BOUNDED COHOMOLOGY AND TOTALLY REAL SUBSPACES IN COMPLEX HYPERBOLIC GEOMETRY

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1. Introduction

To any representation $\pi : \Gamma \to PSU(n,1)$ of a finitely generated group Γ, we associate an invariant lying in the second bounded cohomology group $H_b^2(\Gamma,\mathbb{R})$ of Γ and establish relations between its vanishing and properties of the representation π. Recall that, as far as classical group cohomology is concerned, the second continuous cohomology group of $PSU(n,1)$ is one dimensional, with generator k corresponding to the Kähler form ω on complex hyperbolic space, via the Van Est isomorphism [11]. Essential to our purposes is that the class k admits a bounded representative, given by

$$k_b(g_0,g_1,g_2) = \int_{\Delta(g_0,g_1,g_2)} \omega$$

where $\Delta(g_0,g_1,g_2)$ is a C^1-simplex in the n-dimensional complex hyperbolic space $\mathbb{H}^n_{\mathbb{C}}$ with geodesic sides and with vertices g_0x, g_1x, g_2x, x being a base point [6]. We shall refer to $k_b \in H^2_{b,c}(PSU(n,1))$ as the bounded Kähler class. The object of our interest is then the bounded class $\pi^*(k_b) \in H_b^2(\Gamma,\mathbb{R})$, which of course corresponds via the comparison map $H_b^2(\Gamma,\mathbb{R}) \longrightarrow H^2(\Gamma,\mathbb{R})$ to the class $\pi^*(k)$.

Both cohomology classes give information about the representation π. The class $\pi^*(k)$ represents exactly the obstruction for the action of Γ via π to be lifted to the universal cover $\widetilde{PSU(n,1)}$ but is a rather coarse invariant, being constant on connected components of the representation variety $\mathrm{Rep}(\Gamma, PSU(n,1))$. In particular, while $\pi^*(k) = 0$ whenever $H^2(\Gamma,\mathbb{R}) = 0$, its bounded counterpart $\pi^*(k_b) \in H_b^2(\Gamma,\mathbb{R})$ still has a chance not to vanish. The content of the next theorem is to pin down exactly that the class $\pi^*(k_b)$ does indeed vanish only when the image of π is "small" in the following sense:

Theorem 1.1. *Let Γ be a finitely generated group, $\pi : \Gamma \to PSU(n,1)$ a homomorphism and $k_b \in H^2_{b,c}(PSU(n,1),\mathbb{R})$ the bounded Kähler class. Then the following are equivalent:*

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(1) *$\pi^*(k_b) \in H^2_b(\Gamma, \mathbb{R})$ vanishes;*
(2) *either $\pi(\Gamma)$ fixes a point in the boundary $\mathbb{H}^n_{\mathbb{C}}(\infty)$ of complex hyperbolic space or it leaves a totally real subspace invariant (see § 2).*

As an immediate consequence we obtain:

Corollary 1.2. *Let $\pi : \Gamma \to PSU(n,1)$ be a homomorphism with Zariski dense image. Then $\pi^*(k_b) \in H^2_b(\Gamma, \mathbb{R})$ does not vanish.*

Recall that the kernel of the comparison map $H^2_b(\Gamma) \longrightarrow H^2(\Gamma)$ is described by the quotient

$$QH(\Gamma)/(\ell^\infty(\Gamma) \oplus \mathrm{Hom}(\Gamma, \mathbb{R}))$$

of the vector space of quasihomomorphisms of Γ, that is functions $f : \Gamma \to \mathbb{R}$ such that $\sup_{a,b\in\Gamma} |f(ab) - f(a) - f(b)| < \infty$, modulo the equivalence relation $f \sim g$ if $f - g$ differs from a homomorphism by a bounded function. Applying Corollary 1.2 to $\Gamma = \mathbb{F}_2$, the free group on two generators, and taking into account that $H^2(\mathbb{F}_2) = 0$, we deduce that any homomorphism $\pi : \mathbb{F}_2 \to PSU(n,1)$ with Zariski dense image gives rise in a geometric way to a quasihomomorphism $f_\pi : \mathbb{F}_2 \to \mathbb{R}$, which is not at bounded distance from a homomorphism.

We turn now to a geometric counterpart of Theorem 1.1. To this end, let M be a quotient of $\mathbb{H}^n_{\mathbb{C}}$, and ω_M the induced Kähler form. Given any C^1-simplex $\sigma : \Delta^2 \to M$, let σ^* be a C^1-simplex with geodesic sides, homotopic to σ via a homotopy fixing the vertices. Then

$$k_M(\sigma) = \int_{\sigma^*} \omega_M$$

defines a bounded singular cohomology class $k_M \in H^2_{s,b}(M)$. We shall see that for compact arithmetic quotients M, the presence of a compact submanifold $V \subset M$ such that the restriction $k_M|_V \in H^2_{s,b}(V)$ vanishes, forces the existence of a totally real, compact submanifold $R \subset M$, that is a submanifold which is the compact quotient in M of a totally real subspace of $\mathbb{H}^n_{\mathbb{C}}$, and, moreover, that V can be homotoped into R. It hence follows that if V is not homotopic to a point or a circle, then $\dim R \geq 2$. More generally:

Corollary 1.3. *Let $M = \Lambda\backslash\mathbb{H}^n_{\mathbb{C}}$ be a compact arithmetic manifold, let V be a compact manifold and $f : V \to M$ a continuous map. Then $f^*(k_M) \in H^2_{s,b}(V)$ vanishes if and only if there exists a compact, totally real immersed submanifold $R \subset M$ such that f is homotopic to a map with image in R.*

2. Complex hyperbolic space

We recall here the main points of complex hyperbolic geometry that we shall need and refer to [1, Ch. II.10] and [7] for details.

