Linear representations and arithmeticity of lattices in products of trees

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0 Prolegomenon

In this paper we continue our study of lattices in the automorphisms groups of products of trees initiated in [BM97], [BM00a], [BM00b], [Moz98] (see also [Gla03], [BG02], [Rat04]). We concentrate here on the interplay between the linear representation theory and the structure of these lattices. Before turning to the main results of this paper it may be worthwhile to put certain concepts and results from [BM00a], [BM00b] in perspective, and explain the motivations for our approach.

A lattice Γ in a locally compact group G is a discrete subgroup such that the quotient space G/Γ carries a finite G-invariant measure. If in addition G/Γ is compact the lattice is called cocompact or uniform. Consider the following special setting: let $\mathbb{Q}_p$ be the field of p-adic numbers and let

$$G = \mathrm{PSL}(2, \mathbb{Q}_p) \times \mathrm{PSL}(2, \mathbb{Q}_q).$$

Lattices in products fall into two natural classes: reducible and irreducible. A lattice $\Gamma < G$ is called reducible if it has a subgroup of finite index which is a direct product $\Gamma_p \times \Gamma_q$ where $\Gamma_p < \mathrm{PSL}(2, \mathbb{Q}_p)$ and $\Gamma_q < \mathrm{PSL}(2, \mathbb{Q}_q)$ are lattices. A lattice $\Gamma < G$ is called irreducible if it is not reducible. Recall that here, as well as for other semisimple Lie groups over local fields the projections of an irreducible lattice on any factor obtained by modding out any noncompact normal subgroup is dense (cf. [Mar91] II Thm. (6.7)). It should be noted that this density result makes essential use of the algebraic geometric structure of the ambient group and in particular of the Borel density

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theorem. Irreducible lattices in G enjoy remarkable algebraic and geometric properties as the following results due to G.A. Margulis show:

- *Arithmeticity:* Any irreducible lattice in G is arithmetic. [Mar91, IX Thm (A)].

- *Superrigidity:* Unbounded linear representations of these lattices arise from rational representations of the ambient group G. [Mar91, VII Thm (5.6)].

- *Normal subgroup Theorem:* Any non trivial normal subgroup of an irreducible lattice in G is either finite and central or of finite index. [Mar91, IV Thm (4.9)].

An important geometric object associated with $\mathrm{PGL}(2,\mathbb{Q}_p)$ is its Bruhat–Tits building $\mathcal{T}_{p+1}$ which is a regular tree of degree $p+1$ on which $\mathrm{PGL}(2,\mathbb{Q}_p)$ acts by automorphisms. The group $\mathrm{Aut}\,\mathcal{T}_{p+1}$, with the topology of pointwise convergence on the set of vertices, is a locally compact group containing $\mathrm{PGL}(2,\mathbb{Q}_p)$ as a closed (cocompact) subgroup. This embedding implies that cocompact lattices in $\mathrm{PGL}(2,\mathbb{Q}_p)$, and more generally in the automorphism group of a regular tree, are virtually free groups (see for example [Ser03]). The similarities and differences between the theory of lattices in $\mathrm{Aut}\,\mathcal{T}_{p+1}$ and the theory of lattices in Lie groups, and in $\mathrm{PGL}(2,\mathbb{Q}_p)$ in particular, are part of an extensive study and we refer to [BL01] for many results and references. Returning to the higher rank situation, and in the same geometric vein, observe that a torsion free cocompact lattice $\Gamma < \mathrm{PGL}(2,\mathbb{Q}_p) \times \mathrm{PGL}(2,\mathbb{Q}_q)$ acts freely on the 2-dimensional square complex $\mathcal{T}_{p+1} \times \mathcal{T}_{q+1}$. The quotient $\Gamma\backslash(\mathcal{T}_{p+1} \times \mathcal{T}_{q+1})$ is a finite square complex for which the link of each vertex is a complete bipartite graph. It is an elementary and useful observation that the universal covering space of a 2-dimensional square complex is a product of trees exactly when the link at every vertex is a complete bipartite graph (cf. [Wis95, Thm 1.5]).

Example 0.1.

An interesting example of an irreducible lattice in $\mathrm{PGL}(2,\mathbb{Q}_p)\times\mathrm{PGL}(2,\mathbb{Q}_q)$ may be explicitly described as follows. Let p, q be two fixed distinct primes both congruent to 1 modulo 4. Let $\mathbb{H}_{\mathbb{Z}}$ denote the ring of integer quaternions. Fix $\epsilon_p \in \mathbb{Q}_p$ and $\epsilon_q \in \mathbb{Q}_q$, square roots of -1. Define a map

$\varphi : \mathbb{H}_{\mathbb{Z}} \setminus \{0\} \to \mathrm{PGL}(2,\mathbb{Q}_p) \times \mathrm{PGL}(2,\mathbb{Q}_q)$ where for $x = x_0 + x_1 i + x_2 j + x_3 k$

$$\varphi(x) = \left(\begin{pmatrix} x_0 + x_1\epsilon_p & x_2 + x_3\epsilon_p \\ -x_2 + x_3\epsilon_p & x_0 - x_1\epsilon_p \end{pmatrix} , \begin{pmatrix} x_0 + x_1\epsilon_q & x_2 + x_3\epsilon_q \\ -x_2 + x_3\epsilon_q & x_0 - x_1\epsilon_q \end{pmatrix} \right)$$

Let $\mathbb{H}_{\mathbb{Z}}(p,q) = \{x \in \mathbb{H}_{\mathbb{Z}} \ : \ |x|^2 = p^k q^l \text{ for some } k,l \in \mathbb{Z}^+\}$. Its image $\varphi(\mathbb{H}_{\mathbb{Z}}(p,q)) < \mathrm{PGL}(2,\mathbb{Q}_p) \times \mathrm{PGL}(2,\mathbb{Q}_q)$ is an irreducible arithmetic lattice. We shall be interested in a congruence sublattice of it, i.e., the image of those quaternions in $\mathbb{H}_{\mathbb{Z}}(p,q)$ which are congruent to 1 modulo 2 (i.e., for which x_0 is odd). Let us denote this lattice by $\Gamma < \mathrm{PGL}(2,\mathbb{Q}_p) \times \mathrm{PGL}(2,\mathbb{Q}_q)$. Using Jacobi's theorem concerning the number of ways of representing a number as sum of 4 squares, one can show that this lattice acts freely transitively on the vertices of the building associated with $\mathrm{PGL}(2,\mathbb{Q}_p) \times \mathrm{PGL}(2,\mathbb{Q}_q)$ namely on the product of the corresponding Bruhat–Tits trees. This allows one to give a nice explicit description of the one vertex square complex obtained as the quotient $\mathcal{X}_{p,q} = \Gamma \backslash (\mathcal{T}_{p+1} \times \mathcal{T}_{q+1})$. Let us define the following two sets

$$L_p = \{x \in \mathbb{H}_{\mathbb{Z}} \ : \ |x|^2 = p, \quad x \equiv 1 (\mathrm{mod}\, 2) \quad x_0 > 0\} \tag{1}$$
$$L_q = \{x \in \mathbb{H}_{\mathbb{Z}} \ : \ |x|^2 = q, \quad x \equiv 1 (\mathrm{mod}\, 2) \quad x_0 > 0\} \tag{2}$$

The 0-skeleton $\mathcal{X}_{p,q}^{(0)}$ consists of a single vertex v_0. The 1-skeleton consists of $\frac{p+1}{2} + \frac{q+1}{2}$ loops based at v_0. We shall partition these loops into two sets E_h and E_v of sizes $\frac{p+1}{2}$ and $\frac{q+1}{2}$ respectively where the elements of E_h will be thought of as horizontal loops and the elements of E_v as vertical loops. We shall label each oriented loop of E_h by an element of L_p where the two orientations of a geometric loop are labeled by conjugate quaternions. Similarly we shall label the oriented loops of E_v by the elements of L_q. Abusing notation we shall refer also to the elements of L_p, L_q as (oriented) loops. To describe the 2-skeleton $\mathcal{X}_{p,q}^{(2)}$, consider all the quadruples $(a, b, \bar{a'}, \bar{b'})$ where $a, a' \in L_p$, $b, b' \in L_q$ and

$$ab = \pm b'a'.$$

These quadruples come in equivalence classes:

$$\{(a, b, \overline{a'}, \overline{b'}), (\overline{a}, b', a', \overline{b}), (a', \overline{b}, \overline{a}, b'), (\overline{a'}, \overline{b'}, a, b)\}$$

For each equivalence class we will have a geometric square in $\mathcal{X}_{p,q}^{(2)}$ attached to the 1-skeleton of $\mathcal{X}_{p,q}$ so that when we orient it and read the labels of the oriented loops along its boundary in one of the 4 possible ways we see an element of the corresponding equivalence class. Observe that (cf. [Dic22]) for each $a \in L_p$ and $b \in L_q$ there are unique $a' \in L_p$ and $b' \in L_q$ so that

$$ab = \pm b'a'.$$

Hence there are $\binom{\frac{p+1}{2}}{} \cdot \binom{\frac{q+1}{2}}{}$ geometric squares and the link of the vertex v_0 is a complete bipartite graph. Thus, indeed, the universal cover $\widetilde{\mathcal{X}}_{p,q}$ is a product of a $(p+1)$-regular tree with a $(q+1)$-regular tree. One can then identify $\widetilde{\mathcal{X}}_{p,q}$ with the Bruhat–Tits building $\mathcal{T}_{p+1} \times \mathcal{T}_{q+1}$ associated with $\mathrm{PGL}(2, \mathbb{Q}_p) \times \mathrm{PGL}(2, \mathbb{Q}_q)$.

