

# The Witt groups of the spheres away from two

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## Abstract

We calculate the Witt groups of the spheres up to 2-primary torsion.

## 1 Introduction

Let  $A$  be a Gorenstein ring of finite Krull dimension with  $\frac{1}{2} \in A$ . Denote the coordinate ring of the  $n$ -sphere over  $A$  by

$$S_A^n := A[X_1, \dots, X_{n+1}]/(X_1^2 + \dots + X_{n+1}^2 - 1),$$

and let

$$\mathbb{P}S_A^n := \text{Proj } A[X_0, \dots, X_{n+1}]/(X_0^2 + \dots + X_n^2 - X_{n+1}^2)$$

be the corresponding projective scheme.

We are interested in determining the (total) coherent Witt groups of these varieties, which are still unknown, in terms of the Witt groups of the base  $A$ . As a first step we obtain the following two theorems, with the hope one day of determining also the 2-primary torsion. We abbreviate:

$$\overline{W}^i(X) := \tilde{W}^i(X) \otimes \mathbb{Z}[1/2], \quad \overline{W}^{\text{tot}}(X) := \bigoplus_{0 \leq i \leq 3} \overline{W}^i(X).$$

**Theorem 1.1.** *For all  $i$ , we have isomorphisms*

$$\overline{W}^i(S_A^n) \simeq \overline{W}^i(\mathbb{P}S_A^n) \simeq \overline{W}^i(A) \oplus \overline{W}^{i-n}(A).$$

If the base  $A$  is regular, then  $S_A^n$  is also regular and its coherent and derived (locally free) Witt groups are isomorphic. The tensor product induces a natural structure of  $\mathbb{Z}/4$ -graded rings on  $W^{\text{tot}}(A)$  and  $W^{\text{tot}}(S^n)$ , denoted by  $\star$  (see [GN]). If  $q : \text{Spec } S^n \rightarrow \text{Spec } A$  denotes the canonical projection, then  $q^*$  makes  $W^{\text{tot}}(S^n)$  into a  $\mathbb{Z}/4$ -graded  $W^{\text{tot}}(A)$ -algebra. All this remains true of course for  $\overline{W}^{\text{tot}}$ . Our geometric proof of Theorem 1.1 lets us easily find this multiplicative structure:

**Theorem 1.2.** *Let moreover  $A$  be regular. Then*

$$\overline{W}^{\text{tot}}(S_A^n) \simeq \overline{W}^{\text{tot}}(A)[\alpha]/(\alpha^2),$$

with  $\alpha$  sitting in degree  $n$ .

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**Remarks.** (a) By Proposition 3.1 below, we see that our calculation produces non-trivial groups only when the base ring  $A$  has characteristic 0 (that is, when  $\mathbb{Z} \subset A$ ).

(b) Theorem 1.1 follows also from the general results of G.W. Brumfiel in [Br1] and [Br2]. In *loc. cit.* Brumfiel sketches the development of a theory  $\mathrm{KO}^{-n}(\mathrm{Sper} A)$  inspired by the usual (real) topological  $K$ -theory of spaces, and which depends only on the real spectrum  $\mathrm{Sper} A$  (equipped with its sheaf of abstract semi-algebraic functions). One of the main results of Brumfiel is that there are natural isomorphisms

$$W_n^K(A) \otimes \mathbb{Z}[1/2] \simeq \mathrm{KO}^{-n}(\mathrm{Sper} A) \otimes \mathbb{Z}[1/2] \quad (n \geq 0),$$

where  $W_n^K$  are Karoubi's Witt groups. By identifying  $\overline{W}^n(A) \simeq W_n^K(A) \otimes \mathbb{Z}[\frac{1}{2}]$  on the left hand side ([HS, Lemma A.3]), and by reducing Brumfiel's theory to the usual  $K$ -theory of some space on the right hand side, one can use topological results to calculate Balmer-Witt groups tensored with  $\mathbb{Z}[\frac{1}{2}]$ . We can avoid all this, our approach here being much simpler and completely geometric in nature. Our main tools will be the 12-periodic localization long exact sequence, homotopy invariance and dévissage (see [Bal] for an overview and references).