Let

$$\langle z, w\rangle = \sum_{k=1}^{n} z_k \overline{w}_k - z_{n+1}\overline{w}_{n+1} \ .$$

be the hermitian form of signature $(n, 1)$ on $\mathbb{C}^{n+1}$; recall that the complex hyperbolic n-space $\mathbb{H}^n_{\mathbb{C}}$ is the set of points $[x] \in \mathbb{P}^n(\mathbb{C})$ in complex projective n-space with $\langle x, x\rangle < 0$, equipped with the distance

$$\cosh^2 d([x], [y]) = \frac{\langle x, y\rangle\langle y, x\rangle}{\langle x, x\rangle\langle y, y\rangle} \, .$$

This distance, which comes from a Kähler metric, turns $\mathbb{H}^n_{\mathbb{C}}$ into a CAT(-1) space with sectional curvature $-4 \leq k \leq -1$. The same construction over the field of the real, rather than the complex, numbers gives rise to the real hyperbolic n-space $\mathbb{H}^n_{\mathbb{R}}$, whose sectional curvature is constant and equal to -1.

Particularly important to us will be subsets of $\mathbb{H}^n_{\mathbb{C}}$ isometric to some $\mathbb{H}^k_{\mathbb{R}}$. Recall that a real vector subspace $V \subset \mathbb{C}^{n+1}$ is *totally real* if $\langle z, w\rangle \in \mathbb{R}$ for all $z, w \in V$, or equivalently, if

$$S(z, w) = 0 \qquad \text{for all } z, w \in V,$$

where S is the symplectic form $S(z, w) = \operatorname{Im}\langle z, w\rangle$. A *totally real subspace of* $\mathbb{H}^n_{\mathbb{C}}$ of dimension k is then the image in $\mathbb{H}^n_{\mathbb{C}}$ of a totally real subspace of $\mathbb{C}^{n+1}$ of real dimension $k + 1$, provided the latter contains a negative vector. The totally real subspaces of dimension k in $\mathbb{H}^n_{\mathbb{C}}$ are precisely those subsets of $\mathbb{H}^n_{\mathbb{C}}$ which are isometric to a real hyperbolic space $\mathbb{H}^k_{\mathbb{R}}$.

Obviously any 1-dimensional real subspace of $\mathbb{C}^{n+1}$ is totally real. Also, given any two vectors $v_1, v_2 \in \mathbb{C}^{n+1}$ it is easy to see that the subspace $\mathbb{R}v_1 \oplus \mathbb{R}\langle v_2, v_1\rangle v_2$ is totally real. However, given three vectors, it is not always the case that some complex multiple spans a totally real subspace. To detect whether this is the case the hermitian triple product is a useful tool. Recall that if $z_1, z_2, z_3 \in \mathbb{C}^{n+1}$, their *hermitian triple scalar product* is defined as

$$\langle z_1, z_2, z_3\rangle = \langle z_1, z_2\rangle\langle z_2, z_3\rangle\langle z_3, z_1\rangle \, .$$

Observe that, by definition, if $\alpha, \beta, \gamma \in \mathbb{C}$, then

$$\langle \alpha z_1, \beta z_2, \gamma z_3\rangle = |\alpha|^2|\beta|^2|\gamma|^2\langle z_1, z_2, z_3\rangle$$

so that, in particular, $\langle z_1, z_2, z_3\rangle \in \mathbb{R}$ if and only if $\langle \alpha z_1, \beta z_2, \gamma z_3\rangle \in \mathbb{R}$.

For any set F, we use the notation $\mathcal{C}_n(F)$ to denote the set of n-tuples of distinct points in F.

Lemma 2.1. *Let $F \subset \mathbb{C}^{n+1}$ be a subset such that $\langle x, y, z\rangle \neq 0$ for all $(x, y, z) \in \mathcal{C}_3(F)$. Then $\langle x, y, z\rangle \in \mathbb{R}$ for any $(x, y, z) \in \mathcal{C}_3(F)$ if and only if there exist a totally real subspace V of $\mathbb{C}^{n+1}$ and a function $\lambda : F \to \mathbb{C}^*$ such that the $\mathbb{R}$-linear span of $\{\lambda_z z : z \in F\}$ is contained in V.*

Proof. ($\Leftarrow$) If $\lambda_x x, \lambda_y y, \lambda_z z \in \mathbb{C}^{n+1}$ are contained into a totally real subspace then all the pairwise hermitian products are real, and hence their triple scalar product is real as well.

($\Rightarrow$) To see the converse, let $\{z_m\}$ be any sequence of points in F and let $F_\ell = \{z_1, \dots, z_\ell\}$ be consisting of the first ℓ points of the sequence. To each F_ℓ, for $\ell \geq 3$, we associate a totally real subspace V_ℓ with the properties that $\dim V_\ell \leq \ell$ and $V_\ell \subseteq V_{\ell+1}$, which we construct by induction as follows.