We turn now to the study of cocompact lattices in a product $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$, where T_1, T_2 are regular trees. As above we shall call a lattice $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ reducible if it has a subgroup of finite index which is a product of lattices in the factors. In particular any reducible cocompact lattice is virtually a product of two free groups. For a general cocompact lattice Γ, let $G_i < \mathrm{Aut}\, T_i$ denote the closure of the projection $\mathrm{pr}_i(\Gamma)$. Let x_i be a vertex of the tree T_i, denote by $G_i(x_i)$ the compact open subgroup which is the stabilizer of x_i in G_i. Observe that $G_i(x_i)$ is topologically finitely generated. One has

$$\mathrm{pr}_1(\Gamma) \cap G_1(x_1) = \mathrm{pr}_1(\Gamma \cap (G_1(x_1) \times G_2))$$

The left hand side is dense in $G_1(x_1)$, while the right hand side is a cocompact lattice in the locally compact compactly generated group $G_1(x_1) \times G_2$, hence it is finitely generated. However, for a regular tree T, the stabilizer $\mathrm{Aut}\, T(x)$ of a vertex x is not topologically finitely generated. To see this, one constructs an epimorphism $\mathrm{Aut}\, T(x) \to (\mathbb{Z}/2\mathbb{Z})^{\mathbb{N}}$ obtained by mapping each element $g \in \mathrm{Aut}\, T(x)$ to the sequence of signs of the permutations induced by g on the spheres of radii n around x. In particular, cocompact lattices in $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ never have dense projections. It would be of interest to understand what happens in the case of non-uniform lattices.
The above discussion leads naturally to the following:

Basic Question. *Which groups arise as closures of projections of cocompact lattices in* $\mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$*?*

The interest in this question is partly due to the accumulation of results showing that there are strong connections between the topological and algebraic properties of the groups $G_i < \mathrm{Aut}\, T_i$ and the algebraic and geometric properties of Γ. In particular, results of this type played a role in [BM00a], [BM00b] and led to a family of finitely presented, torsion free, simple groups. More recently, such connections have been established in a remarkable degree of generality. Let us mention some recent result: Monod [Mon03] and Monod–Shalom [MS02] have proved superrigidity results for the actions on CAT(0)-spaces of cocompact lattices $\Gamma < G_1 \times G_2$ in a product of locally compact groups. These results in particular imply the following theorems which we will use in the sequel. In Theorems 0.2, 0.3, 0.4, the groups G_1, G_2 are assumed to be locally compact and compactly generated.

Theorem 0.2. [**Mon03, Cor. 4**] *Let $\Gamma < G_1 \times G_2$ be a cocompact lattice with dense projections. Let $\mathbb{H}$ be a connected, k-simple, adjoint k-group where k is a local field and let $\pi : \Gamma \to \mathbb{H}(k)$ be a homomorphism with unbounded Zariski dense image. Then π extends continuously to $G_1 \times G_2$ factoring via one of the factors G_i.*

In certain situations this result will imply that the linear representation theory of Γ over any field is controlled by the continuous representation theory of $G_1 \times G_2$ over local fields.
The next result shows that the various decompositions of Γ as non trivial amalgams are completely described by the decompositions of G_i, $i = 1, 2$ as non trivial amalgams.

Theorem 0.3. [**MS02, Thm. 1.5**] *Let $\Gamma < G = G_1 \times G_2$ be a lattice with dense projections. Let Γ act non elementarily on a countable tree T. Then there exists an invariant subtree on which the Γ action extends continuously to a G-action factoring via one of the G_i's.*

Finally the following result of Bader and Shalom (generalizing Margulis' Normal Subgroup Theorem for the case of products) shows that normal subgroups of Γ are in a sense controlled by the closed normal subgroups of $G_1 \times G_2$.

Theorem 0.4. [**BS03, Thm. 1.1**] *Let $\Gamma < G = G_1 \times G_2$ be a cocompact lattice with dense projections. Assume that $\mathrm{Hom}_c(G, \mathbb{R}) = (0)$. Let $N \triangleleft \Gamma$ be a normal subgroup. The group Γ/N is finite if and only if the groups $G_i/\overline{\mathrm{pr}_i(N)}$ are compact.*

Remark 0.5. *The idea to extend the scope of the rigidity phenomena of lattices in higher rank Lie groups to the general locally compact setting by considering lattices in general products goes at least back to [BnK70] and is* *present in the work of G. Margulis in the seventies. When G_1 and G_2 are Lie groups over local fields, Theorems 0.2, 0.3 and 0.4 are special cases of results of G. Margulis proven respectively in [Mar74], [Mar81] and [Mar79]. Superrigidity theorems of varying degrees of generality can be found in the book [Mar91] (and references therein), [Bur95], [Mar94], [BM96], [Gao97], [Sha00], [MS02], [Bon], and [Mon03].*

In [BM00a] various classes of closed subgroups of $\operatorname{Aut} T$ defined via local conditions were introduced. It was shown there how these local conditions imply certain global structure results. Let us recall some of these.

Let $T = (X, Y)$ be a locally finite tree, X its set of vertices and Y its set of edges. Given a subgroup $H < \operatorname{Aut} T$ and a vertex $x \in X$, its stabilizer $H(x)$ acts as a finite permutation group on the set $E(x) \subset Y$ of edges whose origin is the vertex x, and we say that H is locally quasiprimitive (respectively, locally primitive, locally 2-transitive, etc.) if this finite permutation group is quasiprimitive (respectively primitive, 2-transitive, etc.) for every vertex x. Recall that a finite permutation group $F < \operatorname{Sym} \Omega$ acting on a finite set Ω is called *quasiprimitive* if every non trivial normal subgroup of F act transitively on Ω. It is called *primitive* if any equivariant factor of Ω is trivial (i.e. either a point or Ω). The action is called 2-transitive if F acts transitively on $\Omega^2 \setminus \Delta_\Omega$ where Δ_Ω is the diagonal. Quasiprimitive and primitive groups have a rich structure theory, exemplified by results such as the O'Nan–Scott theorem (cf., [DM96], [Pra97]). Following the classification of the finite simple groups one has also a classification of 2-transitive groups (cf. the survey paper by P. Cameron [Cam95]).

Given a totally disconnected group H let

$$H^{(\infty)} = \cap_{L<H} L$$

where the intersection is taken over all open finite index subgroups. Let

$$\mathrm{QZ}(H) = \{h \in H \ : \ Z_H(h) \text{ is open } \}$$

be the quasi-center of H. Both are topologically characteristic subgroups of H. The subgroup $H^{(\infty)}$ is closed, and any normal discrete subgroup of H is contained in $\mathrm{QZ}(H)$. To motivate these definitions observe that when $H = \mathbb{G}(\mathbb{Q}_p)$, where $\mathbb{G}$ is a connected semisimple $\mathbb{Q}_p$-group, then $H^{(\infty)}$ coincides with the subgroup $\mathbb{G}^+$ generated by all one parameter unipotent subgroups of $\mathbb{G}(\mathbb{Q}_p)$, and $\mathrm{QZ}(H)$ is the kernel of the adjoint representation of the Lie group $\mathbb{G}(\mathbb{Q}_p)$. We recall next a few basic results established in [BM00a] concerning the structure of these subgroups.

Theorem 0.6. ([BM00a] Prop. 1.2.1) *Let $H < \operatorname{Aut} T$ be a closed non discrete locally quasiprimitive group. Then*

1. *$H/H^{(\infty)}$ is compact.*
2. *$\operatorname{QZ}(H)$ is a discrete not cocompact subgroup of H.*
3. *Any closed normal subgroup of H either contains $H^{(\infty)}$ or is contained in $\operatorname{QZ}(H)$.*

The subgroup $\operatorname{QZ}(H)$ acts freely on the set of vertices of T. In particular if $\operatorname{QZ}(H)$ is non-trivial then the subgroup (of index at most 2) of $\operatorname{QZ}(H)$ consisting of those elements whose translation length is even is a free group on infinitely many generators. In analogy with semisimple Lie groups one would expect $H^{(\infty)}$ to have a relatively simple structure. Indeed we have:

Theorem 0.7. ([BM00a] Prop. 1.2.1, 1.5.1) *Let $H < \operatorname{Aut} T$ be a closed non discrete subgroup.*

1. *If H is locally quasiprimitive, then $H^{(\infty)}/\operatorname{QZ}(H^{(\infty)})$ admits minimal closed normal subgroups. These are finitely many, topologically simple, H-conjugate and their product is dense.*
2. *If H is locally primitive, $H^{(\infty)}/\operatorname{QZ}(H^{(\infty)})$ is a finite product of topologically simple groups.*
3. *If H is locally 2-transitive, $H^{(\infty)}/\operatorname{QZ}(H^{(\infty)})$ is topologically simple.*

In all the above cases we have that $\operatorname{QZ}(H^{(\infty)}) = \operatorname{QZ}(H) \cap H^{(\infty)}$ and it acts freely without inversions on T.

We observe that $H^{(\infty)}/\operatorname{QZ}(H^{(\infty)})$ may have an arbitrary large number of simple factors. This is exemplified by Proposition 0.14 below and the subsequent discussion. We conclude from Theorem 0.7.1 that if H is locally quasiprimitive and $\operatorname{QZ}(H^{(\infty)}) = \{e\}$, then $H^{(\infty)}$ is topologically simple. Indeed, assume $\operatorname{QZ}(H^{(\infty)})$ is trivial; we need to show that there is only one minimal closed normal subgroup in $H^{(\infty)}$. Assume that there were 2 or more such groups $M_1, M_2, \ldots$; then as all are conjugate and their product is dense they are unbounded and hence M_1 has a non void limit set in the boundary of the tree. As M_1 is a normal subgroup of $H^{(\infty)}$ which acts cofinitely on the tree, it follows that this limit set must be the whole boundary of T. Since M_1 and M_2 commute this limit set must be pointwise fixed by M_2 which is impossible. We conclude that the unique minimal closed normal subgroup of $H^{(\infty)}$ coincides with $H^{(\infty)}$ showing it is topologically simple. This observation together with Theorem 0.4 leads to:

Corollary 0.8. *Let $H_i < \mathrm{Aut}\, T_i$ be closed non discrete locally quasiprimitive groups and $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ a cocompact lattice such that for each i $H_i^{(\infty)} \subset \overline{\mathrm{pr}_i(\Gamma)} \subset H_i$. Assume that $\mathrm{QZ}(H_i) = \{e\}$. Then any non trivial normal subgroup of Γ is of finite index.*

Remark 0.9. *If Γ satisfies the conditions of Corollary 0.8 then so does also any finite index subgroup of Γ. See the beginning of § 2.*

The quasi center plays, in several ways, an important role for the structure of Γ. In [BM00b], Theorem 0.6 was applied to obtain the following criterion for non-residually finiteness of Γ. Let $\Lambda_1 = \Gamma \cap (\mathrm{Aut}\, T_1 \times e)$, $\Lambda_2 = \Gamma \cap (e \times \mathrm{Aut}\, T_2)$. We have:

