Bounds for Functions of Multivariate Risks

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of n one-period risks on some probability space $(\Omega, \mathfrak{A}, \mathbb{P})$.

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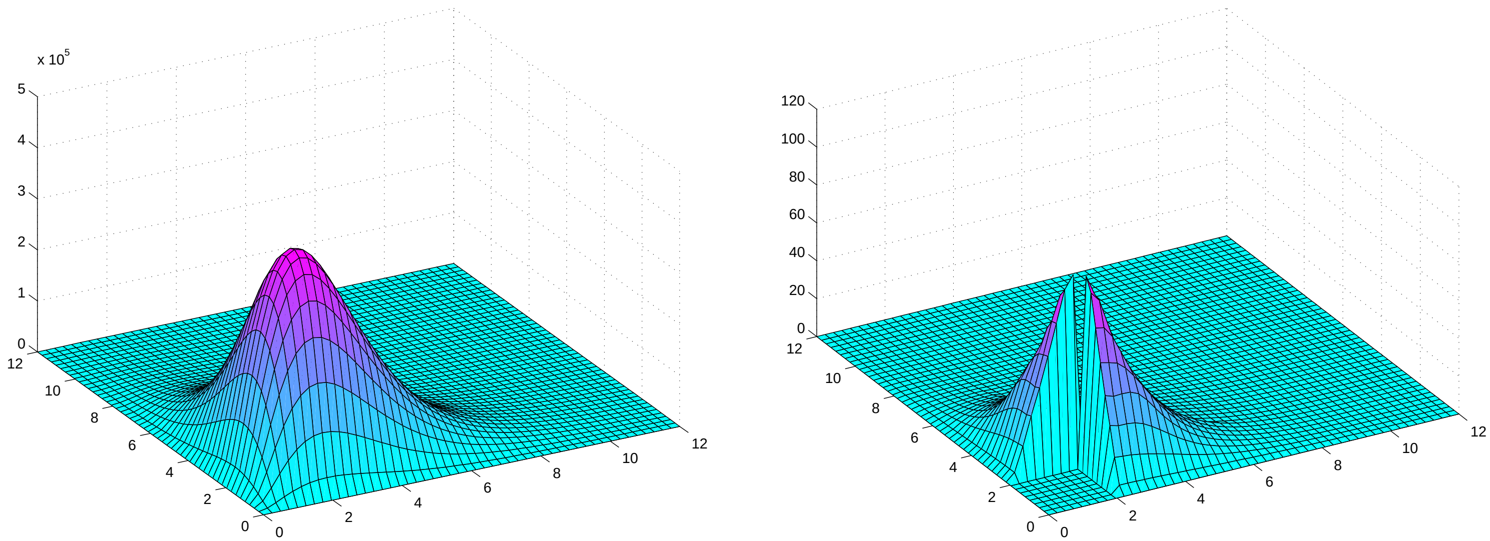
but not about their **joint df**, i.e. the way the risks are interrelated.

Given a measurable function $\psi : \mathbb{R}^n \to \mathbb{R}$,
the aggregate loss which the insurer will bear is

$$\psi(X) = \psi(X_1, \ldots, X_n).$$

The aggregate loss $\psi(X)$



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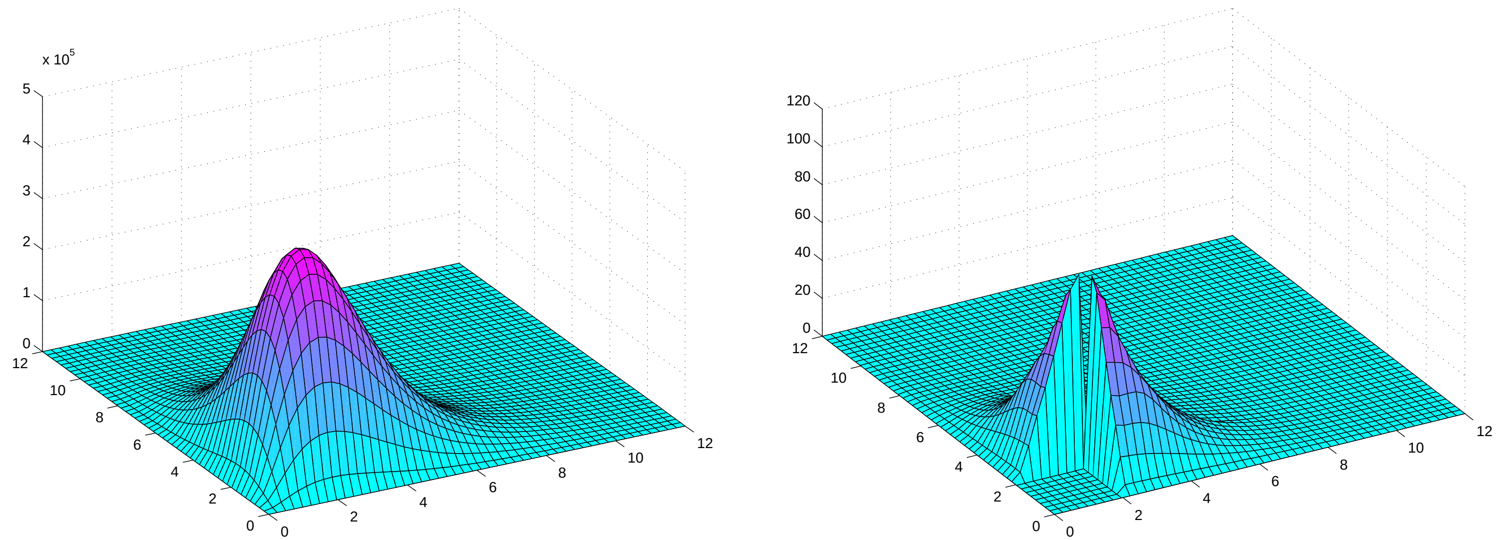

