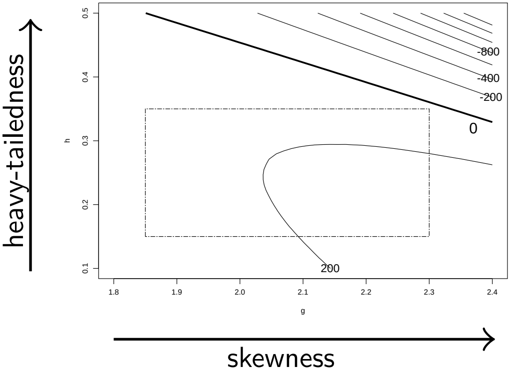
$$\text{delta} = \mathrm{VaR}_\alpha(X_1) + \mathrm{VaR}_\alpha(X_2) - \mathrm{VaR}_\alpha(X_1 + X_2)$$

99.4%



Finite mean:
indeed asymptotically subadditive.
But! VaR superadditive for $\alpha < 0.994$.

Diversification benefit for 99.9%-VaR

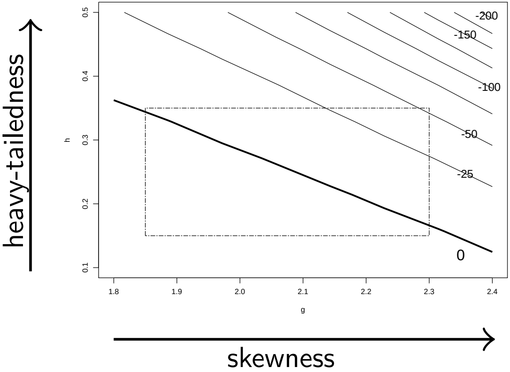


- Typical OpRisk parameters within subadditivity range
- Small changes of parameters ⇒ superadditivity

What happens when we go deeper in the data?

- VaR-estimation at 99.9% and higher: difficult!
- Estimate at lower level (99%, say) and scale.

Diversification benefit for 99%-VaR



- Substantial fraction of parameter rectangle switched regime
- Far from diversification!

Main messages to the OpRisk community

- Questions like *"Do we have diversification?"* or *"How big is the diversification benefit?"* is always related to the VaR-level α.
- We may have a high diversification benefit on a 99.9% level and a small diversification benefit on a 99% level.
- Look at upper left and lower right corner!
- Asymptotically (for positive risks) only the heavy-tailedness matters (finite/infinite mean).
- In the real (and finite!) world also skewness has to be considered.

References

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