For $\ell = 3$, if one chooses $\lambda_1 = 1, \lambda_2 = \langle z_1, z_2\rangle$ and $\lambda_3 = \langle z_1, z_3\rangle$, it is easy to check that the condition $\langle z_1, z_2, z_3\rangle \in \mathbb{R}$ implies that all pairwise hermitian products are real, hence insuring that the subspace V_3 spanned by $\{\lambda_1 z_1, \lambda_2 z_2, \lambda_3 z_3\}$ is totally real. Notice that $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}^*$ because $\langle \lambda_2 z_2, \lambda_3 z_3\rangle = \langle z_1, z_2, z_3\rangle$ and, by hypothesis, we know that $\langle z_1, z_2, z_3\rangle \neq 0$.

Let us now assume that if $\langle z_i, z_j, z_k\rangle \in \mathbb{R}$ for all $1 \leq i, j, k \leq \ell$ with $i \neq j \neq k \neq i$, then $\{\lambda_1 z_1, \dots, \lambda_\ell z_\ell\}$ span a totally real subspace V_ℓ, with $\lambda_1 = 1$ and $\lambda_j = \langle z_1, z_j\rangle \in \mathbb{C}^*$, and $V_3 \subseteq \dots \subseteq V_{\ell-1} \subseteq V_\ell$.

Define now $\lambda_{\ell+1} = \langle z_1, z_{\ell+1}\rangle$. As before, $\lambda_{\ell+1} \in \mathbb{C}^*$. Moreover, by definition,

$$\langle \lambda_1 z_1, \lambda_{\ell+1} z_{\ell+1}\rangle = |\langle z_1, z_{\ell+1}\rangle| \in \mathbb{R}$$

and, by inductive hypothesis, if $j > 1$

$$\begin{aligned}\langle \lambda_j z_j, \lambda_{\ell+1} z_{\ell+1}\rangle =& \langle z_1, z_j\rangle \overline{\langle z_1, z_{\ell+1}\rangle} \langle z_j, z_{\ell+1}\rangle \\ =& \langle z_1, z_j, z_{\ell+1}\rangle \in \mathbb{R},\end{aligned}$$

which shows that the real subspace $V_{\ell+1}$ generated by $\lambda_1 z_1, \dots, \lambda_\ell z_\ell$, $\lambda_{\ell+1} z_{\ell+1}$ is totally real and $V_\ell \subseteq V_{\ell+1}$. Denote by $V_{\{z_m\}}$ the totally real subspace spanned by $\{\lambda_m z_m : m \geq 1\}$ is totally real.

Now fix any $\{z_m\}$ such that $\dim V_{\{z_m\}}$ is maximal; we claim that $V := V_{\{z_m\}}$ satisfies the conclusion of Lemma 2.1. In fact, if there were $z \in F$ such that for all $\lambda_z \in \mathbb{C}^*$ $\lambda_z z \notin V$, then, with an argument like above, we could construct a totally real subspace V' of dimension strictly larger that the dimension of V, which would contradict maximality. $\square$

3. The Cartan invariant as a bounded cohomology class

Let $\mathbb{H}^n_{\mathbb{C}}(\infty)$ be the sphere at infinity of the CAT(-1) space $\mathbb{H}^n_{\mathbb{C}}$, which can be identified with the image of the null cone $\mathcal{C}^0 = \{z \in \mathbb{C}^{n+1} \setminus \{0\} : \langle z, z\rangle = 0\}$ under the projection $p : \mathbb{C}^{n+1} \smallsetminus \{0\} \to \mathbb{P}^n(\mathbb{C})$.

Since the diagonal action of $PSU(n,1)$ is not transitive on the set $\mathcal{C}_3(\mathbb{H}^n_{\mathbb{C}}(\infty))$, one can associate to distinct triples of points in $\mathbb{H}^n_{\mathbb{C}}(\infty)$ an invariant which plays the role of the crossratio for quadruples of points in the boundary of real hyperbolic space. This is the "invariant angulaire" or *Cartan invariant*, defined by

$$c_n(\xi_1, \xi_2, \xi_3) = \frac{2}{\pi}\operatorname{Arg}(-\langle z_1, z_2, z_3\rangle), \tag{3.1}$$

where $p(z_i) = \xi_i$, with $z_i \in \mathcal{C}^0$ for $i = 1, 2, 3$.

It follows from the fact that the hermitian form has signature $(n, 1)$, that the hermitian triple product has negative real part. If we take the convention (3.1) that $\operatorname{Arg}(z) \in [-\pi/2, \pi/2]$ for $\operatorname{Re} z \geq 0$, it follows that c_n takes values in $[-1, 1]$.