The df F for the aggregate loss $\psi(X)$ cannot be determined without further information.

Therefore, the problem at hand becomes determining

$$LB(s) \leq \mathbb{P}[\psi(X) < s] \leq UB(s) \text{ over } \mathfrak{F}(F_1, \ldots, F_n),$$

the *Fréchet class* of probability dfs for X having $F_1, \ldots, F_n$ as marginals.

Mathematical problems with univariate marginals

For a function $\psi : \mathbb{R}^n \to \mathbb{R}$, we define

$$m_\psi(s) := \inf\{\mathbb{P}[\psi(X_1, \ldots, X_n) < s] : X_i \backsim F_i, 1 \le i \le n\}, s \in \mathbb{R},$$
$$M_\psi(s) := \sup\{\mathbb{P}[\psi(X_1, \ldots, X_n) \ge s] : X_i \backsim F_i, 1 \le i \le n\}, s \in \mathbb{R}.$$

Since

$$m_\psi(s) = 1 - M_\psi(s),$$

the above problems:

- are equivalent
- have received a considerable interest in the literature, see Embrechts and Puccetti (2004) and references therein.

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the above problems:

- are **not** equivalent
- have not been given much attention. In fact, dealing with multivariate marginals causes extra problems.

Problems arising with multivariate marginals

- As shown in Scarsini (1989), the concept of *copula* as a tool to generate dfs from a set of marginals, becomes inadequate when dealing with the product of multivariate spaces.

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see Cohen (1984); Rüschendorf (1985); Sánchez Algarra (1986); Marco and Ruiz-Rivas (1992).

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- Genest, Quesada Molina, and Rodríguez Lallena (1995, Prop. A) state that in the multivariate case the only measure lying in $\mathfrak{F}(F_1, \ldots, F_n)$ for all possible choices of the F_i's is the independence measure $\Pi_{i=1}^n F_i$.

Dealing with multivariate marginals

Assuming multivariate marginals allows not only
to fix the univariate df of every component
of the single multivariate policies,
but also the dependence **within** the single risks.

$$\begin{array}{r} \text{insurance line } 1 \rightarrow \\ \vdots \\ \text{insurance line } k \rightarrow \end{array} \quad \Bigg(\underbrace{\begin{pmatrix} X_1^1 \\ \vdots \\ X_1^k \end{pmatrix}}_{\text{policy } 1}, \ldots, \underbrace{\begin{pmatrix} X_n^1 \\ \vdots \\ X_n^k \end{pmatrix}}_{\text{policy } n} \Bigg)$$

Assumptions on $\psi : (\mathbb{R}^k)^n \to \mathbb{R}^k$

Given k measurable functions $\psi_j : \mathbb{R}^n \to \mathbb{R}$, $j \in K$, we construct the function $\psi : (\mathbb{R}^k)^n \to \mathbb{R}^k$ as follows:

$$\psi(\mathbf{X}_1, \ldots, \mathbf{X}_n) = \psi\left(\begin{pmatrix} X_1^1 \\ \vdots \\ X_1^k \end{pmatrix}, \ldots, \begin{pmatrix} X_n^1 \\ \vdots \\ X_n^k \end{pmatrix}\right) = \begin{pmatrix} \psi_1(X_1^1, \ldots, X_n^1) \\ \vdots \\ \psi_k(X_1^k, \ldots, X_n^k) \end{pmatrix}$$

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Example. *If we assume $\psi_j = +$, the sum operator, for all $j = 1, \ldots, k$, we have*

$$\psi(\mathbf{X}_1, \mathbf{X}_2) = \psi\left(\begin{pmatrix} X_1^1 \\ X_1^2 \end{pmatrix}, \begin{pmatrix} X_2^1 \\ X_2^2 \end{pmatrix}\right) = \begin{pmatrix} X_1^1 + X_2^1 \\ X_1^2 + X_2^2 \end{pmatrix}$$

The function ψ makes sense if the risks are componentwise homogeneous.

