

# The best of all possible worlds

The idea of optimization

Ivar Ekeland

Pacific Institute of Mathematical Sciences

<http://www.pims.math.ca/~ekeland/>

ICIAM 2007

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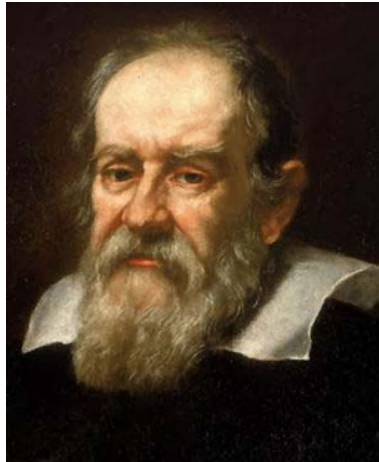
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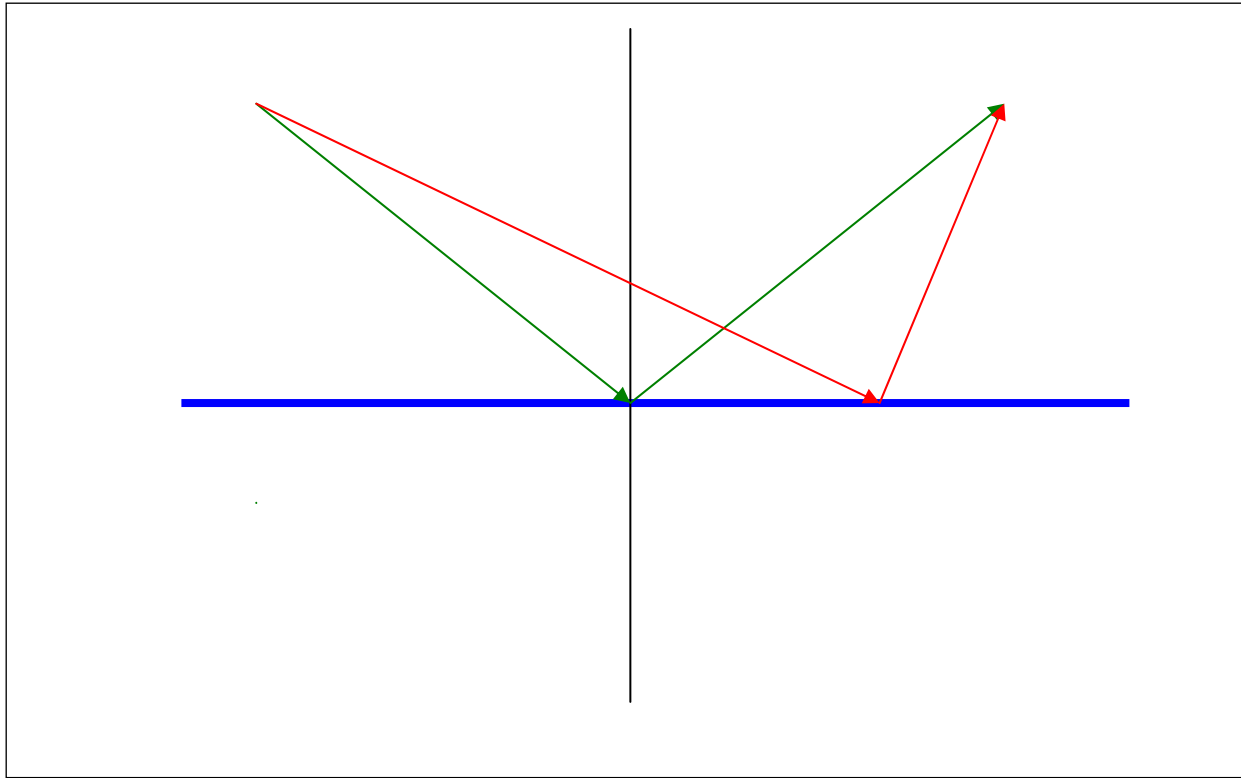
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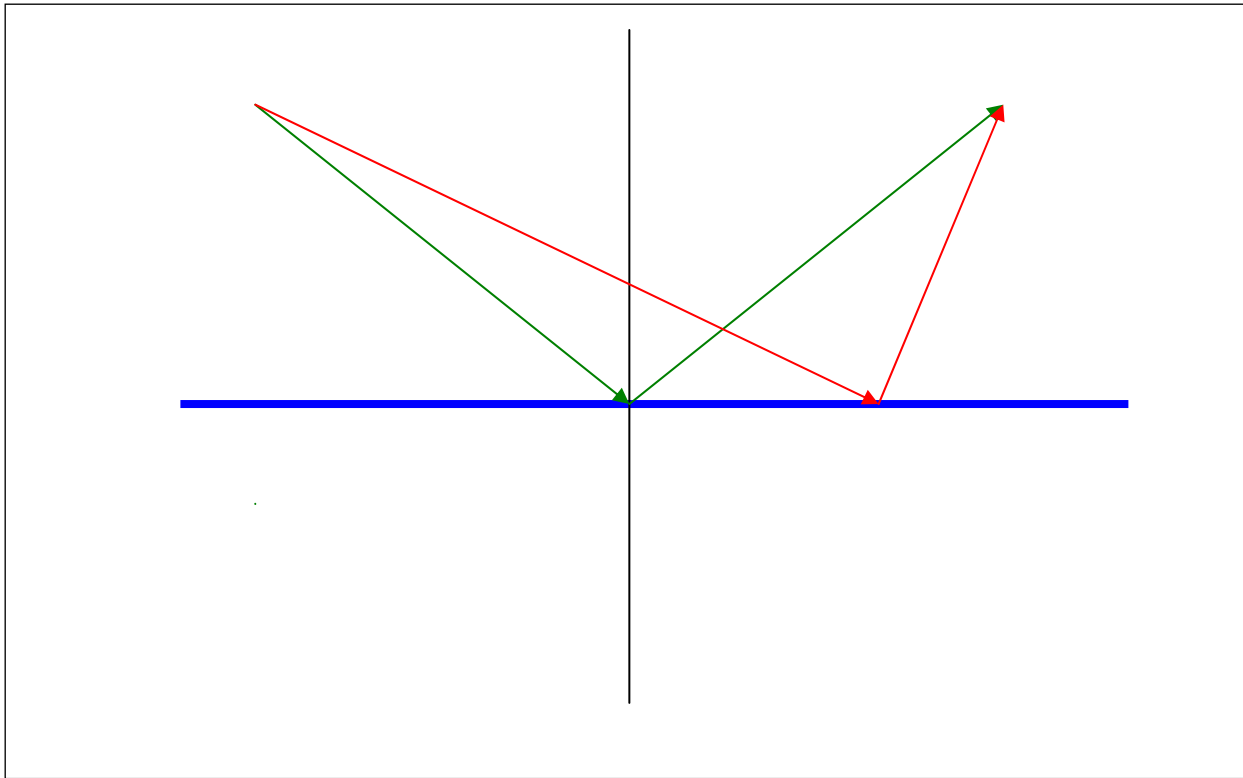