Moreover, the Cartan invariant has the following important properties (see [7]):

- (i) $c_n(\xi_1, \xi_2, \xi_3) = c_n(\eta_1, \eta_2, \eta_3)$ if and only (ξ_1, ξ_2, ξ_3) and (η_1, η_2, η_3) are in the same $PSU(n,1)$-orbit in $\mathcal{C}_3(\mathbb{H}^n_{\mathbb{C}}(\infty))$;
- (ii) c_n is an alternating function on $\mathcal{C}_3(\mathbb{H}^n_{\mathbb{C}}(\infty))$, that is, for all $\sigma \in S_3$, we have that $c_n(\xi_{\sigma(1)}, \xi_{\sigma(2)}, \xi_{\sigma(3)}) = \operatorname{sign}(\sigma) c_n(\xi_1, \xi_2, \xi_3)$;
- (iii) c_n is continuous on $\mathcal{C}_3(\mathbb{H}^n_{\mathbb{C}}(\infty))$;
- (iv) Extending c_n to the whole of $(\mathbb{H}^n_{\mathbb{C}}(\infty))^3$ by setting $c_n(\xi_1, \xi_2, \xi_3) = 0$ if the triple is not distinct, we have the cocycle relation $c_n(\xi_2, \xi_3, \xi_4) - c_n(\xi_1, \xi_3, \xi_4) + c_n(\xi_1, \xi_2, \xi_4) - c_n(\xi_1, \xi_2, \xi_3) = 0$ for any quadruple $(\xi_1, \xi_2, \xi_3, \xi_4) \in (\mathbb{H}^n_{\mathbb{C}}(\infty))^4$;
- (v) Furthermore, $|c_n(\xi_1, \xi_2, \xi_3)| = 1$ if and only if (ξ_1, ξ_2, ξ_3) lie on a chain, that is on the boundary of a complex geodesic.

Actually we shall never use property (v) in this paper, however we chose to point it out here to illustrate, together with next Corollary, what kind of geometric information can be obtained from the maximality or minimality of the (absolute value of the) Cartan invariant.

Corollary 3.1. *Let $\mathcal{L} \subset \mathbb{H}^n_{\mathbb{C}}(\infty)$ be any subset. Then $c_n(\xi_1, \xi_2, \xi_3) = 0$ for all $(\xi_1, \xi_2, \xi_3) \in \mathcal{C}_3(\mathcal{L})$ if and only if $\mathcal{L}$ is contained in the boundary of a totally real subspace of $\mathbb{H}^n_{\mathbb{C}}$.*

Proof. By definition, there exists a subset $F \subset \mathcal{C}^0 \subset \mathbb{C}^{n+1} \setminus \{0\}$ such that $\mathcal{L} = p(F)$. The Corollary is then a restatement of Lemma 2.1 after observing that, again since the hermitian form has signature $(n, 1)$, we have $\langle z_1, z_2, z_3\rangle \neq 0$ for all $(z_1, z_2, z_3) \in \mathcal{C}_3(F)$. □

The extension of c_n defined in (iv) is a bounded measurable alternating function on $(\mathbb{H}^n_{\mathbb{C}}(\infty))^3$, and we want to describe the other essential property that it enjoys, namely how it defines a bounded cohomology class in $H^2_{b,c}(PSU(n,1),\mathbb{R})$ related to the Kähler form on $\mathbb{H}^n_{\mathbb{C}}$. To this end, we move a step back and start defining continuous bounded cohomology for locally compact groups.

If H is a locally compact group, the continuous bounded cohomology of H is defined as the cohomology of the complex of H-invariants of

$$0 \longrightarrow L^\infty(H) \xrightarrow{d} L^\infty(H^2) \xrightarrow{d} \dots$$

where the coboundary operator is given by

$$df(h_0,\dots,h_k) = \sum_{i=0}^{k}(-1)^i f(h_0,\dots,\hat{h}_i,\dots,h_k)$$

However, using the right H-module structure, and appropriate smoothing homotopy operators, the bounded continuous cohomology of H can also be computed as the cohomology of the complex of H-invariants of

$$0 \longrightarrow C_b(H) \xrightarrow{d} C_b(H^2) \xrightarrow{d} \dots$$

where $C_b(H^j)$ is the space of continuous bounded function on the cartesian product of j copies of H ([9]). This complex being a subcomplex of the complex of all continuous functions on H, we obtain a comparison map

$$H^*_{b,c}(H) \xrightarrow{\kappa} H^*_c(H) \tag{3.2}$$

from the bounded continuous cohomology to the continuous cohomology of H.

Recall that continuous cohomology can be computed explicitly in many concrete cases. A prominent example is when H is a connected semisimple Lie group. Denoting by X the associated symmetric space and $\Omega^*(X)$ the complex of differential forms on X, we have the Van Est isomorphism ([11])

$$H^*_c(H) \xrightarrow{\sim} \Omega^*(X)^H \tag{3.3}$$

which identifies the continuous cohomology of H with the complex of invariant differential forms on X.

In the specific situation of interest where $H = PSU(n,1)$, then $X = \mathbb{H}^n_{\mathbb{C}}$ is the complex hyperbolic n-space, and we have therefore that

$$H^2_c(PSU(n,1)) \simeq \Omega^2(\mathbb{H}^n_{\mathbb{C}})^{PSU(n,1)} = \mathbb{R}\cdot\omega \tag{3.4}$$

is one-dimensional and generated by the Kähler form ω.

It is a fact known since Dupont [6], that ω can be represented by a bounded continuous cocycle on $PSU(n,1)$, that is, the comparison map (3.2) is surjective in degree 2 and hence an isomorphism (see [3, Lemma 6.1])

$$H^2_{b,c}(PSU(n,1)) \xrightarrow{\sim} H^2_c(PSU(n,1))\,. \tag{3.5}$$

As the resolutions given above are often intractable, like in the case of continuous cohomology it is desirable to have resolutions at hand which are of "small size" and whose complex of invariants computes continuous bounded cohomology. This will allows us to give an explicit realization of the bounded cohomology group and of the isomorphism in (3.5). This requires the introduction of an appropriate category of H-modules, and appropriate notions of relative injective objects and resolutions. All this is laid out in [4] (or more generally in [9]), from which we borrow some of the essential facts here and some in the next section.