Duality

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Main Duality Theorem (Ramachandran and Rüschendorf (1995)).

$$m_\psi(\mathbf{s}) = \sup\left\{ \sum_{i=1}^{n} \int_{\mathbb{R}^k} f_i dF_i \,:\, f_i \in L^1(F_i), i \in N \text{ with} \right.$$
$$\left. \sum_{i=1}^{n} f_i(\mathbf{x}_i) \leq 1_{(-\infty,\mathbf{s})}(\psi(\mathbf{x})) \text{ for all } \mathbf{x} \in (\mathbb{R}^k)^n \right\},$$

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$M_\psi(\mathbf{s})$ admits an analogous representation.

Every rv $\mathbf{X}^C = (\mathbf{X}_1^C, \dots, \mathbf{X}_n^C)$ with df in $\mathfrak{F}(F_1, \dots, F_n)$ is a *coupling*.

Every set of functions $\hat{\mathbf{f}} = (\hat{f}_1, \dots, \hat{f}_n)$ admissible for the above dual problem is a *dual choice*.

Known solutions

$m_\psi(\mathbf{s})$ and $M_\psi(\mathbf{s})$, as well as their dual counterparts, are very difficult to solve. Solutions under general marginal dfs are known only in few cases.

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- For $\psi = +$, Li, Scarsini, and Shaked (1996) give $m_\psi(\mathbf{s})$ for $n = 2$ and arbitrary k.

- When $n > 2$, the only explicit solution known is given in Rüschendorf (1982) for the sum of risks uniformly distributed on the unit interval.

The basic idea: the coupling-dual approach

If $\mathbf{X}^C$ is a coupling and $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_n)$ and $\hat{\mathbf{g}} = (\hat{g}_1, \ldots, \hat{g}_n)$ are two set of functions which are admissible for the corresponding dual problems, we have

$$\mathbb{P}[\psi(\mathbf{X}^{\mathbf{C}}) < \mathbf{s}] \geq m_\psi(\mathbf{s}) \geq \sum_{i=1}^{n} \int_{\mathbb{R}^k} \hat{f}_i dF_i,$$

$$\mathbb{P}[\psi(\mathbf{X}^{\mathbf{C}}) \geq \mathbf{s}] \leq M_\psi(\mathbf{s}) \leq \sum_{i=1}^{n} \int_{\mathbb{R}^k} \hat{g}_i dF_i.$$

Therefore, even if we do not find optimal couplings, **dual admissible functions provide bounds on the solutions which are conservative from a risk management viewpoint**.

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The best standard bounds on m_ψ and M_ψ are given in Li, Scarsini, and Shaked (1996). In our paper, we correct the second one and prove both using duality.

The standard bound on m_ψ is sharp only in the case of the sum of two risks. The one for M_ψ fails to be sharp also for $n = 2$.