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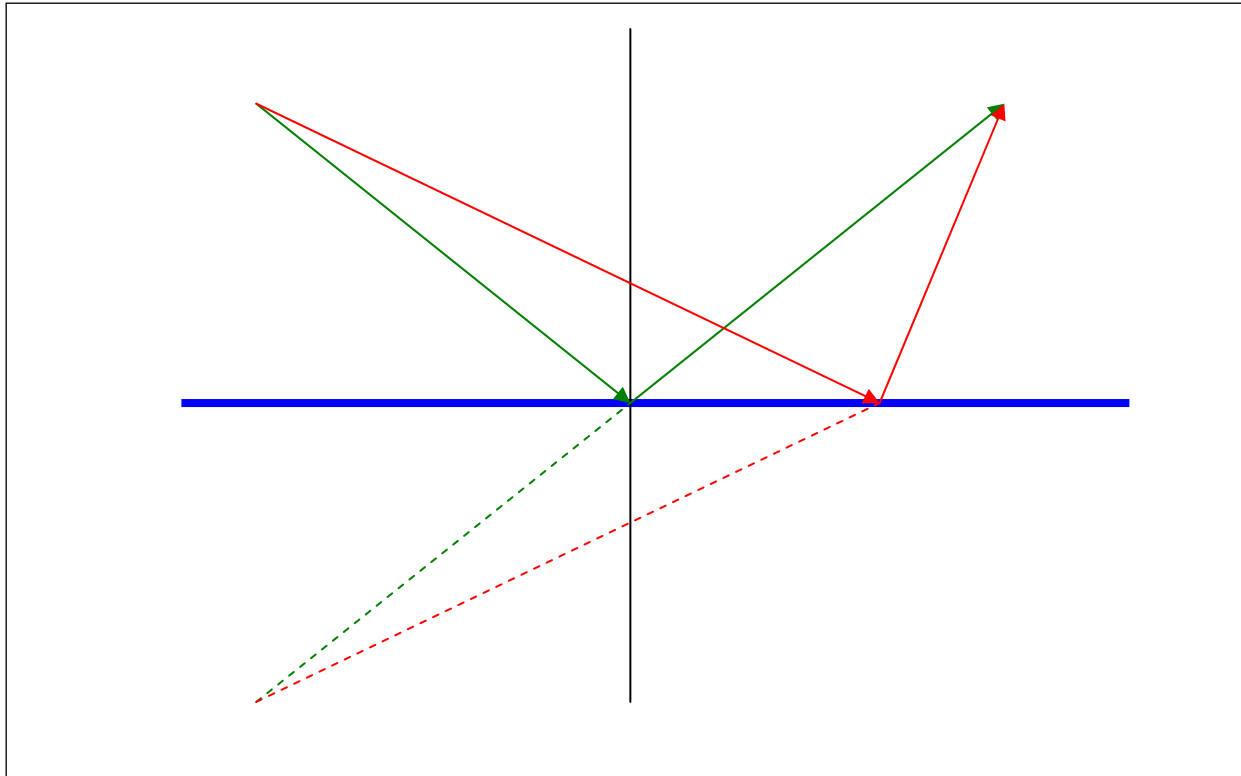
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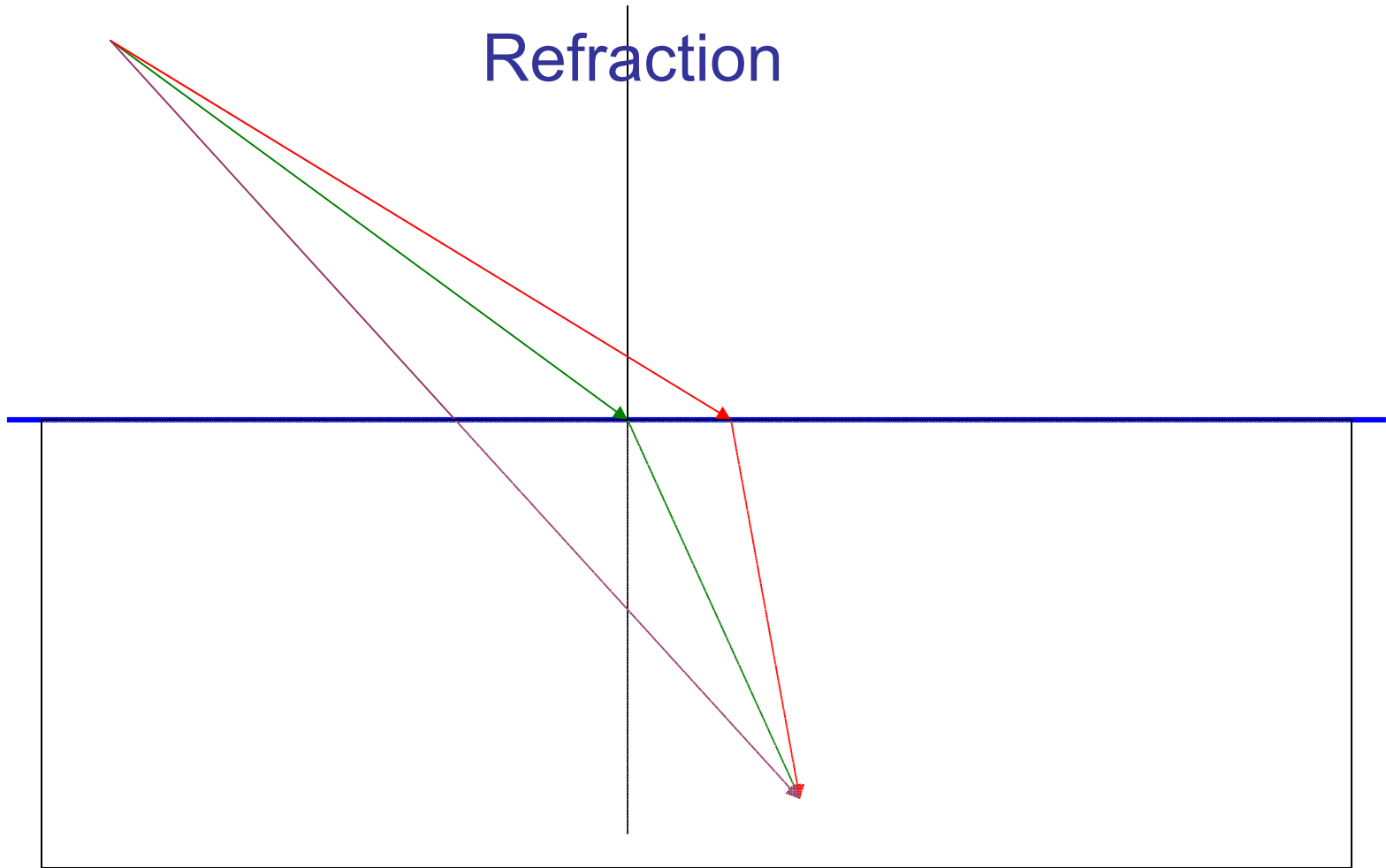