If H is a locally compact group, a *continuous Banach H-module* is a Banach space on which H acts continuously by isometric automorphisms, and *H-morphisms* are linear continuous H-equivariant maps between continuous Banach H-modules. Then a continuous Banach H-module E is *relatively injective* if for every injective H-morphism $\imath : A \hookrightarrow B$ of continuous Banach H-modules A, B which admits a left inverse of norm bounded by 1 and every H-morphism $\alpha : A \to E$, there is a H-morphism $\beta : B \to E$ which extends α and such that $\|\beta\| \leq \|\alpha\|$

$$\begin{array}{ccc} A & \overset{\imath}{\hookrightarrow} & B \\ & \searrow^{\alpha} & \downarrow^{\beta} \\ & & E \end{array}$$

The following theorem is a characterization of amenable actions which provides an essential tool to compute continuous bounded cohomology.

Theorem 3.2. *Let H be a locally compact group, and (B,ν) a regular H-space. The H-action on B is amenable if and only if $L^\infty(B)$ is a relatively injective module. Moreover, the cohomology of the complex*

$$0 \longrightarrow L^\infty(B)^H \longrightarrow L^\infty(B^2)^H \longrightarrow \dots$$

is canonically isomorphic to $H^*_{b,c}(H)$.

We do not recall here the definition of amenable action for which we refer to [12, Ch.4], and we limit ourselves to recall that the actions of $PSU(n,1)$ on $\mathbb{H}^n_{\mathbb{C}}(\infty)$ and of the free group $\mathbb{F}_n$ in r generators on the

Poisson boundary of the regular infinite tree $\mathcal{T}_r$ of valence r (see the proof of Proposition 5.1) are both amenable.

As a particularly important consequence of Theorem 3.2, we record the following fact. Recall that the H-action on B is ergodic if and only if every H-invariant measurable function is essentially constant. Then it is not difficult to see that if H and B are as in Theorem 3.2 and if in addition H acts doubly ergodically on B (that is ergodically on $B \times B$ with respect to the product measure), then

$$H^2_{b,c}(H) = \mathcal{Z}L^\infty_{alt}(B^3)^H \tag{3.6}$$

that is, $H^2_{b,c}(H)$ is identified with the Banach space of bounded, alternating, H-invariant cocycles on B^3 [4].

Taking $H = PSU(n,1)$ and $B = \mathbb{H}^n_{\mathbb{C}}(\infty)$ the boundary of hyperbolic n-space, we see that the Cartan cocycle, defined in (3.1) on $\mathcal{C}_3(B)$ and then extended to B^3, defines via (3.6) a bounded cohomology class; since, using the isomorphisms in (3.5) and (3.4) we deduce that $H^2_{b,c}(PSU(n,1))$ is one-dimensional, we have that

$$H^2_{b,c}(PSU(n,1)) = \mathbb{R} \cdot c_n$$

with the isomorphism and (3.5), sending c_n to $\frac{\omega}{2\pi}$, [4].

4. Functoriality

Given two locally compact groups H_1 and H_2, and a continuous homomorphism $\pi : H_1 \to H_2$, it is plain on the appropriate resolutions by bounded continuous (respectively, continuous) cochains that π induces morphisms

$$\pi^* : H^*_{b,c}(H_2) \to H^*_{b,c}(H_1)$$

and

$$\pi^* : H_c(H_2)^* \to H^*_c(H_1)$$

Though we shall only need the induced map in bounded cohomology, we may record that the diagram

$$\begin{array}{ccc} H^*_{b,c}(H_2) & \xrightarrow{\pi^*} & H^*_{b,c}(H_1) \\ \kappa\downarrow & & \downarrow\kappa \\ H^*_c(H_2) & \xrightarrow{\pi^*} & H^*_c(H_1) \end{array}$$

commutes.

While on the one hand it is much more convenient to compute these cohomology groups using resolutions by L^∞ functions on amenable H_i-regular spaces, on the other, doing so, it is much less clear how the map π^*_b looks in these resolutions. However, the following proposition makes an essential point, obtained using fully the homological algebra approach to bounded cohomology. We refer to [2] for a more detailed discussion.

Proposition 4.1. *[[2]] Let $\pi : H_1 \to H_2$ be a continuous homomorphism of locally compact groups. Let (Y_1, ν_1) be a regular amenable H_1-space, Y_2 a compact metric separable H_2-space on which H_2 acts by homeomorphisms and let $\varphi : Y_1 \to \mathcal{M}(Y_2)$ be a π-equivariant measurable map.*

Then to any strict bounded Borel cocycle $c : Y_2^{n+1} \to \mathbb{R}$ one can canonically associate a bounded class $[c] \in H^n_{b,c}(H_2)$ and $\pi_b^*([c])$ can be represented by the cocycle in $L^\infty(Y_1^{n+1})$ defined by

$$(y_1, \dots, y_{n+1}) \to \varphi(y_1) \otimes \cdots \otimes \varphi(y_{n+1})(c)$$

Once we know that any finitely generated group Γ admits amenable, doubly ergodic standard Γ-spaces, we shall be able to put to use Proposition 4.1 and Theorem 3.2. This will be done in the following section.

5. Boundaries

The following Proposition is a particular case of a theorem in [4, §1]. Since in our setting the proof is very transparent, we recall it here.