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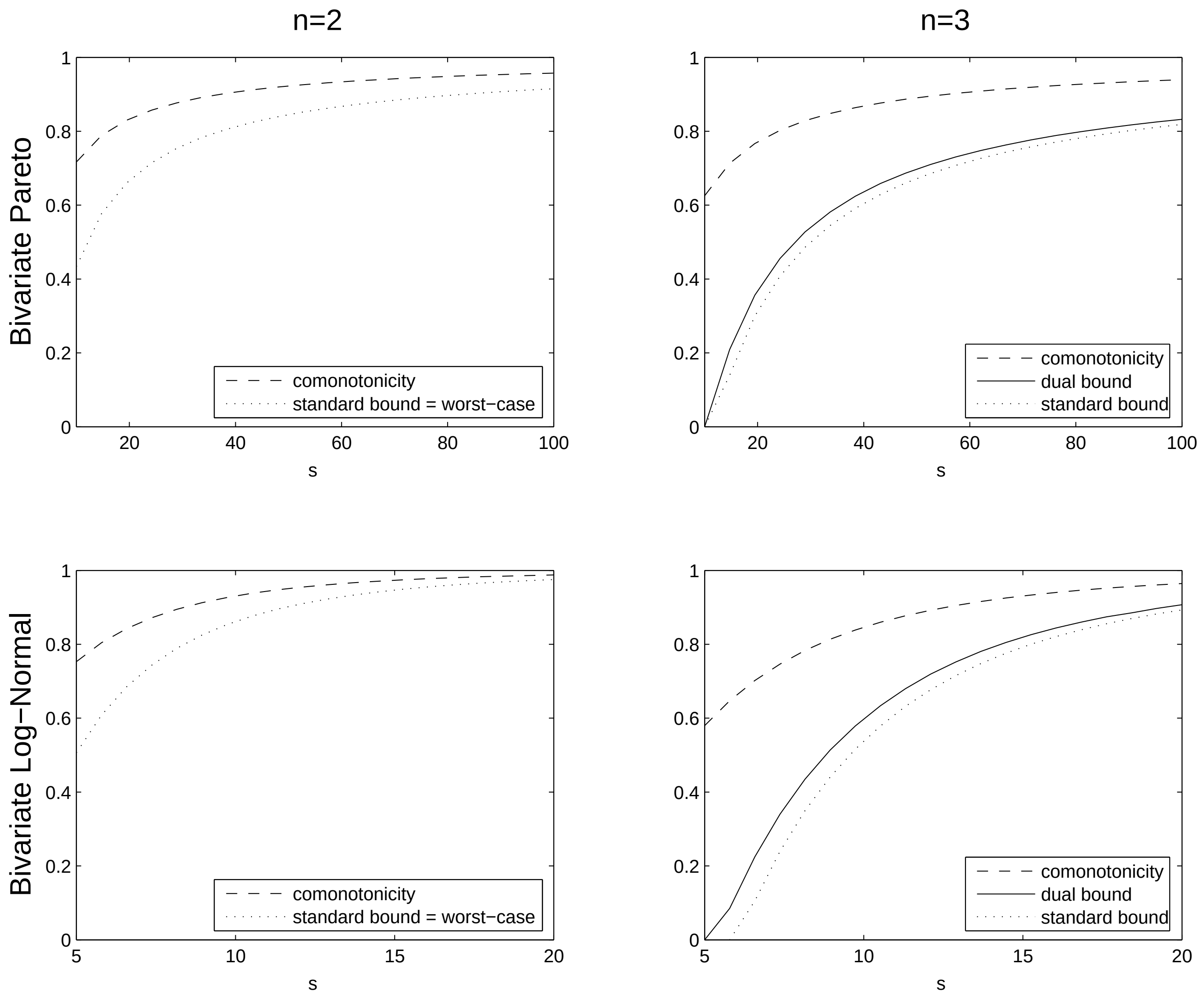
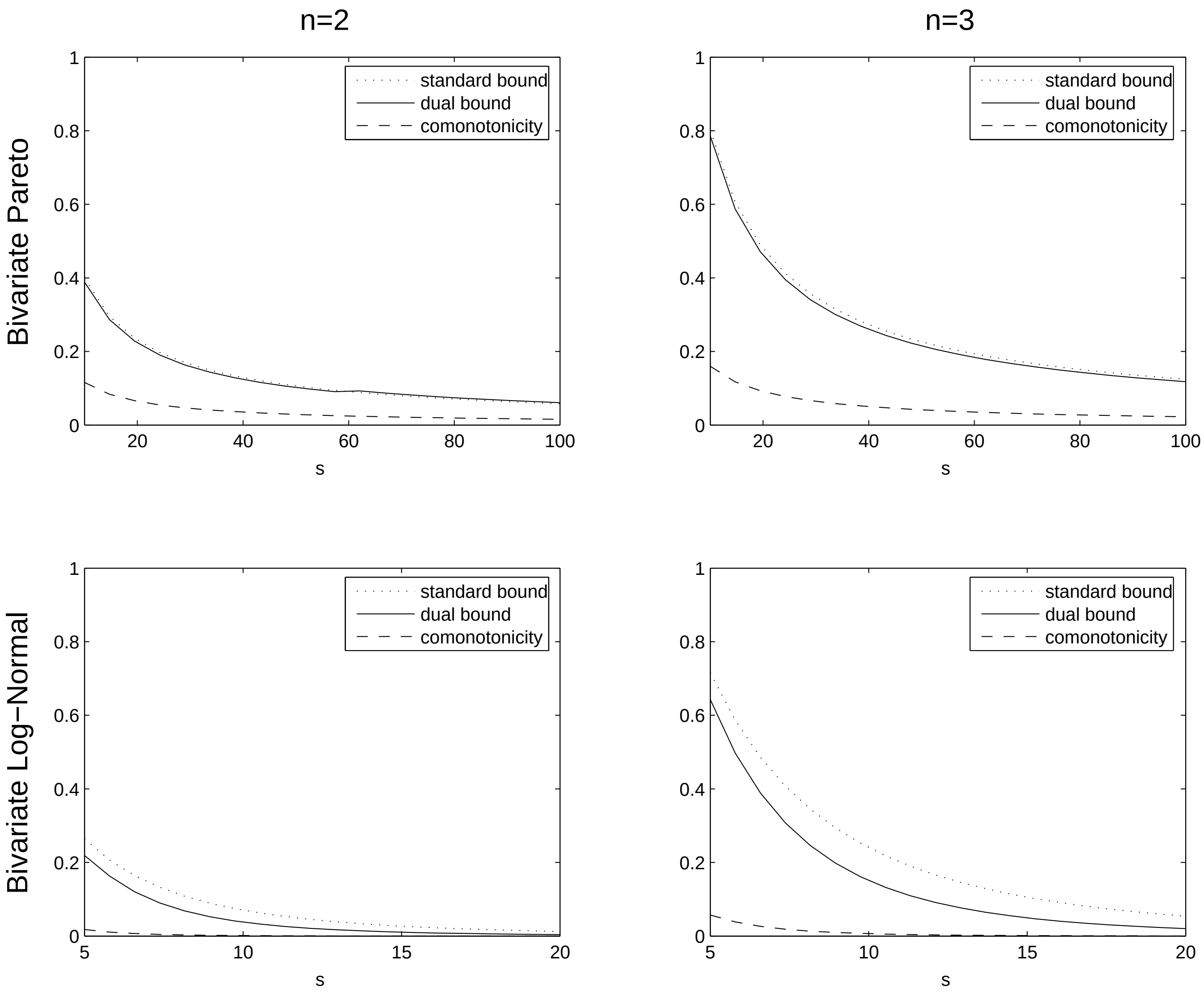
In the univariate-marginal case there is a *natural* choice of the piecewise-linear function yielding the so-called *dual bound*.
In the multivariate setting, instead, that choice is not straightforward.