Proposition 0.10. ([BM00b] Prop. 2.1, 2.2) *Let $H_i < \mathrm{Aut}\, T_i$ be closed, non discrete, locally quasiprimitive groups and $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ be a lattice with $H_i^{(\infty)} \subset \overline{\mathrm{pr}_i(\Gamma)} \subset H_i$. Then*

1. *If $\Lambda_1 \cdot \Lambda_2 \neq \{e\}$ then Γ is not residually finite.*
2. *If $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ then Λ_i is a normal subgroup of $\mathrm{QZ}(H_i)$ and the quotients $\mathrm{QZ}(H_i)/\Lambda_i$ are locally finite groups.*

In the setting of Proposition 0.10.2, we have that Λ_i is non trivial if and only if the quasi center $\mathrm{QZ}(H_i)$ is non trivial. Theorems 0.4, 0.6 and Proposition 0.10.2 can be used to show:

Theorem 0.11. *Let $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ be a cocompact lattice such that $H_i = \overline{\mathrm{pr}_i(\Gamma)}$ are non discrete, locally quasiprimitive. Let $N \lhd \Gamma$ be a normal subgroup. Then either*

1. *$N \subset \mathrm{QZ}(H_1) \times \mathrm{QZ}(H_2)$,*

or

2. *N is of finite index in Γ.*

Theorem 0.10 together with a predecessor (see [BM00b] §4) of the Normal Subgroup Theorem of Bader–Shalom (Theorem 0.4) enabled us to construct a new class of finitely presented simple groups.

Theorem 0.12. **([BM97], [BM00b])** *For every sufficiently large* $n, m \in \mathbb{N}$ *there is a torsion free cocompact lattice in* $\operatorname{Aut} T_{2n} \times \operatorname{Aut} T_{2m}$ *which is a simple group.*

Remark 0.13. *In [Rat04] Diego Rattaggi has constructed many interesting lattices* $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$. *In particular a* $(4, 12)$*-complex ([Rat04, Example 2.26]) whose fundamental group is not residually finite (using Proposition 0.10). Using this example as well as his construction of an* (A_6, A_6)*-square complex described in Example 0.18 and following the same strategy as in [BM00b] he has established the existence of simple groups as above for every* $n \geq 9$ *and* $m \geq 13$*, see [Rat04, Prop. 2.29 (1)].*

An important ingredient in the proof of Theorem 0.12 consists of the identification of the closures of the projections of lattices constructed via geometric methods. This brings us back to the basic question above of which groups arise as closures of projections of uniform lattices in $\operatorname{Aut} T_1 \times \operatorname{Aut} T_2$.

The finitely generated simple groups of Theorem 0.12 are of course non linear. In the rest of the paper we will be interested in studying lattices which admit infinite linear images. We will show that such lattices are essentially extensions of arithmetic lattices in semisimple Lie groups over appropriate local fields.

In view of the superrigidity theorems mentioned above, locally quasiprimitive groups which admit a p-adic analytic structure will play a central role in the discussion of arithmeticity. We have,

Proposition 0.14. *Let* $H < \operatorname{Aut} T$ *be a closed non discrete locally quasiprimitive group. Assume that it admits a* $\mathbb{Q}_p$*-analytic structure. Let* $\mathcal{H}$ *denote the Lie algebra of* H*, let* $\mathbb{G} = \operatorname{Aut}(\mathcal{H} \otimes \overline{\mathbb{Q}_p})$ *a linear algebraic group defined over* $\mathbb{Q}_p$ *and let* $\operatorname{Ad} : H \to \mathbb{G}(\mathbb{Q}_p)$ *be the adjoint representation. Then*

1. $\mathbb{G}$ *is adjoint and semisimple.*
2. $\ker \operatorname{Ad} = \operatorname{QZ}(H)$.
3. $\operatorname{Ad}(H) \supset \mathbb{G}^+$.

As mentioned above $\mathbb{G}^+$ denotes for a $\mathbb{Q}_p$-group $\mathbb{G}$, the subgroup of $\mathbb{G}(\mathbb{Q}_p)$ generated by all the one parameter unipotent $\mathbb{Q}_p$-subgroups.

Thus the group H is an extension by $\mathrm{QZ}(H)$ of the group $\mathrm{Ad}(H)$ which lies between $\mathbb{G}^+$ and $\mathbb{G}$. Recall that $\mathrm{QZ}(H)$ is either trivial or virtually a free group on infinitely many generators. We remark that even though H, being a subgroup of $\mathrm{Aut}\, T$, is a "rank one object" the group $\mathbb{G}$ may be of arbitrarily large rank. Consider a connected simple algebraic group $\mathbb{G}$ defined over $\mathbb{Q}_p$ and let Δ be its affine Bruhat–Tits building. The action of $\mathbb{G}$ on Δ induces an action on any graph $\mathcal{G}$ which is equivariantly drawn on Δ and this leads to the extension of $\mathbb{G}$ by $\pi_1(\mathcal{G})$ acting on the universal covering tree $\tilde{\mathcal{G}}$. In particular if one takes $\mathcal{G}$ to be the subgraph of the 1-skeleton of Δ obtained by considering all edges of a given colour, then the resulting extension H, $1 \to \pi_1(\mathcal{G}) \to H \to \mathbb{G}(\mathbb{Q}_p) \to 1$ is a locally primitive group.

Observe that in the setting of Proposition 0.14 the homomorphism $\mathrm{Ad} : H \to \mathbb{G}(\mathbb{Q}_p)$ sends every vertex stabilizer $H(x)$, $x \in T$, isomorphically to a compact subgroup of $\mathbb{G}(\mathbb{Q}_p)$. In particular $H(x)$ is a virtually pro-p-group, i.e., $H(x)$ contains an open pro-p-subgroup of finite index. However in general it is not clear to which extent the homomorphism is implemented by a geometric mapping of the tree into the Bruhat–Tits building associated to $\mathbb{G}(\mathbb{Q}_p)$, and thus we have no real way of comparing the "congruence" structure on $H(x)$ induced by the tree structure of T and the congruence structure induced by that of a maximal compact subgroup of $\mathbb{G}(\mathbb{Q}_p)$. There is however a powerful tool going back to Thompson and Wielandt:

Proposition 0.15. ([BM00a], Prop. 2.1.2) *Let $H < \mathrm{Aut}\, T$ be a closed locally primitive subgroup and when H is non discrete assume that $H(x)$ is virtually pro-p. Then*

1. *There is a vertex $y \in X$ such that $H_2(y)$ is pro-p.*
2. *If H acts transitively on X, then for any two adjacent vertices x, y the subgroup $H_1(x, y) = H_1(x) \cap H_1(y)$ is pro-p.*

Where as in [BM00a] $H_i(x)$ denotes the stabilizer of the ball of radius i centered at x.

Our main result is:

Theorem 0.16. *Let T_1, T_2 be locally finite trees. Let $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ be a cocompact lattice. Assume*

1. $H^{(\infty)} < \overline{\mathrm{pr}_i(\Gamma)} < H_i$, *where* $H_i < \mathrm{Aut}\, T_i$ *is a closed non discrete, locally quasiprimitive subgroup.*

2. *There is a linear representation* $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ *with infinite image.*

Then there are prime numbers p_1, p_2 *such that* H_i *is* $\mathbb{Q}_{p_i}$*-analytic and we have an exact sequence*

$$1 \to \Lambda_1 \times \Lambda_2 \to \Gamma \to (\mathrm{Ad}_1 \times \mathrm{Ad}_2)(\Gamma) \to 1$$

where

- $\Lambda_i := \Gamma \cap H_i$ *is of finite index in* $\mathrm{QZ}(H_i) = \ker \mathrm{Ad}_i$.

- $(\mathrm{Ad}_1 \times \mathrm{Ad}_2)(\Gamma)$ *is an arithmetic lattice in* $\mathbb{G}_1(\mathbb{Q}_{p_1}) \times \mathbb{G}_2(\mathbb{Q}_{p_2})$, *where* $\mathbb{G}_i$ *is the* $\mathbb{Q}_{p_i}$*-semisimple group given by Proposition 0.14.*

Using the above result we can now characterize the "classical"situation:

Corollary 0.17. *Let* T_1, T_2 *be locally finite trees. Let* $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$ *be a cocompact lattice. Assume*

1. ${H_i}^{(\infty)} < \overline{\mathrm{pr}_i(\Gamma)} < H_i$, *where* $H_i < \mathrm{Aut}\, T_i$ *is a closed non discrete, locally primitive subgroup.*

2. *There is a linear representation* $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ *with infinite image.*

Then the following are equivalent:

1. Γ *is linear over* $\mathbb{C}$.

2. Γ *is residually finite.*

3. $\mathrm{rank}_{\mathbb{Q}_{p_i}}(\mathbb{G}_i) = 1$ *for both* $i = 1, 2$.

In this case the geometric realization $|T_i|$ *is isometric to the Bruhat–Tits tree associated to* $\mathbb{G}_i$.

We turn next to discuss some consequences of the above results formulated in geometric terms, that is expressed in terms of finite square complexes $\Gamma\backslash(T_1 \times T_2)$ rather than in terms of lattices $\Gamma < \mathrm{Aut}\, T_1 \times \mathrm{Aut}\, T_2$. We will concentrate here on the special case of a square complex with a single vertex. Such square complexes correspond to lattices acting freely transitively on the vertices of the product of trees $T_1 \times T_2$. A square complex with one vertex whose universal cover is a product of two trees is given by the following data (see Example 0.1):

- Two finite sets A_1, A_2 each endowed with a fixed point free involution $x \mapsto \overline{x}$.
- A set of "squares" $S \subset A_1 \times A_2 \times A_1 \times A_2$ satisfying the following two conditions:

  1. The elements of S come in equivalence classes:

  $$(a, b, a', b'), (\overline{a}, \overline{b'}, \overline{a'}, \overline{b}), (\overline{a'}, \overline{b}, \overline{a}, \overline{b'}), (a', b', a, b) \in S$$

  2. *(Link condition)* For each pair $(a, b) \in A_1 \times A_2$ there is a unique pair $(a', b') \in A_1 \times A_2$ such that $(a, b, a', b') \in S$.