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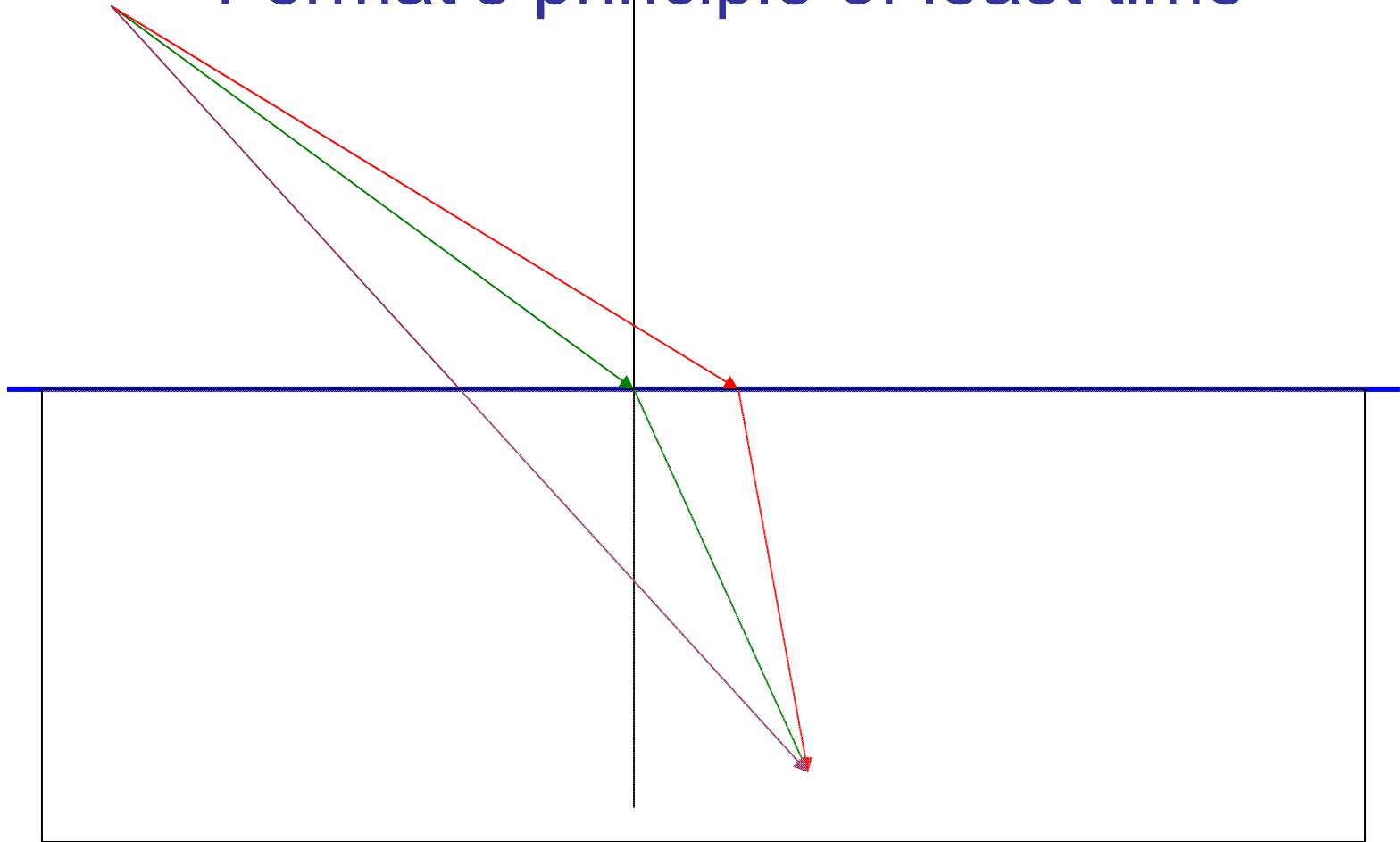
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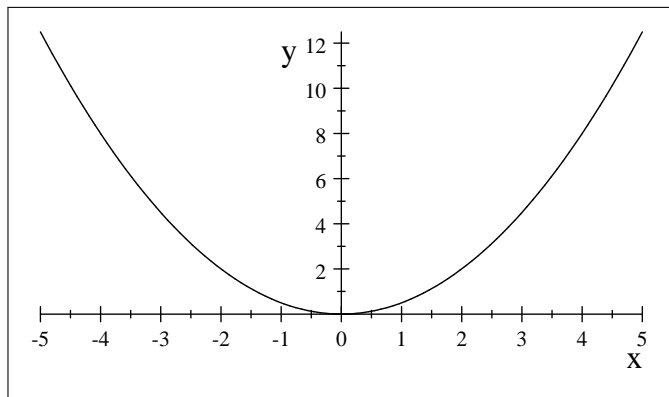
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