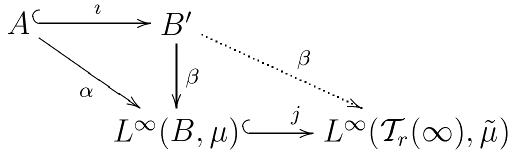
Proposition 5.1 ([4, Theorem 0.2])**.** *Let Γ be a finitely generated group. Then there exists a Γ-space B with a quasi-invariant measure μ, such that the Γ-action on (B, μ) is both amenable and doubly ergodic.*

Proof. Let S be a finite generating set for Γ of cardinality r, and let $\mathbb{F}_r$ be the free group in r generators, so that we have a surjective homomorphism $\rho : \mathbb{F}_r \to \Gamma$ with kernel N. Let $\mathcal{T}_r$ be the regular infinite tree of valence r with automorphism group $\mathrm{Aut}(\mathcal{T}_r)$, so that $\mathbb{F}_r \subset \mathrm{Aut}(\mathcal{T}_r)$.

Let $\mathcal{T}_r(\infty)$ be the natural Poisson boundary of $\mathcal{T}_r$ (consisting of reduced words of infinite length) and let $\tilde{\mu}$ be the natural quasi-invariant probability measure defined by $\tilde{\mu}(E(x)) = (2r(2r-1)^{n-1})^{-1}$, where $|x| = n$ and $E(x) \subset \mathcal{T}_r(\infty)$ consists of the infinite reduced words starting with x (so that $\{E(x) : x \in \mathcal{T}_r\}$ is a basis for the topology of $\mathcal{T}_r(\infty)$). The space (B, μ) will then be realized as the point realization of the algebra of N-invariant L^∞ functions on $(\mathcal{T}_r(\infty), \tilde{\mu})$ (see [10, Theorem 3.3] for details). Namely, let $\mathcal{B}^N$ and $\mathcal{B}$ be the measure algebras generated respectively by $L^\infty(\mathcal{T}_r(\infty), \tilde{\mu}))^N$ and $L^\infty(\mathcal{T}_r(\infty), \tilde{\mu}))$. Since $\mathcal{B}^N \subset \mathcal{B}$, corresponding to $\mathcal{B}^N$ there exists a factor (B, μ) of $(\mathcal{T}_r(\infty), \tilde{\mu})$, namely a measure space (B, μ) with a probability measure μ and a measurable map $p : (\mathcal{T}_r(\infty), \tilde{\mu}) \to (B, \mu)$ such that $\mu = p_*(\tilde{\mu})$ (where, if $A \subset B$ is a measurable set, $p_*(\tilde{\mu})$ is defined by $p_*(\tilde{\mu})(A) = \tilde{\mu}(p^{-1}(A))$. The space (B, μ) carries a Γ-action (since it carries an action of $\mathbb{F}_r$ which factors through the action of N), with respect to which the projection map is $\mathbb{F}_r$-equivariant.

Now that the space (B, μ) has been constructed, we need to show the properties of its Γ-action. Observe first of all that the action of $\mathrm{Aut}(\mathcal{T}_r)$ on $\mathcal{T}_r(\infty)$ is doubly ergodic, and so is the action of $\mathbb{F}_r$ on $\mathcal{T}_r(\infty)$ ([4, Proposition 1.5 and Proposition 1.6]). Since p is $\mathbb{F}_r$-equivariant, it follows that the action of Γ on (B, μ) is doubly ergodic as well.

To prove that the Γ-action on (B, μ) is amenable, we shall use the characterization of amenable actions given in Theorem 3.2, that is we shall prove that the Banach Γ-module $L^\infty(B, \mu)$ is relatively injective. Let A, B two continuous Banach Γ-modules with an injective Γ-morphism $\imath : A \hookrightarrow B$, and let $\alpha : A \to L^\infty(B, \mu)$ a Γ-morphism. If j is the inclusion $j : L^\infty(B, \mu) = L^\infty(\mathcal{T}_r(\infty), \tilde{\mu})^N \hookrightarrow L^\infty(\mathcal{T}_r(\infty), \tilde{\mu})$ and if we think of A and B as continuous Banach $\mathbb{F}_r$-modules (with a trivial N-action), then we have an $\mathbb{F}_r$-morphism $\alpha' = j \circ \alpha : A \to L^\infty(\mathcal{T}_r(\infty), \tilde{\mu})$.



Since the $\mathbb{F}_r$-action is amenable on $(\mathcal{T}_r(\infty), \tilde{\mu})$, by Theorem 3.2 there exists an $\mathbb{F}_r$-morphism $\beta' : B \to L^\infty(\mathcal{T}_r(\infty), \tilde{\mu})$ which extends α' and such that $\|\beta'\| \leq \|\alpha'\|$. Since the N-action on A and B was trivial, the image of β' lies in $L^\infty(\mathcal{T}_r(\infty), \tilde{\mu})^N$, hence defining the desired extension $\beta : B \to L^\infty(\mathcal{T}_r(\infty), \tilde{\mu})^N = L^\infty(B, \mu)$ with $\|\beta\| \leq \|\alpha\|$. □

6. Proofs

Proof of Theorem 1.1. ($\Leftarrow$) It is immediate since the restriction of the Kähler form to a totally real subspace vanishes identically.