When several dual choices are available,
an overall better bound is produced by taking the pointwise minimum/maximum among the corresponding bounds.

Range for $\mathbb{P}[\sum_{i=1}^{n} \mathbf{X}_i < (s, s)]$

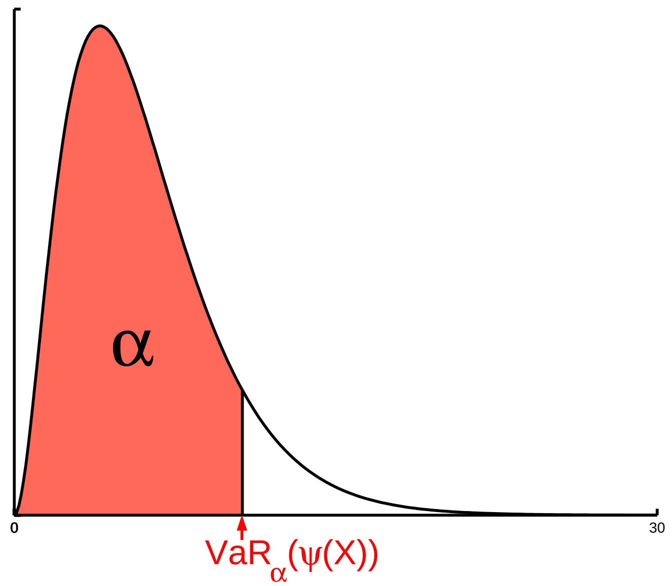


Range for $\mathbb{P}[\sum_{i=1}^{n} \mathbf{X}_i \geq (s, s)]$



Applications: Value-at-Risk

The Value-at-Risk (or *quantile*) at probability level α for $\psi(X)$ is the maximum aggregate loss which can occur with probability α, $\alpha \in [0, 1]$.



$$\mathbb{P}[\psi(X) \geq s] < 1 - \alpha \text{ for all } s > \mathrm{VaR}_\alpha(\psi(X))$$

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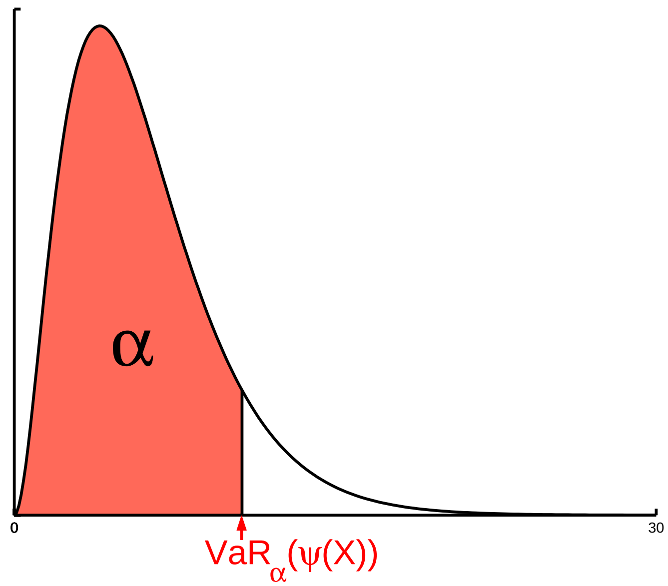

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With univariate marginals, we have: $\mathrm{VaR}_\alpha(\psi(X)) \leq m_\psi^{-1}(\alpha), \alpha \in [0,1]$.