As in Example 0.1 these data determines a 2-dimensional 1-vertex square complex X with a single vertex denoted x_0, $n_1 = \frac{1}{2}|A_1|$ horizontal geometric loops, $n_2 = \frac{1}{2}|A_2|$ vertical geometric loops and $n_1 \cdot n_2 = \frac{1}{4}|S|$ geometric squares. The complex X is the quotient $\Gamma \backslash (T_1 \times T_2)$ where each T_i is a $2n_i$ regular tree and $\Gamma = \pi_1(X, x_0)$. The complex X provides also a presentation for Γ, namely:

$$\Gamma = \langle A_1 \cup A_2 \mid aba'b',\ (a, b, a', b') \in S,\ a\overline{a}, a \in A_1,\ b\overline{b},\ b \in A_2 \rangle.$$

The local transitivity properties of Γ on each factor can be easily read off the complex X by considering the holonomy action of elements corresponding to oriented horizontal loops on vertical edges and vice versa. Fixing a vertex $(x_1, x_2) \in T_1 \times T_2$ the group $\langle A_1 \rangle$, which is a free group on $\frac{1}{2}|A_1|$ generators, may be identified with the stabilizer $\mathrm{Stab}_\Gamma(x_2)$. The covering map $T_1 \times T_2 \to X$ induces a labeling of the oriented 1-skeleton of $T_1 \times T_2$ by the elements of $A_1 \cup A_2$. We may identify T_1 with the tree consisting of the connected component of the "horizontal 1-skeleton" of $T_1 \times T_2$ containing the chosen vertex (x_1, x_2). This induces a labeling of the oriented edges of T_1 by the elements of A_1 and similarly we have a labeling of the oriented edges of T_2 by the elements of A_2. Observe that paths of length k starting at the vertex $x_2 \in T_2$ correspond to $A_2^{(k)}$ =irreducible words of length k over A_2 (a word is reducible if it contains $x\overline{x}$ or $\overline{x}x$). The action of $\langle A_1 \rangle$ on $A_2^{(k)}$, that is the permutation representation $\rho_{12}^{(k)} : \langle A_1 \rangle \to \mathrm{Sym}\, A_2^{(k)}$ can be read directly from the complex by observing that for $a \in A_1$ and $w \in A_2^{(k)}$ there exist a unique pair: $a' \in A_1$, $w' \in A_2^{(k)}$ such that $a \cdot w' = w \cdot a'$. We have

$\rho_{12}^{(k)}(a)(w) = w'$. Exchanging the roles of the trees and the generating sets we obtain: $\rho_{21}^{(k)} : \langle A_2 \rangle \to \mathrm{Sym}\, A_1^{(k)}$. Let us define

$$P_2^{(k)} = \mathrm{Im}\, \rho_{12}^{(k)} < \mathrm{Sym}\, A_2^{(k)}$$
$$P_1^{(k)} = \mathrm{Im}\, \rho_{21}^{(k)} < \mathrm{Sym}\, A_1^{(k)}$$

We shall omit the sup-script (k) when $k = 1$. We will say that X is a (P_1, P_2)-complex when we want to emphasize the local permutation groups $P_1 < \mathrm{Sym}\, A_1$ and $P_2 < \mathrm{Sym}\, A_2$.

Example 0.18. ([Rat04] Theorem 2.3). *The* (A_6, A_6)*-complex* X *given by:*

$$\left\{a_1b_1\overline{a_1}\overline{b_1}, a_1b_2\overline{a_1}\overline{b_3}, a_1b_3a_2\overline{b_2}, a_1\overline{b_3}\overline{a_3}b_2, a_2b_1\overline{a_3}\overline{b_2}, a_2b_2\overline{a_3}\overline{b_3}, a_2b_3\overline{a_3}b_1, a_2\overline{b_3}a_3b_2, a_2\overline{b_1}\overline{a_3}\overline{b_1}\right\}$$

Satisfies the following:

1. *Any non trivial normal subgroup of* $\pi_1(X)$ *is of finite index.*
2. *Any linear representation of* $\pi_1(X)$ *in characteristic zero has finite image.*
3. $\mathrm{Out}(\pi_1(X)) \equiv \mathbb{Z}/2\mathbb{Z}$.

We will say that X is irreducible if Γ is irreducible, namely when no finite covering of X is a product of two graphs. It is a fundamental problem whether there is an algorithm for deciding whether a given X is irreducible. There is however a sufficient condition based on the Thompson-Wielandt Theorem, see Proposition 0.15, which we describe next. Assume that $P_i < \mathrm{Sym}\, A_i$ is transitive, $i = 1, 2$. Fix an edge e_i in T_i and let $B_r(e_i) \subset T_i$ denote the neighbourhood of radius r around e_i, let Q_i denote the restriction to $B_2(e_i)$ of the subgroup of $P_i^{(3)}$ consisting of elements pointwise fixing $B_1(e_i)$. It follows from the Thompson-Wielandt theorem 0.15 that if for some $i = 1, 2$ $Q_i \neq \{e\}$ and is not a p-group then the lattice is irreducible. In fact as a corollary of the arithmeticity theorem a much stronger assertion holds:

Corollary 0.19. *Let* X *be a* (P_1, P_2)*-complex where* $P_i < \mathrm{Sym}\, A_i$ *are primitive permutation groups,* $i = 1, 2$*, and assume that* $Q_1 \neq \{e\}$ *and is not a* p *group. Then any linear representation of* $\pi_1(X)$ *over a field of characteristic zero has finite image.*

In applying the criterion for non residually finiteness (Proposition 0.10) one needs to detect whether the projection of Γ on one of the factor, say $\operatorname{Aut} T_2$, has a non trivial kernel. In other words whether there exists an element $w \in \langle A_1 \rangle$ whose action on T_2 is trivial. Observe that while verifying whether a given element $w \in \langle A_1 \rangle$ acts trivially on T_2 is a (easily) computable question, we do not know whether the existence of such an element is decidable. We also do not know an algorithm for deciding whether the quasi center of $H_1 = \overline{\mathrm{pr}_1(\Gamma)}$ is trivial or not. There is however, a construction - called fibered product, see [BM00b] §2.2, 2.3 - which starting with a 1-vertex square complex X produces a new 1-vertex square complex $X \otimes X$ which in suitable setting has non-residually finite fundamental group as well as closure of projections with non trivial quasi-centers. Let X be given by the data: A_1, A_2 and $S \subset A_1 \times A_2 \times A_1 \times A_2$ then $X \otimes X$ is defined by the data: $A_1 \times A_1$, $A_2 \times A_2$ and $R = \{((a_1, a_2), (b_1, b_2), (a_1', a_2'), (b_1', b_2')) \ : \ (a_i, b_i, a_i', b_i') \in S, \ i = 1, 2\}$. If X is a (P_1, P_2)-complex then $X \otimes X$ is a $(P_1 \times P_1, P_2 \times P_2)$-complex and hence $\pi_1(X \otimes X)$ can never have locally primitive projections. However, see [BM00b] §2.2, it is contained as an index 2 subgroup in a group which is locally primitive when P_1 and P_2 are primitive. The natural homomorphism

$$\pi_1(X \otimes X) \to \pi_1(X) \times \pi_1(X) \tag{3}$$

has image of index 1, 2 or 4 and its kernel defines an infinite covering which is of the form $\mathcal{D}_1 \times \mathcal{D}_2$ where each $\mathcal{D}_i$ is a graph whose fundamental group is an infinitely generated free group. Using [BM00b] Prop. 2.1 we showed:

Proposition 0.20. *If X is an irreducible (P_1, P_2)-square complex where each P_i is a 2-transitive permutation group with 2-transitive socle then $\pi_1(X \otimes X)$ is not residually finite; in fact $\pi_1(X \otimes X)^{\infty)} \supset \pi_1(\mathcal{D}_1) \times \pi_1(\mathcal{D}_2)$.*

Composing the homomorphism in (3) with the projection on either factor we obtain a surjective homomorphism, $\pi_1(X \otimes X) \to \pi_1(X)$ showing that rigidity fails in a strong way since not even the universal covering spaces of $X \otimes X$ and X are isomorphic. However in § 3 we will show:

Corollary 0.21. *Let X be an irreducible (P_1, P_2)-complex and Y a (Q_1, Q_2)-complex. Assume that P_1 and P_2 are primitive permutation groups. Then any surjective homomorphism $\varphi : \pi_1(X) \to \pi_1(Y)$ is induced by an isomorphism $f : X \to Y$. In particular φ is an isomorphism and each Q_i is permutation isomorphic to P_i (up to reindexing).*

1 Locally quasiprimitive groups and p-adic structure

The basic objective of this chapter is to show that a locally quasiprimitive group H which admits a continuous representation into $\mathrm{GL}(n, \mathbb{Q}_p)$ with unbounded image is p-adic analytic, and to investigate this p-adic structure more closely. First we establish a general fact about continuous representations of totally disconnected groups. This will rest on the following consequence of the Howe-Moore Theorem [HM79]. We have:

Theorem 1.1. *Let $\mathbb{G}$ be a $\mathbb{Q}_p$-almost simple group and $O < \mathbb{G}(\mathbb{Q}_p)$ an open unbounded subgroup. Then $O \supset \mathbb{G}^+$*

Proof. We may clearly assume that $\mathbb{G}$ is connected. Then $\delta_O \in \ell^2(\mathbb{G}(\mathbb{Q}_p)/O)$ is a vector which is invariant under the unbounded group O and hence by the Howe-Moore theorem is $\mathbb{G}^+$-invariant. □

Remark 1.2. *To prove Theorem 1.1 one needs in fact only the local field analogue of Mautner phenomenon as proven by C. Moore for semisimple real Lie groups in [Moo66, Theorem 1]. The arguments of Howe–Moore in [HM79, Proposition 5.5] can be used to give a direct proof of the local field analogue of [Moo66, Theorem 1] from which Theorem 1.1 follows.*

Lemma 1.3. *Let H be a locally compact totally disconnected group, $K < H$ an open compact subgroup and $\mathbb{G}$ a $\mathbb{Q}_p$-almost simple group. Let $\pi : H \to \mathbb{G}(\mathbb{Q}_p)$ be a continuous homomorphism with Zariski dense image. Then one of the following holds:*

1. *$\pi(K)$ is finite.*
2. *$\pi(K)$ is open and $\pi(H)$ is compact.*
3. *$\pi(K)$ is open and $\pi(H) \supset \mathbb{G}^+$.*

Proof. If $\pi(K)$ is infinite, its Lie algebra $\mathcal{K} \subset \mathcal{G}(\mathbb{Q}_p)$ is of positive dimension. As $\pi(H)$ commensurates $\pi(K)$ it follows that $\mathrm{Ad}\,\pi(H)$ leaves $\mathcal{K}$ invariant. Hence by the Zariski density of $\pi(H)$ it follows that $\mathcal{K} = \mathcal{G}(\mathbb{Q}_p)$, implying that $\pi(K)$ is open. If $\pi(H)$ is not compact it is open and unbounded and hence by Theorem 1.1 we have $\pi(H) \supset \mathbb{G}^+$. □

Before proceeding we need a few more results from [BM00a] concerning locally quasiprimitive groups.