($\Rightarrow$) We may assume that $\pi(\Gamma)$ is not elementary. In fact, if this is not the case the conclusion is immediate since either $\pi(\Gamma)$ fixes a point in $\mathbb{H}^n_{\mathbb{C}}(\infty)$ or a point in $\mathbb{H}^n_{\mathbb{C}}$ or a geodesic.

Let (B, μ) be the amenable doubly ergodic Γ-space in Proposition 5.1 and let $\mathcal{L}_{\pi(\Gamma)} = \overline{\pi(\Gamma) \cdot x} \cap \mathbb{H}^n_{\mathbb{C}}(\infty)$ be the limit set of $\pi(\Gamma)$ (which is independent of $x \in \mathbb{H}^n_{\mathbb{C}}$). Then, since $\pi(\Gamma)$ is not elementary, there exists a Γ-equivariant measurable map $\varphi : B \to \mathcal{L}_{\pi(\Gamma)}$, [5, Corollary 3.2]. By Proposition 4.1 with $H_1 = \Gamma$, $H_2 = PSU(n, 1)$, $(Y_1, \mu) = (B, \nu)$, $Y_2 = \mathbb{H}^n_{\mathbb{C}}(\infty)$, and where we think of $\mathcal{L}_{\pi(\Gamma)}$ as embedded in

$\mathcal{M}(\mathbb{H}^n_{\mathbb{C}}(\infty))$ as Dirac masses, the cocycle

$$\begin{aligned} c^\pi : B \times B \times B &\to [-1,1] \\ (b_1, b_2, b_3) &\mapsto c_n(\varphi(b_1), \varphi(b_2), \varphi(b_3)) \end{aligned}$$

is a representative of $\pi^*(k_b) \in H^2_b(\Gamma)$. Since Γ acts ergodically on $B \times B$, c^π is an alternating 2-cocycle and $\pi^*(k_b) = 0$, it follows from (3.6) and from the properties of the Cartan invariant, that $c^\pi = 0$ almost everywhere, that is that $c_n(\varphi(b_1), \varphi(b_2), \varphi(b_3)) = 0$ for almost every $(b_1, b_2, b_3) \in B \times B \times B$ with respect to the product measure. The rest of the argument will consist of showing that, in fact,

Claim 6.1. c_n is identically zero on $\mathcal{C}_3(\mathcal{L}_{\pi(\Gamma)})$.

Then, Corollary 3.1, with $\mathcal{L} = \mathcal{L}_{\pi(\Gamma)}$, shows that $\mathcal{L}_{\pi(\Gamma)}$ is contained in the boundary of a totally real subspace of $\mathbb{H}^n_{\mathbb{C}}$. The intersection of all totally real subspaces of $\mathbb{H}^n_{\mathbb{C}}$ containing $\mathcal{L}_{\pi(\Gamma)}$ is then a totally real subspace left invariant by Γ.

To prove the claim, let $\lambda = \varphi_*\mu$ be the measure on $\mathcal{L}_{\pi(\Gamma)}$. Since φ is Γ-equivariant and μ is quasi-invariant, $\operatorname{supp}\lambda$ is a closed $\pi(\Gamma)$-invariant subset of $\mathcal{L}_{\pi(\Gamma)}$. Since $\pi(\Gamma)$ is not elementary, it acts minimally on $\mathcal{L}_{\pi(\Gamma)}$, which implies that $\operatorname{supp}\lambda = \mathcal{L}_{\pi(\Gamma)}$. Now let $\nu = \lambda \times \lambda \times \lambda$ be the product measure on $(\mathcal{L}_{\pi(\Gamma)})^3$. Then we have so far that $c_n(x_1, x_2, x_3) = 0$ for all triples of points $(x_1, x_2, x_3) \in \operatorname{supp}(\nu) = (\operatorname{supp}\lambda)^3 = (\mathcal{L}_{\pi(\Gamma)})^3$. Now let $(a, b, c) \in \mathcal{C}_3(\mathcal{L}_{\pi(\Gamma)})$, and let U_a, U_b, U_c be small neighborhoods in $\mathcal{L}_{\pi(\Gamma)}$ of a, b, c respectively which are pairwise disjoint. Since $\operatorname{supp}\lambda = \mathcal{L}_{\pi(\Gamma)}$, the measure λ of an open non-void set is positive. Hence $\operatorname{supp}(\nu) \cap (U_a \times U_b \times U_c) \neq \emptyset$ so that for $(a', b', c') \in \operatorname{supp}(\nu) \cap (U_a \times U_b \times U_c)$ we have $c_n(a', b', c') = 0$. Then, by continuity of c_n, $c_n(a, b, c) = 0$ as well, hence completing the proof. □

Proof of Corollary 1.3. ($\Leftarrow$)

($\Rightarrow$) Let $\Lambda = \pi_1(V)$, $\Gamma = \pi_1(M)$, $\pi : \Lambda \to \Gamma$ the homomorphism induced by f, and $f^* : H^*_{s,b}(M) \to H^*_{s,b}(V)$ the map in singular bounded cohomology induced by f. According to Gromov [8], there is a natural isomorphism $H^*_b(\pi_1(X)) \simeq H^*_{s,b}(X)$, for any countable CW-complex X. In particular, in our case we have that this isomorphism sends the class k_M to the class k_b, so that the commutativity of the square,

$$\begin{array}{ccc} H^*_{s,b}(M) & \xrightarrow{f^*_b} & H^*_{s,b}(V) \\ \wr\downarrow & & \downarrow\wr \\ H^*_b(\Gamma) & \xrightarrow{\pi^*_b} & H^*_b(\Lambda) \end{array}$$

together with the hypothesis $f^*(k_M) = 0$, implies that $\pi^*(k_b) \in H^2_b(\Lambda)$ vanishes.