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## 2 Transfer and dévissage

We recall here some technical facts, for the convenience of the reader. Given a finite dimensional Gorenstein scheme  $X$  with  $\frac{1}{2} \in \mathcal{O}_X(X)$  and a line bundle  $\mathcal{L}$  on it, the coherent Witt groups  $\tilde{W}^i(X, \mathcal{L})$  depend on  $X$ ,  $\mathcal{L}$  and also on a choice of an injective resolution of  $\mathcal{L}$  ([Gi1]). Thus when we write  $\tilde{W}^i(X) := \tilde{W}^i(X, \mathcal{O}_X)$  we tacitly choose such a resolution. Given a finite morphism  $f : Y \rightarrow X$  of pure relative dimension  $d$ , Gille constructed transfer (push-forward) morphisms

$$f_* : \tilde{W}^i(Y, \overline{f}^* \mathrm{Ext}_{\mathcal{O}_X}^d(f_* \mathcal{O}_Y, \mathcal{O}_X)) \rightarrow \tilde{W}^{i+d}(X)$$

(see *e.g.* [Gi4, §2]). Transfers are functorial: for two composable morphisms  $f$  and  $g$ , one can choose the injective resolutions so that  $(gf)_* = g_* f_*$ .

Let now  $Y$  be a connected Gorenstein closed subscheme of  $X$  of codimension  $d$ , and let  $f : Y \hookrightarrow X$  be the inclusion. Then the transfers induce “dévissage” isomorphisms

$$f_* : \tilde{W}^i(Y, \overline{f}^* \mathrm{Ext}_{\mathcal{O}_X}^d(f_* \mathcal{O}_Y, \mathcal{O}_X)) \rightarrow \tilde{W}_Y^{i+d}(X),$$

for all  $i$  ([Gi4, Thm. 3.2]), where the groups on the right are the Witt groups with support appearing in the localization sequence.

It happens sometimes that the sheaf  $\overline{f}^* \mathrm{Ext}_{\mathcal{O}_X}^d(f_* \mathcal{O}_Y, \mathcal{O}_X)$  is isomorphic to  $\mathcal{O}_Y$ , *e.g.* when  $Y = \mathrm{Spec} A$  is affine and  $f : \mathrm{Spec} A/I \rightarrow \mathrm{Spec} A$  is a closed immersion whose defining ideal  $I$  can be given by a regular sequence  $(a_1, \dots, a_d)$  in  $A$ . In this case, every choice of a sequence gives an isomorphism  $\mathcal{O}_Y \simeq \overline{f}^* \mathrm{Ext}_{\mathcal{O}_Y}^d(j_* \mathcal{O}_X, \mathcal{O}_Y)$  by sending  $1 \in \mathcal{O}_Y$  to the Koszul complex of  $A/(a_1, \dots, a_d)$ , and therefore an isomorphism of Witt groups

$$\phi : \tilde{W}^i(Y) \rightarrow \tilde{W}^i(Y, \overline{f}^* \mathrm{Ext}_{\mathcal{O}_X}^d(f_* \mathcal{O}_Y, \mathcal{O}_X)).$$

In what follows we will make this choice tacitly and will still write  $f_* : \tilde{W}^i(Y) \simeq \tilde{W}_Y^{i+d}(X)$  for the composite isomorphism  $f_* \circ \phi$ .

Observe also that for flat morphisms  $g : X \rightarrow Y$  the usual contravariant functoriality of derived Witt groups, written  $g^* : W^i(Y) \rightarrow W^i(X)$ , works just as well for coherent Witt groups.

### 3 Proof of Theorem 1.1

The case  $n = 0$  is trivial. We have isomorphisms

$$\mathrm{Spec} S^0 \simeq \mathrm{Proj} A[X_0, X_1]/(X_0^2 - X_1^2) \simeq \mathrm{Spec}(A \times A)$$

and thus  $\tilde{W}^i(S^0) \simeq \tilde{W}^i(\mathbb{P}S^0) \simeq \tilde{W}^i(A) \oplus \tilde{W}^i(A)$ .

For the rest of the proof we can assume  $n \geq 1$ . The following Proposition is the main component of our calculation.