Applications: Value-at-Risk

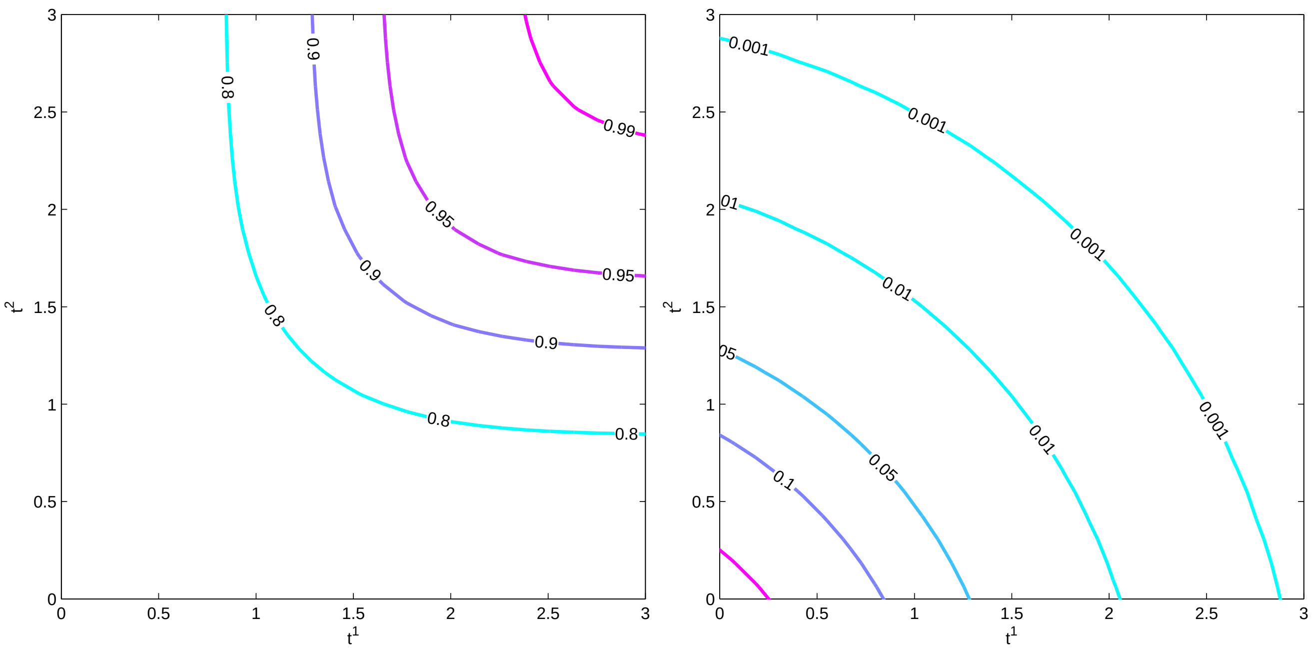
If F is increasing, $\mathrm{VaR}_\alpha(\psi(X))$ is the unique threshold t at which $F(t) = \alpha$.

Note that with multivariate marginals, the definition of VaR does not make sense since, even for a continuous df F, there are possibly infinitely many vectors $\mathbf{s} \in \mathbb{R}^k$ at which $F(\mathbf{s}) = \alpha$.

An intuitive and immediate measure of the risk involved in a multivariate loss df F is represented by the α-level sets

of its df and of its tail $\overline{F}$.

Multivariate Value-at-Risk



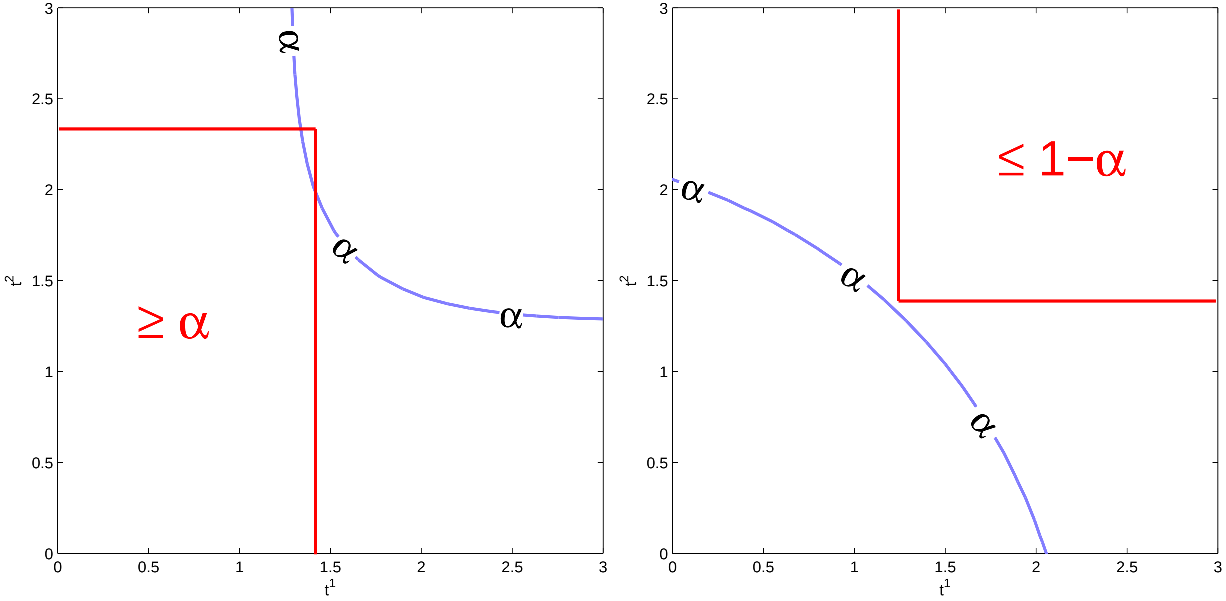
Level sets for the df (left) and the tail (right)
for a bivariate standard normal random vector.