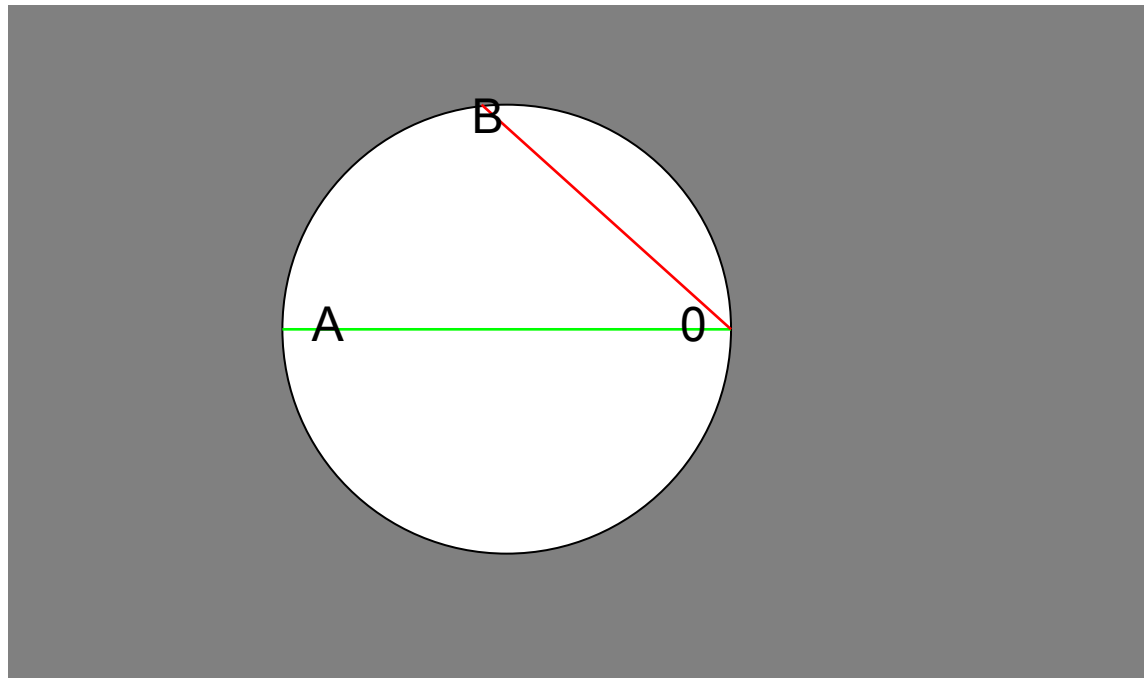
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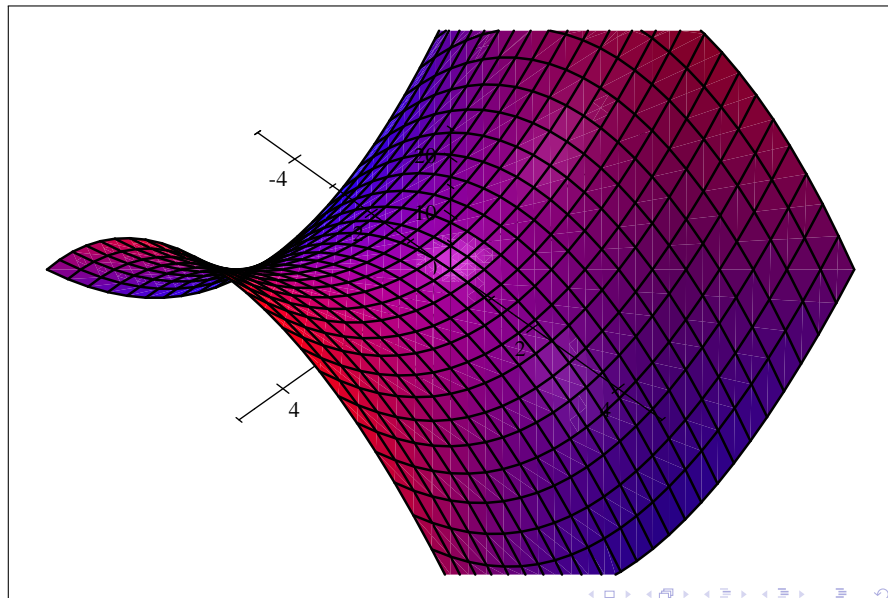
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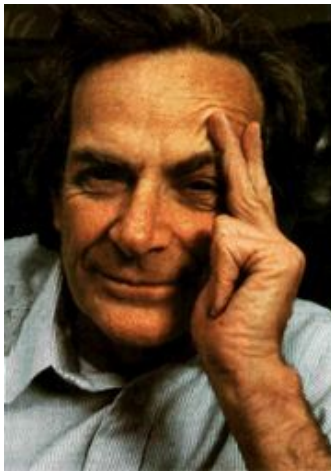