**Proposition 3.1.** *Let  $X$  be a finite dimensional Gorenstein scheme with  $\frac{1}{2} \in \mathcal{O}_X(X)$ , such that none of its residue fields admits a total ordering (i.e., it has no ‘real points’). Let  $Z \subset X$  be a closed subset, and let  $\mathcal{L}$  be a line bundle over  $X$ . Then all the coherent Witt groups  $\tilde{W}_Z^i(X, \mathcal{L})$  of  $X$  with support in  $Z$  and values in  $\mathcal{L}$  are 2-primary torsion groups.*

*Proof.* For any field  $K$ , the classical Pfister local-global principle (see e.g. [Lam, VIII Thm. 3.2]) implies that if  $K$  is nonreal (i.e., iff  $K$  doesn’t admit a total ordering, iff  $-1$  is a sum of squares in  $K$ ) then  $W(K)$  is a 2-primary torsion group. The groups in the initial page of the Gersten-Witt spectral sequence for  $(X, \mathcal{L})$  with support in  $Z$  (see Balmer-Walter [BW, Thm. 7.2], Gille [Gi1, Thm. 3.14]) consist of sums of Witt groups  $W(k(x))$  of the residue fields of  $X$ , thus with the above hypothesis they are 2-primary torsion. But the spectral sequence converges to the groups  $\tilde{W}_Z^i(X, \mathcal{L})$  because  $X$  is finite dimensional.  $\square$

**Corollary 3.2.** *Let  $X$  be a finite dimensional separated Gorenstein scheme with  $\frac{1}{2} \in \mathcal{O}_X(X)$ , and let  $X_{\mathbb{Q}} := X \times_{\mathrm{Spec} \mathbb{Z}} \mathrm{Spec} \mathbb{Q}$ . Then the canonical projection  $X_{\mathbb{Q}} \rightarrow X$  induces isomorphisms  $\overline{W}^i(X) \simeq \overline{W}^i(X_{\mathbb{Q}})$  for all  $i$ .*

*Proof.* Let first  $X = \mathrm{Spec} R$  be affine. Notice that a ring  $R$  satisfies the non-reality hypothesis of Proposition 3.1 if  $-1$  is a sum of squares in  $R$ . Thus we can assume that the characteristic of  $R$  is 0, the other case being trivial. Consider for  $m \geq 2$  the localization long exact sequence

$$\cdots \longrightarrow \tilde{W}_{(m)}^i(A) \longrightarrow \tilde{W}^i(A) \longrightarrow \tilde{W}^i(A[1/m]) \longrightarrow \cdots .$$

Proposition 3.1 implies that  $\tilde{W}_{(m)}^i(A) \simeq 0$ , so we obtain  $\overline{W}^i(A) \simeq \overline{W}^i(A[1/m])$ . Now  $A \otimes \mathbb{Q} \simeq \mathrm{colim}_m A[1/m]$  and Witt groups commute with filtering colimits ([Gi2, Thm. 1.6]), therefore  $\overline{W}^i(A) \simeq \overline{W}^i(A \otimes \mathbb{Q})$ . For  $X$  separated, the global case can be obtained with Mayer-Vietoris.  $\square$

**Notation.** In the following, we will write

$$C^n = C_A^n := \frac{A[X_1, \dots, X_n]}{(1 + X_1^2 + \cdots + X_n^2)}.$$

An immediate corollary of Proposition 3.1 is that  $\overline{W}^i(C^n) \simeq 0$  for all  $i$ .

Now, the intersection of the projective sphere  $\mathbb{P}S^n = V(\sum_k X_k^2 - X_{n+1}^2) \subset \mathbb{P}^{n+1}$  with the affine open  $D(X_{n+1}) = \{X_{n+1} \neq 0\} \subset \mathbb{P}^{n+1}$  is isomorphic to the affine sphere  $\text{Spec } S^n$ . We have

**Lemma 3.3.** *The open immersion  $\text{Spec } S^n \simeq D(X_{n+1}) \hookrightarrow \mathbb{P}S^n$  induces isomorphisms  $\overline{W}^i(\mathbb{P}S^n) \simeq \overline{W}^i(S^n)$ .*