Proposition 1.4. (see [BM00a, Prop. 1.2.1]) *Let T be a tree and $H < \operatorname{Aut} T$ a closed non-discrete locally quasiprimitive subgroup. we have:*

1. $\mathrm{QZ}(H^{(\infty)})$ *is discrete, in fact,* $\mathrm{QZ}(H^{(\infty)}) = H^{(\infty)} \cap \mathrm{QZ}(H)$.
2. *for any open normal subgroup* $N \lhd H^{(\infty)}$ *we have* $N = H^{(\infty)}$.
3. $H^{(\infty)} = \overline{[H^{(\infty)}, H^{(\infty)}]}$.

Lemma 1.3 applied to locally quasiprimitive groups implies the following:

Lemma 1.5. *Let $H < \operatorname{Aut} T$ be a closed non-discrete locally quasiprimitive group. Let L be a closed subgroup, $H^{(\infty)} < L < H$, $\mathbb{G}$ an almost $\mathbb{Q}_p$-simple group and $\pi : L \to \mathbb{G}(\mathbb{Q}_p)$ a continuous homomorphism with Zariski dense image. Then either*

1. $\pi(L)$ *is compact.*

   or

2. (a) $\pi(H^{(\infty)}) = \mathbb{G}^+$
   (b) $\pi(\mathrm{QZ}(H^{(\infty)})) \subset Z(\mathbb{G}(\mathbb{Q}_p))$
   (c) $\pi(K)$ *is open for any compact open subgroup* $K < H^{(\infty)}$.

Proof. Assume that $\pi(L)$ is not compact, then $\pi(H^{(\infty)})$ is not finite (Theorem 0.6.1) in particular not central and being normal in a Zariski dense subgroup it is itself Zariski dense. Now we apply the trichotomy of Lemma 1.3 to π, $H^{(\infty)}$ and K. We have:

1. If $\pi(K)$ is finite then $\ker \pi \lhd H^{(\infty)}$ is open and hence (by Prop. 1.4.2) we have $\ker \pi = H^{(\infty)}$ and (by Theorem 0.6.1) hence $\pi(L)$ is compact, contradiction.

2. If $\pi(K)$ is open and $\pi(H^{(\infty)})$ is compact then $\pi(H^{(\infty)})$ is profinite; by Proposition 1.4.2 any profinite quotient is trivial, hence $\pi(H^{(\infty)}) = \{e\}$, thus $\pi(L)$ is compact, contradiction.

3. We have that $\pi(K)$ is open and $\pi(H^{(\infty)}) \supset \mathbb{G}^+$. Since $\mathbb{G}/\mathbb{G}^+$ is finite one has $\pi(H^{(\infty)}) = \mathbb{G}^+$. Finally $\pi(\mathrm{QZ}(H^{(\infty)}))$ is a countable subgroup of $\mathbb{G}(\mathbb{Q}_p)$ normalized by $\mathbb{G}^+$ hence contained in $Z(\mathbb{G}(\mathbb{Q}_p))$.

□

The preceding lemma gives useful information about homomorphisms of locally quasiprimitive groups into almost simple groups. The next lemma says that in a continuous representation there is always a semisimple part.

Lemma 1.6. *Let H be a non discrete, locally quasiprimitive group. Let $\pi : H^{(\infty)} \to \mathrm{GL}(n, \mathbb{Q}_p)$ be a continuous non trivial representation and $\mathbb{L}$ be the Zariski closure of $\pi(H^{(\infty)})$. Then*

1. *$\mathbb{L}$ is connected.*
2. *$\mathbb{L}/\operatorname{Rad}(\mathbb{L})$ is of positive dimension.*

Proof. Since $H^{(\infty)}$ does not have non trivial continuous finite images it follows that $\mathbb{L}^\circ = \mathbb{L}$ and $\mathbb{L}$ is connected. Assertion 2 follows from the fact that $\mathbb{L}$ is connected and $H^{(\infty)}$ is topologically perfect (see Prop. 1.4.3). □

It follows that if there exists a non-trivial continuous representation $\pi : H^{(\infty)} \to \mathrm{GL}(n, \mathbb{Q}_p)$ then there exists a $\mathbb{Q}_p$-simple, connected, adjoint group $\mathbb{G}$ of positive dimension and a continuous homomorphism $\rho_0 : H^{(\infty)} \to \mathbb{G}(\mathbb{Q}_p)$ with Zariski dense image and hence with the following properties (see Lemma 1.5)

- $\rho_0(H^{(\infty)}) = \mathbb{G}^+$
- $\mathrm{QZ}(H^{(\infty)}) \subset \ker \rho_0$

We recall a few results from [BM00a] concerning normal subgroups of $H^{(\infty)}$. Let $\mathcal{M}$ be the set of closed normal subgroups M of $H^{(\infty)}$, which are minimal with respect to the property $M \supsetneq \mathrm{QZ}(H^{(\infty)})$. Then (see Theorem 0.7.1) The set $\mathcal{M}$ is non void and contains finitely many elements. These elements are H-conjugate and their product is dense in $H^{(\infty)}$. In our setting notice that since $\mathrm{QZ}(H^{(\infty)}) \subset \ker \rho_0 \subsetneq H^{(\infty)}$ it follows from Theorem 0.7.1) that there exists $M_1 \in \mathcal{M}$, such that $M_1 \not\subset \ker \rho_0$. Let also $h_1 = e, h_2, \ldots, h_n \in H$ be such that

$$\mathcal{M} = \{h_i M_1 h_i^{-1} \;\; : \;\; 1 \leq i \leq n\}$$

and define $\rho_i : H^{(\infty)} \to \mathbb{G}(\mathbb{Q}_p)$ by $\rho_i(h) = \rho_0(h_i^{-1} h h_i)$. Then the following holds:

Lemma 1.7. *The homomorphism* $\rho : H^{(\infty)} \to (\mathbb{G}^+)^n$ *given by* $\rho(h) = (\rho_i(h))_{1\le i\le n}$ *induces a topological isomorphism*

$$H^{(\infty)}/\operatorname{QZ}(H^{(\infty)}) \to (\mathbb{G}^+)^n$$

Proof. Since $\mathbb{G}^+$ is simple and $\rho_0(M_1)$ is a non trivial normal subgroup of $\rho_0(H^{(\infty)}) = \mathbb{G}^+$ we have that $\rho_0(M_1) = \mathbb{G}^+$ and hence $M_1 \ker \rho_0 = H^{(\infty)}$. We claim that $M_2 \cdots M_n \subset \ker \rho_0$. Since $M_1 \cap M_i = \operatorname{QZ}(H^{(\infty)})$ (by minimality), it follows that for $2 \le i \le n$, $[M_1, M_i] \subset \operatorname{QZ}(H^{(\infty)})$ and hence $\rho_0(M_1) = \mathbb{G}^+$ commutes with $\rho_0(M_i)$ which implies $\rho_0(M_i) = e$, $2 \le i \le n$, and the claim is proved. We conclude that $\rho(M_1 \cdots M_n) = (\mathbb{G}^+)^n$. Note also $\ker \rho \supset \operatorname{QZ}(H^{(\infty)})$ and in fact one has equality since otherwise (see [BM00a] Prop. 1.5.1(3)) $\ker \rho$ would contain one of the M_i's ☐

The following is one of the main results of this chapter:

Theorem 1.8. *Let* $H < \operatorname{Aut} T$ *be a closed non-discrete locally quasiprimitive subgroup, and let* $H^{(\infty)} < L < H$ *be a closed subgroup. Assume that there exists a continuous, unbounded representation:*

$$\pi : L \to \operatorname{GL}(m, \mathbb{Q}_p).$$

Then

1. *H is a $\mathbb{Q}_p$-analytic group and $H^{(\infty)}$ is of finite index in H.*
2. *Its Lie algebra is a product* $\operatorname{Lie}(H) = \mathcal{G}^n$ *where $\mathcal{G}$ is a $\mathbb{Q}_p$-simple Lie algebra.*
3. *The adjoint representation* $\operatorname{Ad} : H \to \operatorname{Aut}(\mathcal{G}^n)$ *has kernel equal* $\operatorname{QZ}(H)$ *and induces an isomorphism:* $H^{(\infty)}/\operatorname{QZ}(H^{(\infty)}) \to (\mathbb{G}^+)^n$ *where $\mathbb{G}$ is the connected $\mathbb{Q}_p$-simple adjoint group attached to $\mathcal{G}$.*

Proof. Applying lemmas 1.5, 1.6 we are in the setting of lemma 1.7 and we obtain via ρ a continuous action of H on $(\mathbb{G}^+)^n$, which we denote $\epsilon : H \to \operatorname{Aut}((\mathbb{G}^+)^n)$, with $\epsilon(H^{(\infty)}) = \operatorname{Inn}((\mathbb{G}^+)^n)$. The group $H' = \epsilon^{-1}(\operatorname{Inn}((\mathbb{G}^+)^n))$ is a closed subgroup of finite index in H. Then $H^{(\infty)} \cdot \ker \epsilon = H'$, but since $\ker \epsilon$ is a closed normal subgroup which is not cocompact, we deduce (using prop. 0.6.1) that $\ker \epsilon \subset \operatorname{QZ}(H)$, in particular $H^{(\infty)}$ is of countable index in H and hence (by Baire category) of finite index. Assertion (1) is proved. The remaining statements follow by repeated application of lemma 1.7. ☐