The main point is now to show that there is a totally real subspace $T \subset \mathbb{H}^n_{\mathbb{C}}$ such that:

(1) $\pi(\Lambda) \subset Stab_\Gamma(T)$;
(2) $Stab_\Gamma(T)$ acts cocompactly on T.

Indeed, setting $R = pr(T)$, where $pr : \mathbb{H}^n_{\mathbb{C}} \to M$ is the canonical projection, we have that $R = T/Stab_\Gamma(T)$ is a compact immersed submanifold. Let $p_T : \mathbb{H}^n_{\mathbb{C}} \to T$ be the orthogonal projection, and for every pair $x, y \in \mathbb{H}^n_{\mathbb{C}}$ of points, let $g_{x,y} : [0,1] \to \mathbb{H}^n_{\mathbb{C}}$ be the constant speed geodesic connecting x to y. Define

$$\tilde{f}_t(x) = g_{\tilde{f}(x), p_T(\tilde{f}(x))}(t).$$

Clearly $\tilde{f}_t$ is Λ-equivariant and thus descends to a homotopy $t \to f_t$ between $f_0 = f$ and f_1 which has image in R.

Thus we turn to the construction of T. Because of our opening remarks, we know that $\pi^*_b(k_b) = 0$ and we are hence in the position of applying Theorem 1.1.

There are two cases. First, assume that $\pi(\Lambda)$ is elementary. Since Γ is torsion free and cocompact, this implies that either $\pi(\Lambda) = \{e\}$, in which case we take $T = \{pt\}$, or $\pi(\Lambda)$ is infinite cyclic, in which case we take as T the axis of a generator of $\pi(\Lambda)$. In both cases, T satisfies the properties (1) and (2) and we are done.

Assume now that $\pi(\Lambda) := \Delta$ is non-elementary. Let $\mathcal{L}_\Delta$ be its limit set and T the minimal totally real subspace of $\mathbb{H}^n_{\mathbb{C}}$ such that $T(\infty)$ contains $\mathcal{L}_\Delta$. From Theorem 1.1 we know that T is Δ-invariant and what remains to show is that $Stab_\Gamma(T)$ acts cocompactly on T. Here we bring in the hypothesis that Γ is arithmetic. Namely, let $\mathbb{G}$ be a connected, semisimple adjoint group defined over $\mathbb{Q}$ such that $\mathbb{G}(\mathbb{R}) = PSU(n,1) \times K$, where K is compact and $\Gamma' = pr_1(\mathbb{G}(\mathbb{Z}))$ is commensurable with Γ. Define $\mathbb{H}$ to be the connected component of the Zariski closure of

$$\{\gamma \in \mathbb{G}(\mathbb{Z}) : pr_1(\gamma) \text{ leaves } T \text{ invariant}\}.$$

Then $\mathbb{H}$ is a $\mathbb{Q}$-subgroup of $\mathbb{G}$; let $H = pr_1(\mathbb{H}(\mathbb{R}))$, which is closed and with a finite number of connected components. We have $H \supset \Delta \cap \Gamma'$, where the latter is of finite index in Δ and non-elementary; hence

$$\mathcal{L}_H \supset \mathcal{L}_{\Delta\cap\Gamma'} = \mathcal{L}_\Delta.$$

Since H is non-elementary as well, $\mathcal{L}_H = \mathcal{L}_{H^\circ}$, so that finally

$$T(\infty) \supset \mathcal{L}_{H^\circ} \supset \mathcal{L}_\Delta.$$

Observe that if S is the image of H° in $Iso(T)$ under the restriction map, then

$$\mathcal{L}_{H^\circ} = \mathcal{L}_S.$$

We claim now that S is reductive with compact center. Indeed, let $\mathcal{R}$ be the (connected) radical of S. Then the fixed point set of $\mathcal{R}$ in $\mathcal{L}_S$ is non-void, S-invariant, and hence equal to $\mathcal{L}_S$. Since $|\mathcal{L}_S| \geq 3$, $\mathcal{R}$ is compact and hence central. Now we use a theorem of Mostow which guarantees the existence of a point $t \in T$ such that the orbit $S \cdot t \subset T$ is totally geodesic, and hence coincides with the symmetric space associated to S; but then $T' = S \cdot t$ is totally real, with $T' \supset \mathcal{L}_S$ which by minimality of T implies that $T' = T$ and hence S acts transitively on T. Since $\mathbb{H}(\mathbb{R})^\circ$ is a compact extension of S, we conclude firstly that $\mathbb{H}$ has no $\mathbb{Q}$-rational characters, and hence $\mathbb{H}(\mathbb{Z})$ is a (cocompact) lattice in $\mathbb{H}(\mathbb{R})$; secondly, that, T being the symmetric space associated to $\mathbb{H}(\mathbb{R})$, $pr_1(\mathbb{H}(\mathbb{Z}))$ acts cocompactly on T. Thus $Stab_{\Gamma'}(T)$, and hence $Stab_\Gamma(T)$, act cocompactly on T. □

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