Aggregating Risk Capital, with an Application to Operational Risk

2007 version

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The present talk is mainly based on the following papers:

JMVA Embrechts, P. and G. Puccetti (2006). Bounds for functions of multivariate risks. *J. Mult. Analysis*, *97*(2), 526–547.

F&S Embrechts, P. and G. Puccetti (2006b). Bounds for functions of dependent risks. *Finance Stoch. 10*(3), 341–352.

GRIR Embrechts, P. and G. Puccetti (2006c). Aggregating risk capital, with an application to operational risk. *Geneva Risk. Insur. Rev.*, *31*(2), 71–90.

The problem at hand

On some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$, consider a random vector

$$X := (X_1, \ldots, X_n)$$

of n one-period financial losses or insurance claims,

and fix its marginal dfs $F_1, \ldots, F_n$.

The **joint** df of the random vector X
is **not** completely determined by the F_i's.

There are infinitely many distributions for the vector X which are consistent with the initial choice of the marginals.

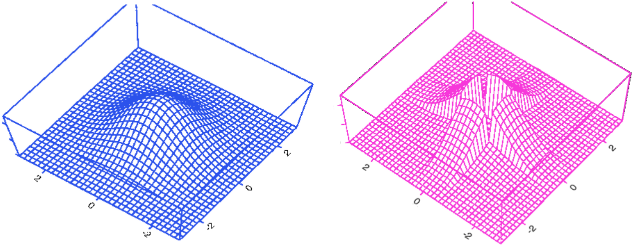
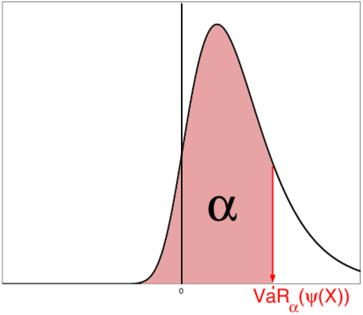


Figure: Two bivariate dfs having $N(0,1)$-marginals and the same correlation

Given an aggregating function $\psi : \mathbb{R}^n \to \mathbb{R}$, which is the df giving the **worst-possible Value-at-Risk (VaR)** for $\psi(X)$?

The Value-at-Risk at probability level α for $\psi(X)$ is the maximum aggregate loss which can occur with probability $\alpha, \alpha \in [0, 1]$.



If G (the df of $\psi(X)$) is strictly increasing,
$\text{VaR}_\alpha(\psi(X))$ is the unique threshold t at which $F(t) = \alpha$, i.e. $F^{-1}(\alpha)$.

Searching for the worst-possible VaR means looking for

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \dots, n\}, s \in \mathbb{R}.$$

Indeed, according to the definition of VaR, we have

$$\mathrm{VaR}_\alpha(\psi(X)) \leq m_\psi^{-1}(\alpha), \alpha \in [0, 1].$$

$m_\psi(s)$ is a **linear problem** over a convex feasible space of measures.
Therefore, it admits a **dual representation**.

Main Duality Theorem (Rüschendorf (1982))

$$m_\psi(s) = \inf\{\mathbb{P}[\psi(X) < s] : X_i \backsim F_i, i = 1, \ldots, n\}$$
$$= 1 - \inf\left\{ \sum_{i=1}^{n} \int f_i dF_i : f_i \in L^1(F_i), i \in N \; s.t. \right.$$
$$\left. \sum_{i=1}^{n} f_i(x_i) \geq 1_{[s,+\infty)}(\psi(x)) \; for \; all \; x \in \mathbb{R}^n \right\}.$$

Some remarks on the dual problem

$m_\psi(s)$, as well as its dual counterpart, is very difficult to solve. Solutions are known only in few cases.

- When $n = 2$; see Rüschendorf (1982).
- When $n > 2$, the only explicit solution we know is given in Rüschendorf (1982) for the sum of risks uniformly distributed on the unit interval

Even if we do not solve the dual problem,
dual admissible functions provide bounds on the solutions which are conservative from a risk management viewpoint

We call *standard bounds* those bounds obtained by choosing *piecewise-constant* dual choices.

- Standard bounds are those typically obtained from elementary probability; see: Denuit, Genest, and Marceau (1999). They are sharp when $n = 2$.

We call *dual bounds* those bounds obtained by choosing *piecewise-linear* dual choices.

- Dual bounds are **better** than standard bounds when $n > 2$ but are available only for $\psi = +$.
- Extensions are stated for portfolios of vectors; see [JMVA]

Bounds on Value-at-Risk

	$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *exact*		$\mathrm{VaR}_\alpha(\sum_{i=1}^3 X_i)$, *upper bound*	
α	independence	comonoton.	dual	standard
0.90	7.54	8.85	14.44	15.38
0.95	9.71	12.73	19.50	20.63
0.99	16.06	25.16	35.31	37.03
0.999	29.78	53.99	69.98	73.81

Table: Range for VaR for a Log-Normal(-0.2,1)-portfolio.

Bounds on Value-at-Risk (large portfolios!)

	$\mathrm{VaR}_\alpha(\sum_{i=1}^{10} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{100} X_i)$		$\mathrm{VaR}_\alpha(\sum_{i=1}^{1000} X_i)$	
α	dual	standard	dual	standard	dual	standard
0.90	0.669	1.485	11.039	149.850	150.162	14998.500
0.95	1.353	2.985	22.227	229.850	301.823	29998.500
0.99	2.985	14.985	111.731	1499.850	1515.111	149998.500
0.999	68.382	149.985	1118.652	14999.850	15164.604	1499998.500

Table: Upper bounds for $\mathrm{VaR}_\alpha(\sum_{i=1}^{n} X_i)$ of three Pareto portfolios of different dimensions. Data in thousands.