*Proof.* We have a localization sequence

$$\cdots \longrightarrow \tilde{W}_{(X_{n+1})}^i(\mathbb{P}S^n) \longrightarrow \tilde{W}^i(\mathbb{P}S^n) \longrightarrow \tilde{W}^i(S^n) \longrightarrow \cdots \quad (1)$$

Denote by  $\mathbb{P}C^n$  the closed projective subscheme  $V(X_{n+1}) \subset \mathbb{P}S^n$ , i.e.  $\mathbb{P}C^n = V(X_0^2 + \cdots + X_n^2) \subset \mathbb{P}^n$ . By dévissage we have isomorphisms

$$\tilde{W}_{(X_{n+1})}^i(\mathbb{P}S^n) \simeq \tilde{W}^{i-1}(\mathbb{P}C^n, \mathcal{L}_n), \quad (2)$$

where  $\mathcal{L}_n$  is some line bundle over  $\mathbb{P}C^n$  ( $n \geq 1$ ). On the affine opens  $D(X_i) = \{X_i \neq 0\}$  ( $i = 0, \dots, n$ ), the scheme  $\mathbb{P}C^n$  is isomorphic to  $C^n$ ; thus  $\mathbb{P}C^n$  clearly satisfies the hypothesis of Proposition 3.1 and therefore  $\overline{W}^i(\mathbb{P}C^n, \mathcal{L}_n) = 0$ . Then the exactness of (1) implies the claim.  $\square$

**Lemma 3.4.** *Write  $f := 1 + X_1^2 + \cdots + X_n^2 \in A[X_1, \dots, X_n]$ , and denote by  $\mathbb{A}_f^n$  the corresponding open subscheme in the affine plane. Then we have isomorphisms  $S_{1-X_{n+1}}^n \simeq \mathbb{A}_f^n$ .*

*Proof.* The isomorphism is given by the stereographic projection. Indeed, writing  $U := S_{1-X_{n+1}}^n$  and  $V := A[Y_1, \dots, Y_n]_{1+Y_1^2+\dots+Y_n^2}$  for the localized rings, we define a morphism  $V \rightarrow U$  by

$$Y_\ell \mapsto \frac{X_\ell}{1 - X_{n+1}} \quad (\ell = 1, \dots, n).$$

Its inverse is given by

$$X_{n+1} \mapsto \frac{\sum_k Y_k^2 - 1}{\sum_k Y_k^2 + 1}, \quad X_\ell \mapsto \frac{2Y_\ell}{\sum_k Y_k^2 + 1} \quad (\ell = 1, \dots, n).$$

(Recall that 2 is invertible in  $A$ .)  $\square$

**Lemma 3.5.** *The projection  $p : \mathbb{A}_f^n \rightarrow \text{Spec } A$  induces an isomorphism  $p^* : \overline{W}^i(A) \simeq \overline{W}^i(\mathbb{A}_f^n)$ .*

*Proof.* We use the localization long exact sequence:

$$\cdots \longrightarrow \overline{W}_{(f)}^i(\mathbb{A}^n) \longrightarrow \overline{W}^i(\mathbb{A}^n) \longrightarrow \overline{W}^i(\mathbb{A}_f^n) \longrightarrow \cdots$$

Since  $f = 1 + X_1^2 + \cdots + X_n^2$  is a regular element, by dévissage we have

$$\overline{W}_{(f)}^i(\mathbb{A}^n) \simeq \overline{W}^i(C^n).$$

Because of Proposition 3.1, the latter is zero. Now we use homotopy invariance of coherent Witt groups to conclude.  $\square$

**Lemma 3.6.** *There are split short exact sequences*

$$0 \longrightarrow \overline{W}_{(1-X_{n+1})}^i(S^n) \longrightarrow \overline{W}^i(S^n) \longrightarrow \overline{W}^i(A) \longrightarrow 0.$$