2 Arithmeticity

Our goal in this section is to prove Theorem 0.16. A cocompact lattice $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ is said to be *LQP-sandwiched* if there exist closed non discrete, locally quasiprimitive subgroups $H_i < \operatorname{Aut} T_i$ such that

$$H_i{}^{(\infty)} < \overline{\mathrm{pr}_i(\Gamma)} < H_i$$

Observe that since $H_i{}^{(\infty)}$ has no proper closed subgroup of finite index it follows that every finite index subgroup of Γ is also LQP-sandwiched. Recall

Proposition 2.1. [BM00b, Cor. 3.3] *Let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be a cocompact lattice which is LQP-sandwiched. Then*

1. *Its abelianization Γ_{ab} is finite.*
2. *For any unitary representation $\omega : \Gamma \to \mathrm{U}(n)$, $H^1(\Gamma, \omega) = 0$*

Combining Theorem 0.2 and Proposition 2.1 we obtain:

Corollary 2.2. *Let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be a LQP-sandwiched lattice. Then:*

1. *Any homomorphism $\pi : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ has bounded image.*
2. *Let k be a field of characteristic zero and $\pi : \Gamma \to \mathrm{GL}(n, k)$ a homomorphism, then the Zariski closure $\overline{\pi(\Gamma)}^Z$ is semisimple.*

Proof. In 2. we may replace the field k by $\mathbb{C}$ and we will prove both 1. and 2. simultaneously. Let $\mathbb{L}$ be the connected component of the Zariski closure of the image of Γ and let $\Gamma' < \Gamma$ be a subgroup of finite index with $\overline{\pi(\Gamma')}^Z = \mathbb{L}$. Since Γ'_{ab} is finite and its image in $\mathbb{L}/[\mathbb{L}, \mathbb{L}]$ is Zariski dense, the latter is trivial and hence the radical $\operatorname{Rad}(\mathbb{L})$ coincides with the unipotent radical $\operatorname{Rad}_u(\mathbb{L})$. Let $\mathbb{S} = \mathbb{L}/\operatorname{Rad}(\mathbb{L})$ and $\overline{\pi} : \Gamma' \to \mathbb{S}$ be the composition of π with the canonical projection $\mathbb{L} \to \mathbb{S}$. If $\overline{\pi}(\Gamma)$ were unbounded then there would be a simple quotient $\mathbb{S} \to \mathbb{S}_1$ such that composing with $\overline{\pi}$ we would have a homomorphism $\pi_1 : \Gamma' \to \mathbb{S}_1$ with Zariski dense unbounded image. Setting $P_i = \overline{\mathrm{pr}_i(\Gamma')}$, we are in the setting of Theorem 0.2 and π_1 extends to a continuous homomorphism factoring via, say, P_1. Let $\pi_{1,ext} : P_1 \to \mathbb{S}_1$ be this continuous homomorphism. Since P_1 is totally disconnected, $\ker \pi_{1,ext} \triangleleft P_1$ is open. Since $P_1 \supset H_1{}^{(\infty)}$, we have $\ker \pi_{1,ext} \supset H_1{}^{(\infty)}$ (Proposition 1.4.2) and hence the image of $\pi_{1,ext}$ is compact implying that $\pi_1(\Gamma')$ is bounded, a contradiction. Consider now $\mathbb{L}/[\mathrm{Rad}_u(\mathbb{L}), \mathrm{Rad}_u(\mathbb{L})] = \mathbb{S} \cdot V$ a semidirect product of $\mathbb{S}$ with a vector space V. The corresponding homomorphism $\Gamma' \to \mathbb{S} \cdot V$ has the form$\gamma \mapsto \overline{\pi}(\gamma) \cdot c(\gamma)$, where $\overline{\pi}(\Gamma')$ is bounded and $c : \Gamma \to V$ is a 1-cocycle with values in a bounded representation of Γ'. It follows from Prop 2.1.2 that this cocycle is trivial and hence that the image of Γ' in $\mathbb{S} \cdot V$ is contained in a conjugate of $\mathbb{S}$. Since the image is Zariski dense it follows that $V = 0$. I.e., $\mathrm{Rad}_u(\mathbb{L}) = [\mathrm{Rad}_u(\mathbb{L}), \mathrm{Rad}_u(\mathbb{L})]$ and hence we have $\mathrm{Rad}_u(\mathbb{L}) = 0$. □

We introduce the following terminology: Let k be a local field. A *k-triple* is a triple $(\Gamma', \pi, \mathbb{H})$ where $\Gamma' < \Gamma$ is a subgroup of finite index, $\mathbb{H}$ is a connected k-simple adjoint group of positive dimension and $\pi : \Gamma' \to \mathbb{H}(k)$ is a homomorphism with Zariski dense image. We say that it is of *unbounded type* if $\pi(\Gamma')$ is unbounded in $\mathbb{H}(k)$.

Lemma 2.3. *Let $(\Gamma', \pi, \mathbb{H})$ be a k-triple. Then there exist a finite extension $\mathbb{Q} \subset K \subset k$ of $\mathbb{Q}$ and a K-structure on $\mathbb{H}$ so that $\pi(\Gamma') \subset \mathbb{H}(K)$.*

Proof. Consider $\mathbb{H}$ as a subgroup of $\mathrm{GL}(n, k)$ via the adjoint representation, fix an embedding $\iota : k \to \mathbb{C}$ and let us abuse notation and denote by ι also the induced group homomorphism $\iota : \mathrm{GL}(n, k) \to \mathrm{GL}(n, \mathbb{C})$. The homomorphism $\iota \circ \pi : \Gamma' \to \mathrm{GL}(n, \mathbb{C})$ as well as any twist of it by an automorphism of $\mathbb{C}$ have bounded image (Corollary 2.2). It follows that $\mathrm{tr}(\iota \circ \pi(\gamma)) = \iota(\mathrm{tr}\, \pi(\gamma))$ are algebraic numbers. Since ι is the identity on $\mathbb{Q}$, it follows that $\mathrm{tr}\, \pi(\gamma)$ are in $\overline{\mathbb{Q}} \cap k$ for all $\gamma \in \Gamma'$. Since Γ' is finitely generated it follows that the field generated by all these traces is a finite extension of $\mathbb{Q}$ which we denote by K. It follows now by a standard argument using the Zariski density of $\pi(\Gamma')$ that there is a faithful k-rational representation $\rho : \mathbb{H} \to \mathrm{GL}(V)$ such that $\rho(\mathbb{H})$ is defined over K and $\rho(\pi(\Gamma')) \subset \rho(\mathbb{H})(K)$ □

Lemma 2.4. *Assume that there is a representation $\rho : \Gamma \to \mathrm{GL}(n, \mathbb{C})$ with infinite image. Then there exists a $\mathbb{Q}_p$-triple of unbounded type.*

Proof. By corollary 2.2.2 the Zariski closure $\overline{\rho(\Gamma)}^Z$ is semisimple. Passing to a simple quotient we get a $\mathbb{C}$-triple $(\Gamma', \pi, \mathbb{H})$. Endow $\mathbb{H}$ with the K-structure given by Lemma 2.3. Then in particular $\pi(\Gamma') \subset \mathbb{H}(K)$. Let $\mathbb{L} = \mathrm{Res}_{K/\mathbb{Q}} \mathbb{H}$ and let $\Delta : \mathbb{H}(K) \to \mathbb{L}(\mathbb{Q})$ be the “diagonal” isomorphism. Let S be the set of primes p for which $\Delta(\pi(\Gamma'))$ is unbounded in $\mathbb{L}(\mathbb{Q}_p)$. Using Corollary 2.2.2 as above it suffices to show that S is non empty. Assume that $S = \emptyset$, then $\Delta(\pi(\Gamma'))$ is up to finite index, contained in $\mathbb{L}(\mathbb{Z})$, hence discrete. On the other hand $\Delta(\pi(\Gamma')) \subset \mathbb{L}(\mathbb{R})$ is bounded (Corollary 2.2.1) which implies that $\Delta(\pi(\Gamma'))$ is finite contradicting the fact that $\pi(\Gamma')$ is Zariski dense in the non trivial connected group $\mathbb{H}$. □

The following observation will be useful in the sequel:

Lemma 2.5. *Let $\mathbb{G}_i$, $1 \leq i \leq n$, be connected $\mathbb{Q}_p$-almost simple groups. Let $\Lambda < \prod_{i=1}^n \mathbb{G}_i(\mathbb{Q}_p)$ be a subgroup such that*

1. $\mathrm{pr}_i(\Lambda) \supset \mathbb{G}_i^+$.
2. *Λ is Zariski dense.*

Then $\Lambda \supset \prod_{i=1}^n \mathbb{G}_i^+$.

2.1 Proof of the arithmeticity theorem

Let P_Γ denote the set of primes p for which there exists a $\mathbb{Q}_p$-triple of unbounded type. According to Lemma 2.4, P_Γ is non-void. On the other hand let $(\Gamma', \pi, \mathbb{H})$ be a $\mathbb{Q}_p$-triple of unbounded type and denote $L_i = \overline{\mathrm{pr}_i(\Gamma')}$. Theorem 0.2 implies that π extends continuously to $L_1 \times L_2$ factoring via, say, L_1. Thus we obtain a continuous unbounded linear representation $L_1 \to \mathbb{H}(\mathbb{Q}_p)$. By Theorem 1.8 it follows that L_1 is $\mathbb{Q}_p$-analytic. This implies that $|P_\Gamma| \in \{1, 2\}$. We treat the case where $|P_\Gamma| = 1$ (of the two cases this requires a slightly more involved argument, hence we leave the second case for the reader). Let $P_\Gamma = \{p\}$ and let $(\Gamma', \pi, \mathbb{H})$ be any $\mathbb{Q}_p$-triple of unbounded type.