We call these curves *lower-orthant (LO-)* Value-at-Risk and *upper-orthant (UO-)* Value-at-Risk at probability level α and $1 - \alpha$, respectively.

Of course the same definitions hold for general monotone functions.

Multivariate Value-at-Risk

The LO-VaR$_\alpha$ for m_ψ (left) and the UO-VaR$_\alpha$ for M_ψ (right) provide conservative estimates of the α-VaRs for the aggregate loss $\psi(\mathbf{X})$ over $\mathfrak{F}(F_1, \ldots, F_n)$.



$$\mathbb{P}[\psi(\mathbf{X}) < \mathbf{s}] \geq \alpha \text{ for every } \mathbf{s} > \mathbf{x}_1 \in \underline{\mathrm{VaR}}_\alpha(m_+),$$

$$\mathbb{P}[\psi(\mathbf{X}) \geq \mathbf{s}] \leq 1 - \alpha \text{ for every } \mathbf{s} > \mathbf{x}_2 \in \overline{\mathrm{VaR}}_\alpha(M_+).$$

Worst-possible VaR

We refer to $\underline{\mathrm{VaR}}_\alpha(m_\psi)$ and $\overline{\mathrm{VaR}}_\alpha(M_\psi)$ as the *worst-possible* Value-at-Risks for the risky position $\psi(\mathbf{X})$.

Recall that if $\hat{\mathbf{f}} = (\hat{f}_1, \ldots, \hat{f}_n)$ and $\hat{\mathbf{g}} = (\hat{g}_1, \ldots, \hat{g}_n)$ are two set of functions which are admissible for the corresponding dual problems, we have

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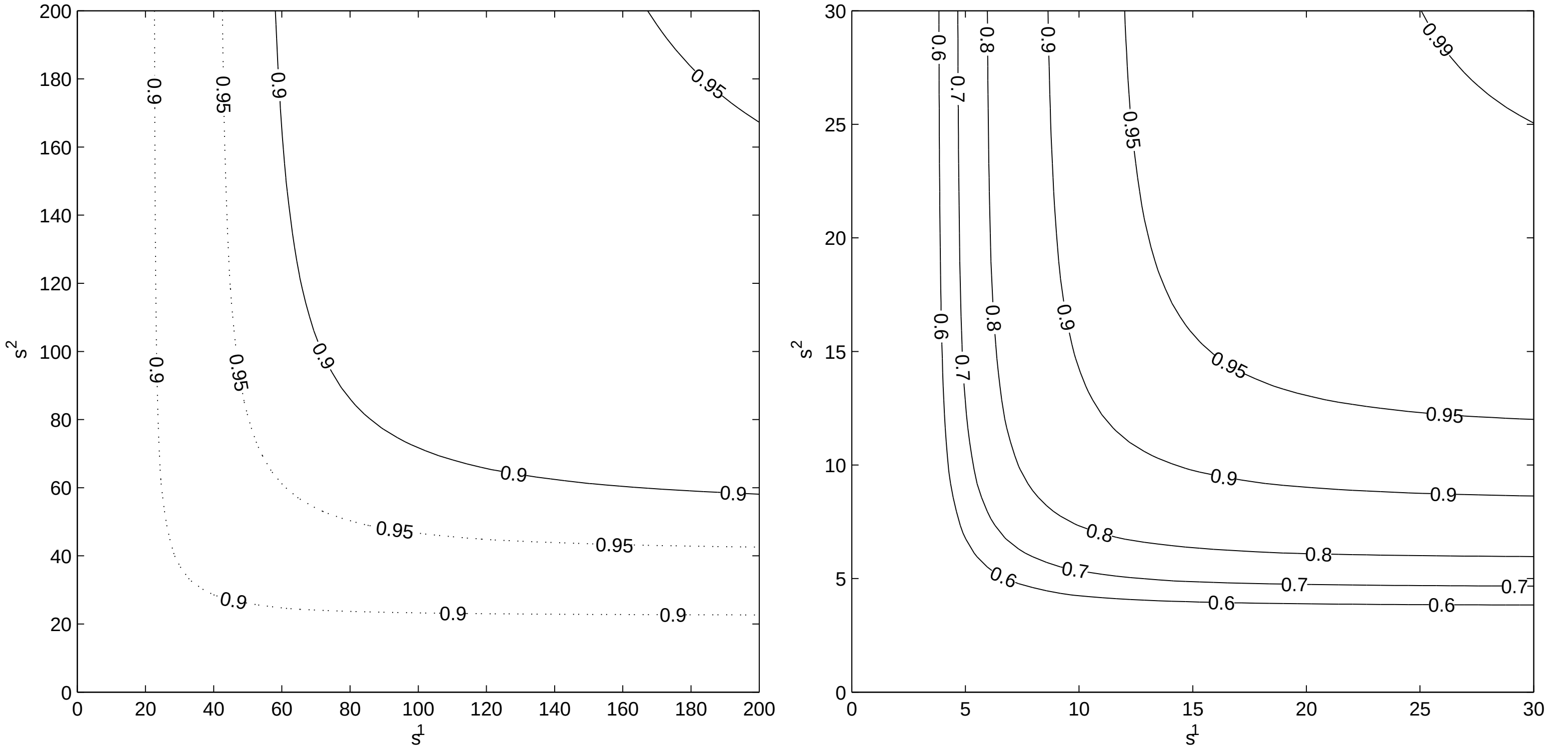
When it is not possible to compute m_ψ and M_ψ exactly, the α-VaRs for the corresponding dual bounds still provide conservative estimates.