*Proof.* Consider the long exact sequence associated to the localization  $S^n \rightarrow S_{1-X_{n+1}}^n$ :

$$\cdots \rightarrow \overline{W}_{(1-X_{n+1})}^i(S^n) \rightarrow \overline{W}^i(S^n) \rightarrow \overline{W}^i(S_{1-X_{n+1}}^n) \rightarrow \overline{W}_{(1-X_{n+1})}^{i+1}(S^n) \rightarrow \cdots,$$

and consider the projection  $q : \text{Spec } S^n \rightarrow \text{Spec } A$ . We then have a commutative diagram

$$\begin{array}{ccc} \text{Spec } S^n & \xleftarrow{i} & \text{Spec } S_{1-X_{n+1}}^n \\ & \searrow q & \downarrow p \\ & & \text{Spec } A \end{array}$$

where  $i$  is the inclusion. Using Lemma 3.5 and the fact that  $\text{Spec } S_{1-X_{n+1}}^n \simeq \mathbb{A}_f^n$  (Lemma 3.4), we obtain a commutative diagram

$$\begin{array}{ccc} \overline{W}^i(S^n) & \xrightarrow{i^*} & \overline{W}^i(\mathbb{A}_f^n) \\ & \swarrow q^* & \uparrow p^* \simeq \\ & & \overline{W}^i(A) \end{array}$$

The above long exact sequence yields finally the split sequences

$$0 \longrightarrow \overline{W}_{(1-X_{n+1})}^i(S^n) \longrightarrow \overline{W}^i(S^n) \longrightarrow \overline{W}^i(A) \longrightarrow 0.$$

□

**Corollary 3.7.** *For any  $i$ , we have  $\overline{W}^i(S^n) \simeq \overline{W}^i(A) \oplus \overline{W}_{(1-X_{n+1})}^i(S^n)$ .*

Next we compute  $\overline{W}_{(1-X_{n+1})}^i(S^n)$ . In order to do this, we introduce some more notation:

**Notation.**

$$B^n = B_A^n := \frac{A[X_1, \dots, X_n]}{(X_1^2 + \dots + X_n^2)}.$$

Thus we have isomorphisms  $S^n/(1-X_{n+1}) \simeq B^n$ . For  $n \geq 1$ , the element  $1-X_{n+1} \in S_n$  is regular and we can use dévissage to obtain:

$$\tilde{W}_{(1-X_{n+1})}^i(S^n) \simeq \tilde{W}^{i-1}(B^n). \quad (3)$$

Notice *en passant* that the rings  $B^n$  are singular. This makes the use of coherent (rather than derived) Witt groups necessary for our calculation, even when  $A$  is regular.

**Lemma 3.8.** *For  $n \geq 1$  and any  $i$ , we have  $\overline{W}^i(B^n) \simeq \overline{W}^{i-n+1}(A)$ .*

*Proof.* We will exploit the recursive property  $B^n/(X_n) \simeq B^{n-1}$ . Consider the long exact sequence

$$\cdots \longrightarrow \tilde{W}_{(X_n)}^i(B^n) \longrightarrow \tilde{W}^i(B^n) \longrightarrow \tilde{W}^n(B_{X_n}^n) \longrightarrow \cdots$$

associated to the localization  $B^n \rightarrow B_{X_n}^n$ . Notice that  $B_{X_n}^n \simeq C^{n-1}[X_n, X_n^{-1}]$ . It is a result of Gille that  $\tilde{W}^i(R[T, T^{-1}]) \simeq \tilde{W}^i(R) \oplus \tilde{W}^i(R)$  for  $R$  a finite dimensional Gorenstein ring (see [G1, Thm. 5.6]). Thus we have

$$\overline{W}^i(B_{X_n}^n) \simeq \overline{W}^i(C^{n-1}) \oplus \overline{W}^i(C^{n-1}) = 0,$$

where the vanishing is due to Proposition 3.1. For  $n \geq 2$ , the element  $X_n \in B^n$  is regular, so dévissage yields

$$\tilde{W}_{(X_n)}^i(B^n) \simeq \tilde{W}^{i-1}(B^n/X_n) = \tilde{W}^{i-1}(B^{n-1}).$$