Claim 2.6. *There are $\mathbb{Q}_p$-triples $(\Gamma'', \pi_i, \mathbb{H}_i)$, $1 \leq i \leq n$, of unbounded type such that $\mathbb{H}_1 = \mathbb{H}$, $\pi_1 = \pi_{|_{\Gamma''}}$ and the product homomorphism $\prod \pi_i : \Gamma'' \to \prod_{i=1}^n \mathbb{H}_i(\mathbb{Q}_p)$has Zariski dense discrete image.*

Endow $\mathbb{H}$ with the K-structure given by lemma 2.3; $\mathbb{Q} \subset K \subset \mathbb{Q}_p$. Let $\Delta : \mathbb{H}(K) \to \mathrm{Res}_{K/\mathbb{Q}} \mathbb{H}(\mathbb{Q})$ be the diagonal isomomorphism, $\mathbb{L}$ the connected component of the Zariski closure of the image of Γ' and $\Gamma'' = (\Delta \circ \pi)^{-1}(\mathbb{L}(\mathbb{Q}))$. Since p is the only prime for which $\Delta \circ \pi(\Gamma'')$ is unbounded in $\mathbb{L}(\mathbb{Q}_p)$, we deduce that, up to a subgroup of finite index, $\Delta\pi(\Gamma'')$ is contained in $\mathbb{L}(\mathbb{Z}[1/p])$ and hence discrete in $\mathbb{L}(\mathbb{Q}_p)$. If one lets now $\mathbb{H}_1, \ldots, \mathbb{H}_n$ be the $\mathbb{Q}_p$-simple adjoint quotients $p_i : \mathbb{L} \to \mathbb{H}_i$ of $\mathbb{L}$ such that $(\Gamma'', \pi_i, \mathbb{H}_i)$, with $\pi_i = p_i \circ \Delta \circ \pi$, is of unbounded type, then it is an easy verification that those fulfill the claim.

Let now $(\Gamma', \pi, \mathbb{H})$ be a $\mathbb{Q}_p$-triple of unbounded type and let $(\Gamma'', \pi, \mathbb{H}_i)$ be the triples given by the above claim. Each $\pi_i : \Gamma'' \to \mathbb{H}_i(\mathbb{Q}_p)$ extends continuously to $L_1 \times L_2$ (where $L_i = \overline{\mathrm{pr}_i(\Gamma'')}$) factoring via L_1 or L_2. WLOG, assume that for $1 \leq i \leq r$, the extension factors via L_1 and let $\overline{\pi}_i : L_1 \to \mathbb{H}_i(\mathbb{Q}_p)$ be this continuous homomorphism and for $r+1 \leq i \leq n$ it factors via L_2 giving rise to a $\overline{\pi}_i : L_2 \to \mathbb{H}_i(\mathbb{Q}_p)$. Let $\alpha_1 = \prod_{i=1}^r \overline{\pi}_i$, $\alpha_2 = \prod_{i=r+1}^n \overline{\pi}_i$. It follows from Lemma 1.5.2 and Lemma 2.5 that $\alpha({H_1}^{(\infty)}) = \prod_{i=1}^r \mathbb{H}_i^+$ and $\alpha_2({H_2}^{(\infty)}) = \prod_{i=r+1}^n \mathbb{H}_i^+$.

Claim 2.7. *$r < n$*

Indeed if $r = n$ then $\prod_{i=1}^n \pi_i : \Gamma'' \to \prod_{i=1}^n \mathbb{H}_i(\mathbb{Q}_p)$ extends continuously to $L_1 \times L_2$, the extension being given by $(\ell_1, \ell_2) \mapsto \alpha_1(\ell_1)$. This implies that $\alpha_1(\mathrm{pr}_1(\Gamma''))$ is discrete, in particular closed and hence contains $\alpha_1(\overline{\mathrm{pr}_1(\Gamma'')}) = \alpha_1(L_1) \supset \prod_{i=1}^n \mathbb{H}_i^+$ which is impossible.

Thus $r < n$ and it follows from Theorem 1.8 that both H_1 and H_2 are p-adic analytic groups. Let $\mathcal{G}_i$, $\mathbb{G}_i$ and $\mathrm{Ad}_i : H_i \to \mathrm{Aut}(\mathcal{G}_i^{n_i})$ be as in Theorem 1.8 and

$$\begin{aligned} \mathrm{Ad} = \mathrm{Ad}_1 \times \mathrm{Ad}_2 : H_1 \times H_2 &\to \mathrm{Aut}(\mathcal{G}_1^{n_1}) \times \mathrm{Aut}(\mathcal{G}_2^{n_2}) \\ {H_1}^{(\infty)} \times {H_2}^{(\infty)} &\twoheadrightarrow (\mathbb{G}_1^+)^{n_1} \times (\mathbb{G}_2^+)^{n_2} \end{aligned}$$

Claim 2.8. $\mathrm{Ad}(\Gamma)$ *is discrete.*

Since ${H_i}^{(\infty)}$ is of finite index in H_i (Theorem 1.8.1) we may replace Γ by $\Gamma' = \Gamma \cap ({H_1}^{(\infty)} \times {H_2}^{(\infty)})$. Let $\mathbb{H}$ be the j'th factor of $\mathbb{G}_1^{n_1}$, and let π be the composition of Ad with the projection on $\mathbb{H}$. Then $(\Gamma', \pi, \mathbb{H})$ is a $\mathbb{Q}_p$-triple of unbounded type to which Claim 2.6 and the subsequent construction applies. In particular we obtain

$$\begin{aligned} \alpha_1 : {H_1}^{(\infty)} &\to \prod_{i=1}^r \mathbb{H}_i, \qquad & \alpha_1({H_1}^{(\infty)}) &= \prod_{i=1}^r \mathbb{H}_i^+ \\ \alpha_2 : {H_2}^{(\infty)} &\to \prod_{i=r+1}^n \mathbb{H}_i, \qquad & \alpha_2({H_2}^{(\infty)}) &= \prod_{i=r+1}^n \mathbb{H}_i^+ \end{aligned}$$

Arguing at the level of Lie algebras we obtain quotients maps $q_1 : \mathbb{G}_1^{n_1} \to \prod_{i=1}^r \mathbb{H}_i$, $q_2 : \mathbb{G}_2^{n_2} \to \prod_{i=r+1}^n \mathbb{H}_i$ such that

$$\begin{array}{ccc} H_1^{(\infty)} & \xrightarrow{\mathrm{Ad}_1} & \mathbb{G}_1^{n_1} \\ & \searrow^{\alpha_1} & \downarrow{q_1} \\ & & \prod_{i=1}^{r} \mathbb{H}_i \end{array}$$

and

$$\begin{array}{ccc} H_2^{(\infty)} & \xrightarrow{\mathrm{Ad}_2} & \mathbb{G}_2^{n_2} \\ & \searrow^{\alpha_2} & \downarrow{q_2} \\ & & \prod_{i=r+1}^{n} \mathbb{H}_i \end{array}$$

commute. In particular $(\alpha_1 \times \alpha_2)(\Gamma') = (q_1 \times q_2)(\mathrm{Ad}(\Gamma'))$ is discrete.

Since this applies to any factor of $\mathbb{G}_1^{n_1}$ and similarly for $\mathbb{G}_2^{n_2}$, we deduce that $\mathrm{Ad}(\Gamma)$ is discrete. We deduce from the facts that $\mathrm{Ad} : {H_1}^{(\infty)} \times {H_2}^{(\infty)} \to (\mathbb{G}_1^+)^{n_1} \times (\mathbb{G}_2^+)^{n_2}$ is surjective, Γ' is a cocompact lattice and $\mathrm{Ad}(\Gamma')$ is discrete that $\mathrm{Ad}(\Gamma')$ is a cocompact lattice in $\mathbb{G}_1(\mathbb{Q}_p)^{n_1} \times \mathbb{G}_2(\mathbb{Q}_p)^{n_2}$ for which the closure of the projection on the i'th factor contains $(\mathbb{G}_i^+)^{n_2}$, for $i = 1, 2$. This implies that $\mathrm{Ad}(\Gamma')$ is an arithmetic lattice and completes the proof in the case $P_\Gamma = \{p\}$. □

Proof of Corollary 0.17

1 $\Rightarrow$ 2 is just the well known fact that finitely generated linear groups are residually finite.

2 $\Rightarrow$ 3. Since Γ is residually finite, Proposition 0.10.1 implies that $\Lambda_i = \{e\}$, $i = 1, 2$ and hence (by Theorem 0.16) we have $\mathrm{QZ}(H_i) = \ker \mathrm{Ad}_i = \{e\}$. Thus $\mathrm{Ad}_i : H_i^{(\infty)} \to \mathbb{G}_i^+$ is a topological isomorphism.By considering the action on T_i (via $(\mathrm{Ad}_i)^{-1}$) of a split Cartan subgroup of $\mathbb{G}_i$, one deduces that $\mathbb{G}_i$ has $\mathbb{Q}_{p_i}$-rank 1.

3 $\Rightarrow$ 1. The locally primitive group H_i acts via the adjoint map Ad_i on the Bruhat–Tits tree Δ_i associated with $\mathbb{G}_i$. By Lemma 3.7 it follows that this action is implemented via an isometry between the geometric realizations $|T_i|$ and $|\Delta_i|$. This shows that Ad_i is injective and hence Γ is linear. □

2.2 Proof of Theorem 0.11

Theorem 0.11 follows from Theorem 0.4 and the following:

Lemma 2.9. *Let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be a cocompact lattice such that $H_i = \overline{\operatorname{pr}_i(\Gamma)}$ are non discrete, locally quasiprimitive. Let $N \lhd \Gamma$ be a normal subgroup. Then either*