Worst-possible VaR

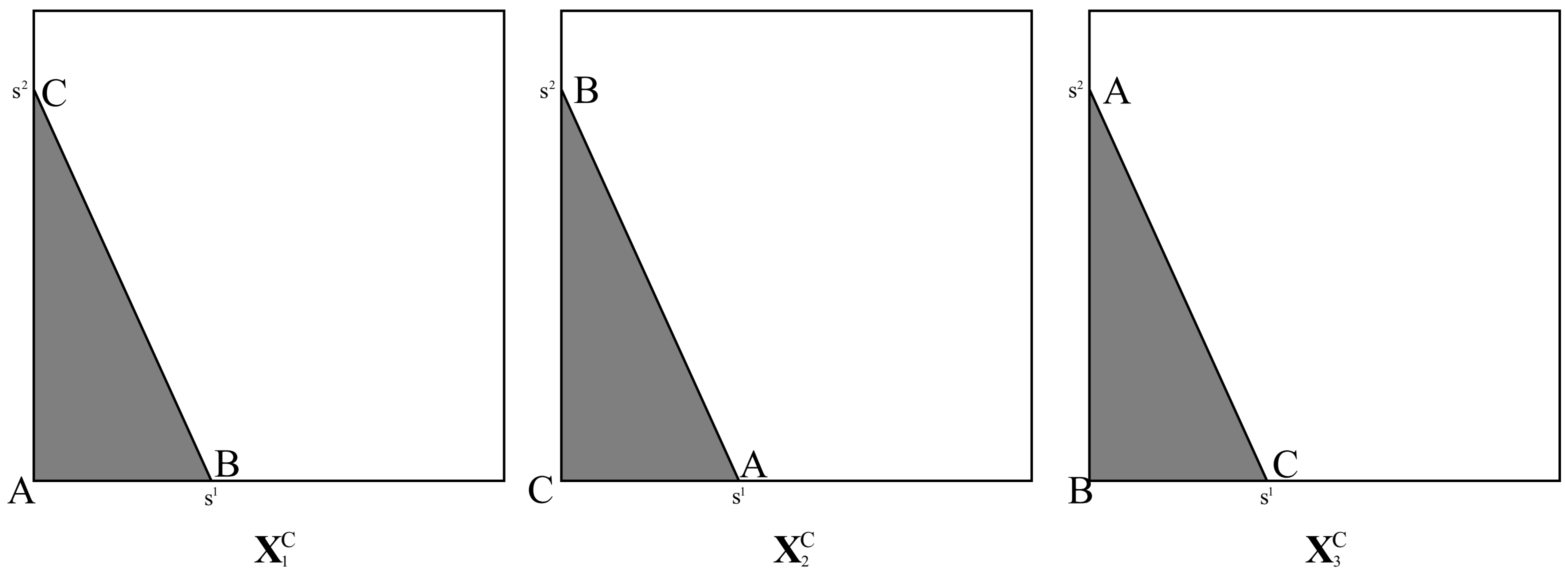


Worst-possible LO-VaRs for the sum of two bivariate Pareto ($\theta = 1.2$ for the dotted line) (left) and Log-Normal (right) distributed risks.

Optimal couplings

We extend Rüschendorf (1982, Th. 1) by providing **optimal couplings** for the sum of risks uniformly distributed on the k-dimensional hypercube when

- $n = 2, k \in \mathbb{N}$
- $n = k + 1, k \in \mathbb{N}$



Optimal coupling for $\sup\{\mathbb{P}[\sum_{i=1}^{3} \mathbf{X}_i \leq \mathbf{s}] : \mathbf{X}_i \backsim U(\mathbb{I}^2)\}$.

Conclusions

Embrechts and Puccetti (2004) propose a dual approach for the problem of determining bounds for functions of dependent risks having fixed **univariate** marginals.

In this paper we give an extension of all results contained in the latter article to **multivariate** marginals.

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using the dual formulation we can improve the standard bounds obtained from elementary probability

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