Altogether, we obtain the formula  $\overline{W}^i(B^n) \simeq \overline{W}^{i-1}(B^{n-1})$  ( $n \geq 2$ ). We finish the proof by remarking that  $\tilde{W}^j(B^1) \simeq \tilde{W}^j(A)$  for all  $j$ . (For example, one can use the generalization of affine dévissage to zero dimensional ideals, see [Gi2, Thm. 3.5]:  $\tilde{W}^j(A[X]/X^2) = \tilde{W}_{(X)}^j(A[X]/X^2) \simeq \tilde{W}^j((A[X]/X^2)/X) = \tilde{W}^j(A)$ .)  $\square$

Finally, we have  $\overline{W}^i(S^n) \simeq \overline{W}^i(A) \oplus \overline{W}_{1-X_{n+1}}^i(S^n)$  by Corollary 3.7 and  $\overline{W}_{1-X_{n+1}}^i(S^n) \simeq \overline{W}^{i-n}(A)$  by (3) and the last Lemma. Together with Lemma 3.3, this ends to proof of Theorem 1.1.

## 4 Proof of Theorem 1.2

From now on, the base  $A$  and thus also  $S_A^n$  will be assumed to be regular rings. We will still denote by  $q : \text{Spec } S^n \rightarrow \text{Spec } A$  the structure morphism.

For any ring  $R$ , we will denote by  $e_R \in W^0(R)$  the multiplicative unit of  $W^{\text{tot}}(R)$ ; this is just the diagonal form  $\langle 1 \rangle = [\text{id} : R \rightarrow R]$  (we make the usual identification  $R = R^\vee$ ). In this proof we will also abbreviate

$$P := S^n/(X_1, \dots, X_n)$$

$$\mathfrak{n} := (X_{n+1} - 1) \quad \mathfrak{s} := (X_{n+1} + 1),$$

so that  $P/\mathfrak{n} \simeq A$  and  $P/\mathfrak{s} \simeq A$  are the North Pole and the South Pole of the sphere. Write  $i_P$ ,  $i_N$  and  $i_S$  for the corresponding closed immersions. We will further write

$$\alpha_N := i_{N*}(e_{P/\mathfrak{n}}) \in W^n(S^n), \quad \alpha_S := i_{S*}(e_{P/\mathfrak{s}}) \in W^n(S^n),$$

and we will keep the same notation for the images in  $\overline{W}^n(S^n)$  of these forms.

The next Lemma is just a corollary of the proof of Theorem 1.1.

**Lemma 4.1.**

$$\overline{W}^{\text{tot}}(S^n) = \overline{W}^{\text{tot}}(A) \cdot e_{S^n} \oplus \overline{W}^{\text{tot}}(A) \cdot \alpha_N$$

*Proof.* From the proof of Lemma 3.6 we see that  $q^* : \overline{W}^{\text{tot}}(A) \rightarrow \overline{W}^{\text{tot}}(S^n)$  is injective. Since  $q^*(e_A) = e_{S^n}$ , we recognize the first direct summand. By the proof of Lemma 3.8 and functoriality of the transfer (see §2 above), we see that the other summand is the image of  $i_{N*} : \overline{W}^{\text{tot}}(P/\mathfrak{n}) \rightarrow \overline{W}^{\text{tot}}(S^n)$ . This image is  $\overline{W}^{\text{tot}}(A) \cdot \alpha_N$  by the following Lemma.  $\square$

**Lemma 4.2.** *Let  $i : \text{Spec } A \hookrightarrow \text{Spec } S^n$  be the closed immersion corresponding either to the North or the South pole, and let  $i_* : W^j(A) \rightarrow W^{j+n}(S^n)$  be the induced transfer morphism. Then for every form  $\beta \in W^j(A)$ :*

$$i_*(\beta) = q^*(\beta) \star i_*(e_A) \in W^{j+n}(S^n).$$

*Proof.* This follows from the projection formula for coherent Witt groups (Gille [Gi3, Thm. 5.2]), applied to the finite morphism  $i$ :

$$i_*(\beta) = i_* \left( \underbrace{i^* q^*}_{\text{id}}(\beta) \star e_A \right) = q^*(\beta) \star i_*(e_A).$$

$\square$

**Proposition 4.3.** *The relation  $\alpha_N = -\alpha_S$  holds in  $W^n(S^n)$ .*

As an immediate consequence of this, the form  $\alpha_N^2 = -\alpha_N \star \alpha_S$  is supported on the intersection of the North and the South Poles, which is empty:  $\text{Spec } P/\mathfrak{n} \cap \text{Spec } P/\mathfrak{s} = \emptyset$ ; so it is trivially equal to zero.