Application 2: Aggregation of OpRisk losses

Under the New Basel Capital Accord (**Basel II**) banks are required to set aside capital for the specific purpose of offsetting OpRisk.

OpRisk The risk of losses resulting from inadequate or failed **internal processes**, **people** and **systems**, or **external events**. Included is **legal risk**, excluded are strategic/business and reputational risk.

OpRisk pillar 1 issue: Loss Distribution Approach (LDA)

- Operational losses $L_{i,j}$ are separately modeled in **eight business lines** (rows) and by **seven risk types** (columns) in the 56-cell Basel matrix
- Marginal risks have **non-homogeneous** distributions
- Capital requirement to be calculated as the sum of $\mathrm{VaR}_{0.999}^{1\text{ year}}$ across the cells of the matrix:

$$\sum_{i=1}^{8} \sum_{j=1}^{7} \mathrm{VaR}_{0.999}(L_{i,j})$$

In standard practice (see Moscadelli (2004)), the OR capital charge can be calculated aggregating risks BL-wise yielding capital estimates $\mathrm{VaR}_{1.}, \ldots, \mathrm{VaR}_{8.}$

	RT_1	$\ldots$	RT_j	$\ldots$	RT_7		
BL_1						$\rightarrow$	$\mathrm{VaR}_{1.}$
$\vdots$						$\vdots$	$\vdots$
BL_i			$L_{i,j}$			$\rightarrow$	$\mathrm{VaR}_{i.}$
$\vdots$						$\vdots$	$\vdots$
BL_8						$\rightarrow$	$\mathrm{VaR}_{8.}$

Finally calculate (as indicated in Basel II) the **comonotonic value**:

$$\mathrm{VaR}^R = \sum_{i=1}^{8} \mathrm{VaR}_{i.}$$

How reliable is this procedure?

α	Basel II value	dual bound	standard bound
0.99	2.8924×10^4	1.4778×10^5	2.6950×10^5
0.995	6.7034×10^4	3.3922×10^5	6.1114×10^5
0.999	4.8347×10^5	2.3807×10^6	4.1685×10^6
0.9999	8.7476×10^6	4.0740×10^7	6.7936×10^7

Table: Range for $\mathrm{VaR}_\alpha\left(\sum_{i=1}^{8} \mathrm{BL}_i\right)$ for the OR portfolio given in Moscadelli (2004).

- The computation of standard bounds is non-trivial in the non-homogeneous case; see [GRIR]
- The dual bound is prudential, more realistic and economically advantageous with respect to the standard one.
- There is no mathematical reason to drop the worst-case bounds if no dependence assumptions on the portfolio are explicitly made.

Recall that $\mathrm{VaR}^R = \sum_{i=1}^8 \mathrm{VaR}_{i.}$

The OR capital charge could be calculated aggregating risks also RT-wise, yielding the capital estimate $\mathrm{VaR}^C = \sum_{j=1}^7 \mathrm{VaR}_{.j}$

	RT_1	$\dots$	RT_j	$\dots$	RT_7		
BL_1						$\rightarrow$	$\mathrm{VaR}_{1.}$
$\vdots$						$\vdots$	$\vdots$
BL_i			$L_{i,j}$			$\rightarrow$	$\mathrm{VaR}_{i.}$
$\vdots$						$\vdots$	$\vdots$
BL_8						$\rightarrow$	$\mathrm{VaR}_{8.}$
	$\downarrow$ $\mathrm{VaR}_{.1}$	$\dots$ $\dots$	$\downarrow$ $\mathrm{VaR}_{.j}$	$\dots$ $\dots$	$\downarrow$ $\mathrm{VaR}_{.7}$		

- $\mathrm{VaR}^C \neq \mathrm{VaR}^R$?
- Which factors does $\Delta := \mathrm{VaR}^C - \mathrm{VaR}^R$ depend upon?

A *toy model* approach

- Simplified 2×3 Basel matrix
- $L_{i,j} \sim$ Pareto(4) for all $i = 1, 2, j = 1, 2, 3$
- Dependence among the $L_{i,j}$ is modeled by a six-dimensional **Gumbel copula** with parameter θ
- the Gumbel copula is symmetric and has Gumbel projections, losses are identically distributed

$$\downarrow$$

The only asymmetry in OR aggregation is caused by the fact that the Basel matrix is not square

Conclusions from the toy model approach

θ		VaR^C	VaR^R	Δ
1.00	independence	18.31	14.46	3.85
1.10	dependence	21.32	19.38	1.94
1.25	higher dependence	23.71	22.62	1.09
$+\infty$	comonotonicity	27.74	27.74	0

- Large differences between the two VaR-aggregation methods.
- Starting from a maximum in the independence set-up, Δ becomes smaller as the strength of dependence increases.
- Under the comonotonic assumption, $\Delta = 0$ due to VaR additivity.
- Coming soon: a more sophisticated *soft* model

Conclusions

- The worst-possible VaR for a non-decreasing function of dependent risks can be calculated when the portfolio is **two-dimensional**.
- When dealing with more than two risks, the problem gets much more complicated and we provide a **dual** bound which we prove to be better than the standard one generally used in the literature.
- OpRisk VaR-aggregation leads to problems and diversification effects have to be handled with care.

Extensions (more research is needed!)

- Exact VaR bounds when $n > 2$, calculation of bounds for other portfolio functions ψ
- Basel II has some issues to solve (2008+)
- Problems of scaling when fixing marginal dfs (Market + Credit + Op Risk).
- For a textbook treatment, see

Visit the book zone: `www.ma.hw.ac.uk/~mcneil/book/index.html`

Acknowledgements

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For Further Reading I

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