1. *$N \subset \mathrm{QZ}(H_1) \times \mathrm{QZ}(H_2)$,*

or

2. *$\overline{\operatorname{pr}_i(N)} \supset H_i^{(\infty)}$.*

Proof. The closure of a projection $\operatorname{pr}_i(N)$ is a normal subgroup of H_i and hence (by Theorem 0.6) either contains $H_i^{(\infty)}$ or is contained in $\mathrm{QZ}(H_i)$. Hence assume

$$\operatorname{pr}_1(N) \subset \mathrm{QZ}(H_1) \tag{4}$$

we have to show

$$\operatorname{pr}_2(N) \subset \mathrm{QZ}(H_2) \tag{5}$$

Assume by contradiction that (5) does not hold. Let $\Lambda_i = \Gamma \cap \operatorname{Aut} T_i$ (where we slightly abuse notation). Since $\overline{\operatorname{pr}_2(N)}$ is normal in $\overline{\operatorname{pr}_2(\Gamma)} = H_2$ we have by Theorem 0.6 that

$$\overline{\operatorname{pr}_2(N)} \supset H_2^{(\infty)} \tag{6}$$

But then (using 1.3.2 in [BM00a]) there is a finitely generated group $L \subset N$ such that $\operatorname{pr}_2(L)$ acts co-finitely on the tree T_2. Observe that $(\operatorname{pr}_i(L) \cap \Lambda_i) \lhd \operatorname{pr}_i(L)$ and that $\operatorname{pr}_1(L)/(\operatorname{pr}_1(L) \cap \Lambda_1)$ is isomorphic to $\operatorname{pr}_2(L)/(\operatorname{pr}_2(L) \cap \Lambda_2)$, indeed both are isomorphic to $L/((\ker \operatorname{pr}_1|_L)\cdot(\ker \operatorname{pr}_2|_L))$. Since $\operatorname{pr}_1(L)/(\operatorname{pr}_1(L) \cap \Lambda_1)$ is isomorphic to a subgroup of $\mathrm{QZ}(H_1)/\Lambda_1$ it is locally finite (Proposition 0.10.2). Hence also the finitely generated group $\operatorname{pr}_2(L)/(\operatorname{pr}_2(L) \cap \Lambda_2)$ is locally finite and hence finite. But then $\operatorname{pr}_2(L) \cap \Lambda_2$ would act on T_2 with a finite quotient. Hence Λ_2 and $\mathrm{QZ}(H_2)$ will be cocompact in H_2, which contradicts Theorem 0.6. Hence we conclude that $\operatorname{pr}_2(N) \subset \mathrm{QZ}(H_2)$. □

3 Geometric Rigidity

3.1

Terminology: our trees will be without leaves. Given a tree T let us denote by $|T|$ its geometric realization which is a CAT(-1) space, endowed with the structure of a 1-dimensional simplicial complex. Let $|T|^0$ denote the set of 0-cells (corresponding to the vertices of T) and every 1-cell of $|T|$ is isometric to $[0,1]$. A group action on a tree T is called c-minimal if there is no proper invariant subtree.

Theorem 3.1. *Let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be a cocompact lattice such that $H_i = \overline{\operatorname{pr}_i(\Gamma)}$ are locally primitive and non discrete. Let $\Gamma' < \operatorname{Aut} T_1' \times \operatorname{Aut} T_2'$ be such that $\Gamma' \backslash (T_1' \times T_2')$ is finite and let $\pi : \Gamma \to \Gamma'$ be a surjective homomorphism. Then up to rescaling metrics and exchanging the factors there are isometries $t_i : |T_i| \to |T_i'|$ such that $t := t_1 \times t_2$ induces the homomorphism π, in particular π is an isomorphism.*

Remark 3.2. *If we assume that π is an isomorphism we do not need to impose the condition that H_i is non-discrete.*

Corollary 3.3. *Let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be a cocompact lattice such that each $H_i = \overline{\operatorname{pr}_i(\Gamma)}$ is locally primitive and $X_\Gamma = \Gamma \backslash (T_1 \times T_2)$ be the quotient square complex. Then*

$$\operatorname{Out}(\Gamma) \cong \operatorname{Aut} X_\Gamma$$

and hence is finite.

The above results follow from a general result describing all non-elementary actions of Γ on a tree:

Theorem 3.4. *Let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be a uniform lattice such that each $H_i = \overline{\operatorname{pr}_i(\Gamma)}$ is locally primitive and $\pi : \Gamma \to \operatorname{Aut} T$ be an action of Γ on a countable tree T such that it is non-elementary and c-minimal. Then π extends continuously to $H_1 \times H_2$, factoring via one H_i and the continuous homomorphism $\pi : H_i \to \operatorname{Aut} T$ obtained is realized by an isometry $|T_i| \to |T|$.*

Let us mention one more corollary:

Corollary 3.5. *Let $\Gamma < \operatorname{Aut} T_1 \times \operatorname{Aut} T_2$ be a cocompact lattice and assume that $H_i = \overline{\operatorname{pr}_i(\Gamma)}$ is locally primitive. Let $\Lambda_1 = \Gamma \cap (\operatorname{Aut} T_1 \times e)$ and $\Lambda_2 = \Gamma \cap (e \times \operatorname{Aut} T_2)$. If both $\Lambda_1 \neq \{e\}$ and $\Lambda_2 \neq \{e\}$, then $\Gamma/(\Lambda_1 \cdot \Lambda_2)$ has property FA.*

3.2

In this subsection we shall prove Theorem 3.4. We will need the following lemmas

Lemma 3.6. *Let T be a locally finite tree, $H < \operatorname{Aut} T$ a closed locally primitive subgroup, X a complete CAT(0)-space and $H \times X \to X$ a continuous action with unbounded orbits. Then there is a continuous H-equivariant map $\alpha : |T| \to X$ whose restriction to each 1-cell is isometric and whose restriction to the star of each 0-cell of $|T|$ is injective.*

Proof. Subdividing T once we may assume that H does not contain inversions and hence $H = \langle H_\alpha \cup H_\beta \rangle$ where α, β are any pair of adjacent vertices. Since for every vertex v the subgroup $\pi(H_v)$ has bounded orbits in X, it follows that $X^{\pi(H_v)} \neq \emptyset$ and we may pick $x_\alpha \in X^{\pi(H_\alpha)}$, $x_\beta \in X^{\pi(H_\beta)}$ and define $\varphi : V \to X$ by $\varphi(h\alpha) = \pi(h)x_\alpha$, $\varphi(h\beta) = \pi(h)x_\beta$, $h \in H$, thus obtaining an H-equivariant map from V to X. Pick $v \in V$ and let $\{x_1, \ldots, x_k\}$ be the set of vertices adjacent to v. We show that $\varphi : \{v, x_1, \ldots, x_k\} \to X$ is injective. If $\varphi(v) = \varphi(x_1)$,say, then $\pi(H_v)$ and $\pi(H_{x_1})$ both fix $b := \varphi(v) = \varphi(x_1)$ and hence $\pi(H)$ fixes b, a contradiction. The map $\varphi : \{x_1, \ldots, x_k\} \to \{\varphi(x_1), \ldots, \varphi(x_k)\}$ is H_v-equivariant and hence since the H_v action is primitive, it is either injective or constant. In the latter case, let $b \in X$ denote this constant value of the image. Then each of the subgroups $\pi(H_{x_i})$, $1 \leq i \leq k$ fixes b and so does the subgroup $N = \langle \cup H_{x_i} \rangle$ generated by them. Notice that $\pi(H_v)$ normalizes N and acts with bounded orbits on the subset $X^N \subset X$ of points fixed by N. Since $X^N \ni b$ is non empty it follows that there is a point fixed by $\pi(H_v \cdot N) = \pi(H)$, a contradiction. In the sequel we pick φ such that $[x_\alpha, x_\beta] \cap X^{\pi(H_\alpha)} = \{x_\alpha\}$ and $[x_\alpha, x_\beta] \cap X^{\pi(H_\beta)} = \{x_\beta\}$. Note that such a choice is clearly possible. Let $|\varphi| : |T| \to X$ be the geodesic extension of φ to $|T|$. Let $T_v = \cup_{i=1}^k [v, x_i]$; we have to show that $|\varphi|$ is injective on T_v. We say that $x_i \sim x_j$ if $[\varphi(v), \varphi(x_i)] \cap [\varphi(v), \varphi(x_j)] = [\varphi(v), q]$ with $q \neq \varphi(v)$. Clearly this is an H_v-invariant equivalence relation and hence is either separating points, i.e., $[\varphi(v), \varphi(x_i)] \cap [\varphi(v), \varphi(x_j)] = \{\varphi(v)\}$, $\forall i \neq j$, or consists of one equivalence class, in which case we have $\cap_{i=1}^k [\varphi(v), \varphi(x_i)] = [\varphi(v), q]$ with $q \neq \varphi(v)$. In this case however the point q is $\pi(H_v)$-fixed, contradicting the construction of φ. It follows that $|\varphi|_{|T_v}$ is injective on T_v. □

Lemma 3.7. *Let T and H be as in lemma 3.6. Let T' be a countable tree, with a continuous H-action $\pi : H \to \operatorname{Aut} T'$ which is c-minimal and has unbounded orbits. Then $\pi : H \to \operatorname{Aut} T'$ is realized by an isometry $|T| \to |T'|$ (up to possible rescaling of the distance on T').*

Proof. Apply lemma 3.6 to $X = |T'|$ and let $\alpha : |T| \to |T'|$ be the H-

equivariant map given by lemma 3.6. We may assume that all the vertices of T have degree at least 3. Together with lemma 3.6 this implies that $\alpha(|T|^0) \subset |T'|^0$ hence α is locally distance preserving. Observe that $\alpha(|T|) \subset |T'|$ is an H-invariant subtree and hence using the c-minimality of the action $\alpha(|T|) = |T'|$. A surjective map which is uniformly locally isometric is necessarily a covering map. Hence α is an isometry. □

Theorem 3.4 is an immediate consequence of Lemma 3.7 and Theorem 0.3 of Monod and Shalom.

3.3

Here we complete the remaining assertions. To establish Theorem 3.1 consider $\pi_i := \mathrm{pr}_i \circ \pi : \Gamma \to \mathrm{Aut}\, T_i'$. These are non-elementary, c-minimal actions hence by Theorem 3.4 extend continuously to $\tilde{\pi}_i : H_1 \times H_2 \to \mathrm{Aut}\, T_i'$ factoring via one of the factors. Let us denote $\tilde{\pi} = \tilde{\pi}_1 \times \tilde{\pi}_2$. If both $\tilde{\pi}_1$ and $\tilde{\pi}_2$ factored via, say, H_1 then we will have (abusing notation) that $\tilde{\pi}(H_1 \times H_2) = \tilde{\pi}(H_1) \subset \overline{\tilde{\pi}(\mathrm{pr}_1(\Gamma)} = \Gamma'$. It follows that the image of H_1 is countable and hence must be finite (since the kernel would be an open normal subgroup of a locally primitive group), which is impossible. We conclude that, after possible exchanging the indexes, the map $\tilde{\pi}_i$ factors via H_i, and the resulting homomorphism $\overline{\pi_i} : H_i \to \mathrm{Aut}\, T_i'$ is realized (see Lemma 3.7) by an isometry $t_i : |T_i| \to |T_i'|$ and hence the homomorphism π is realized by the isometry $t = t_1 \times t_2$. □

The remaining corollaries 3.3 and 3.5 follow easily.

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