To prove Proposition 4.3, we first recall straight from [Gi3, §9] some facts about forms on Koszul complexes. For any ring  $R$  and any regular sequence  $(x_1, \dots, x_n)$  in  $R$ , we will denote by  $K_\bullet(x_1, \dots, x_n)$  the Koszul complex for this sequence, and we set it in (homological) degrees from  $n$  to  $0$ . (Below we will specialize to the ring  $S^n$  and the regular sequence  $(X_1, \dots, X_n)$  in it.) For  $1 \leq i \leq n$  and any unit  $r \in R^\times$ , the complex  $K_\bullet(x_i)$  can be equipped by the following symmetric 1-form:

$$\begin{array}{ccccccc} K_\bullet(x_i) : & \cdots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot x_i} & R & \longrightarrow & 0 & \longrightarrow & \cdots \\ & & & \downarrow & & \downarrow \cdot r & & \downarrow \cdot (-r) & & \downarrow & & \\ & & & K_\bullet(x_i) : & \cdots & \longrightarrow & 0 & \longrightarrow & R & \xrightarrow{\cdot (-x_i)} & R & \longrightarrow & 0 & \longrightarrow & \cdots \end{array}$$

For any choice of  $n$  units in  $R$ , the product  $(K_\bullet(x_1), \ell_r) \star \cdots \star (K_\bullet(x_n), \ell_r)$  is a symmetric  $n$ -space on the complex  $K_\bullet(x_1, \dots, x_n) \simeq K_\bullet(x_1) \otimes \cdots \otimes K_\bullet(x_n)$ . We denote its form by  $\ell_{r_1 \dots r_n}$ .

By displaying an explicit Lagrangian, it is easy to see that  $[K_\bullet(x_i), \ell_{r_i}] = 0 \in W^n(R)$ , and therefore  $[K_\bullet(x_1, \dots, x_n), \ell_{r_1 \dots r_n}] = 0 \in W^n(R)$  for all symmetric spaces as above.

**Lemma 4.4.** *If  $\phi : K_\bullet(x_1, \dots, x_n) \rightarrow \text{Hom}_R(K_\bullet(x_1, \dots, x_n), R)[n]$  is a quasi-isomorphism of complexes, then there exists a unit  $r \in R^\times$  such that  $\ell_{r \cdot 1 \dots 1}$  is chain homotopic to  $\phi$ .*

*Proof.* This is a slight generalization of [Gi3, Lemma 9.1]. The same proof goes through.  $\square$

We have an isomorphism  $W^0(P) \simeq W^0(P/\mathfrak{n}) \oplus W^0(P/\mathfrak{s})$ , induced by  $P \simeq P/\mathfrak{n} \times P/\mathfrak{s}$ , which identifies  $e_P$  with  $(e_N, e_S)$ . Under this isomorphism, the transfer  $i_{P^*} : W^0(P) \rightarrow W^n(S^n)$  identifies with  $(i_{N^*}, i_{S^*})$ , and in particular

$$\alpha_N + \alpha_S = i_{N^*}(e_N) + i_{S^*}(e_S) = i_{P^*}(e_P).$$

But the last term is zero in  $W^n(S^n)$ . In fact,  $i_{P^*}(e_P)$  can be represented by  $(F_\bullet, \psi)$ , where  $F_\bullet$  is a projective resolution of the  $S^n$ -module  $P = S^n/(X_1, \dots, X_n)$ , and where  $\psi$  is a symmetric quasi-isomorphism between  $F_\bullet$  and its  $n$ -shifted dual, lying above the morphism  $\text{id} : A \rightarrow A$ . Since  $(X_1, \dots, X_n)$  is a regular sequence in  $S^n$ , we can take  $F_\bullet$  to be the Koszul complex for this sequence. By the above Lemma (or by direct inspection) we have  $i_{A^*}(e_A) = [K_\bullet(X_1, \dots, X_n), \ell_1] = 0 \in W^n(S^n)$ .

This ends the proof of our second Theorem.

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