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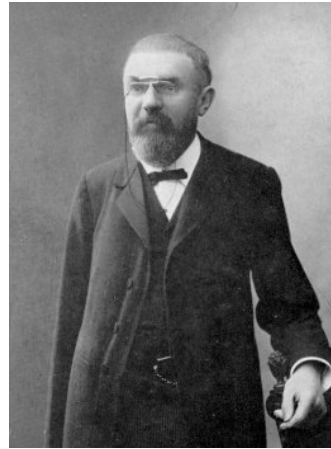
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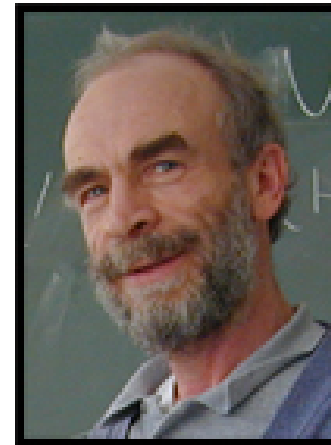
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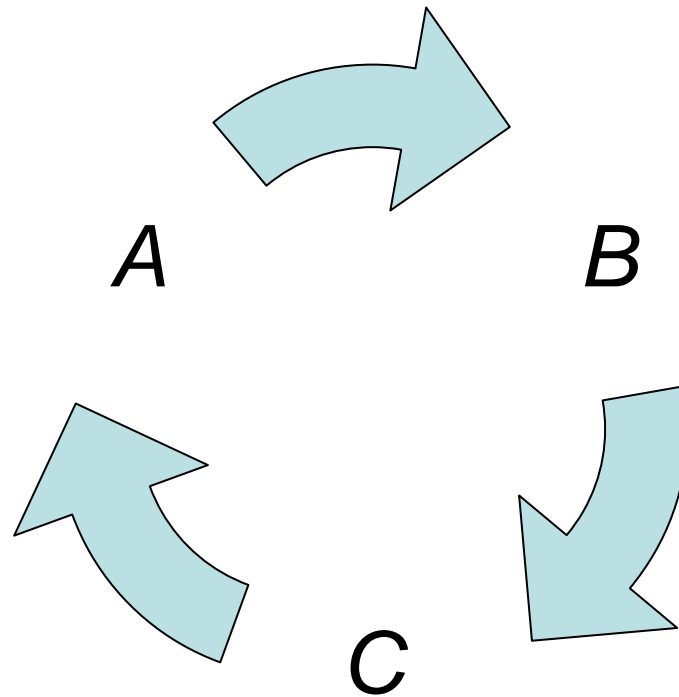
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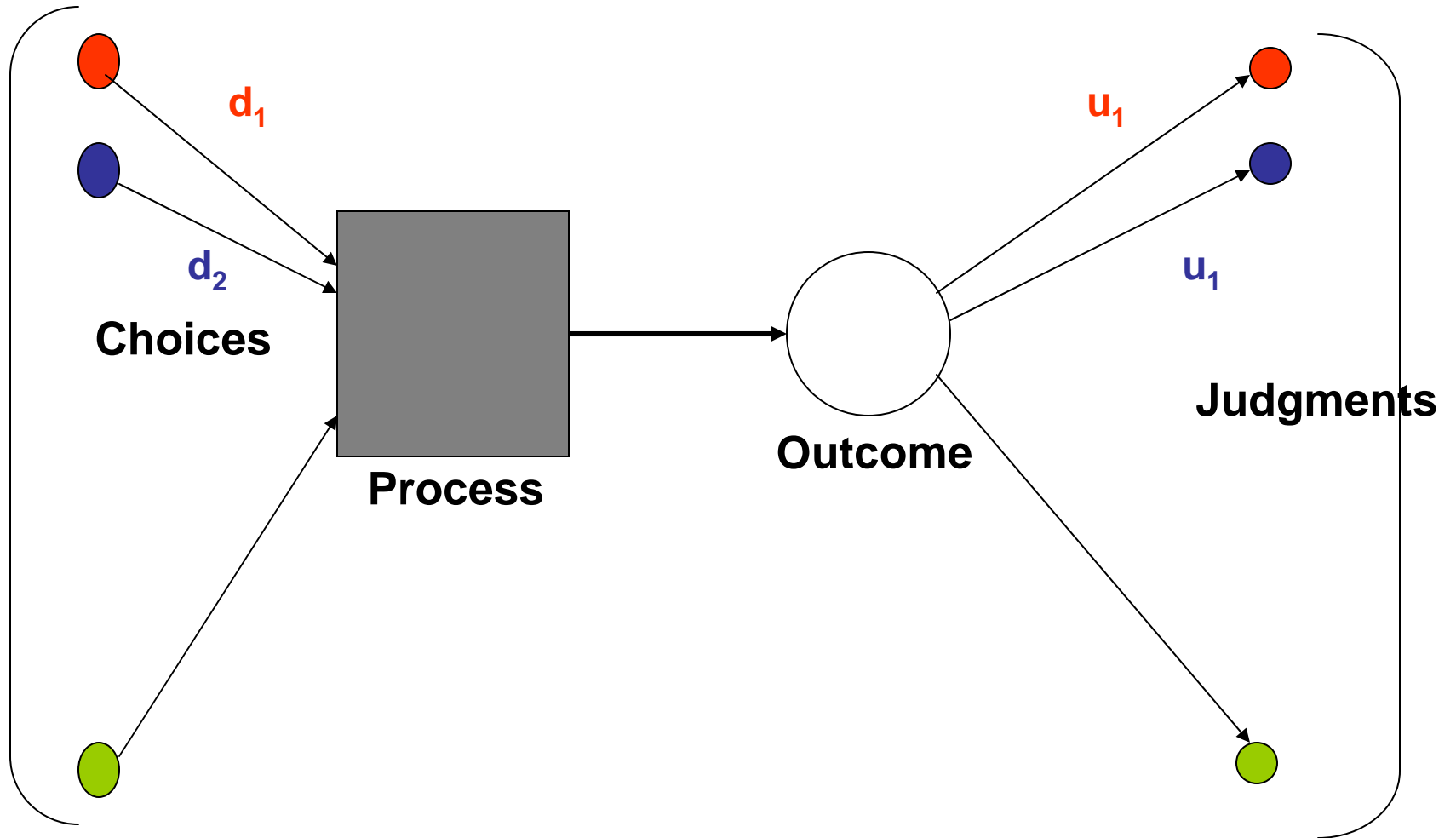
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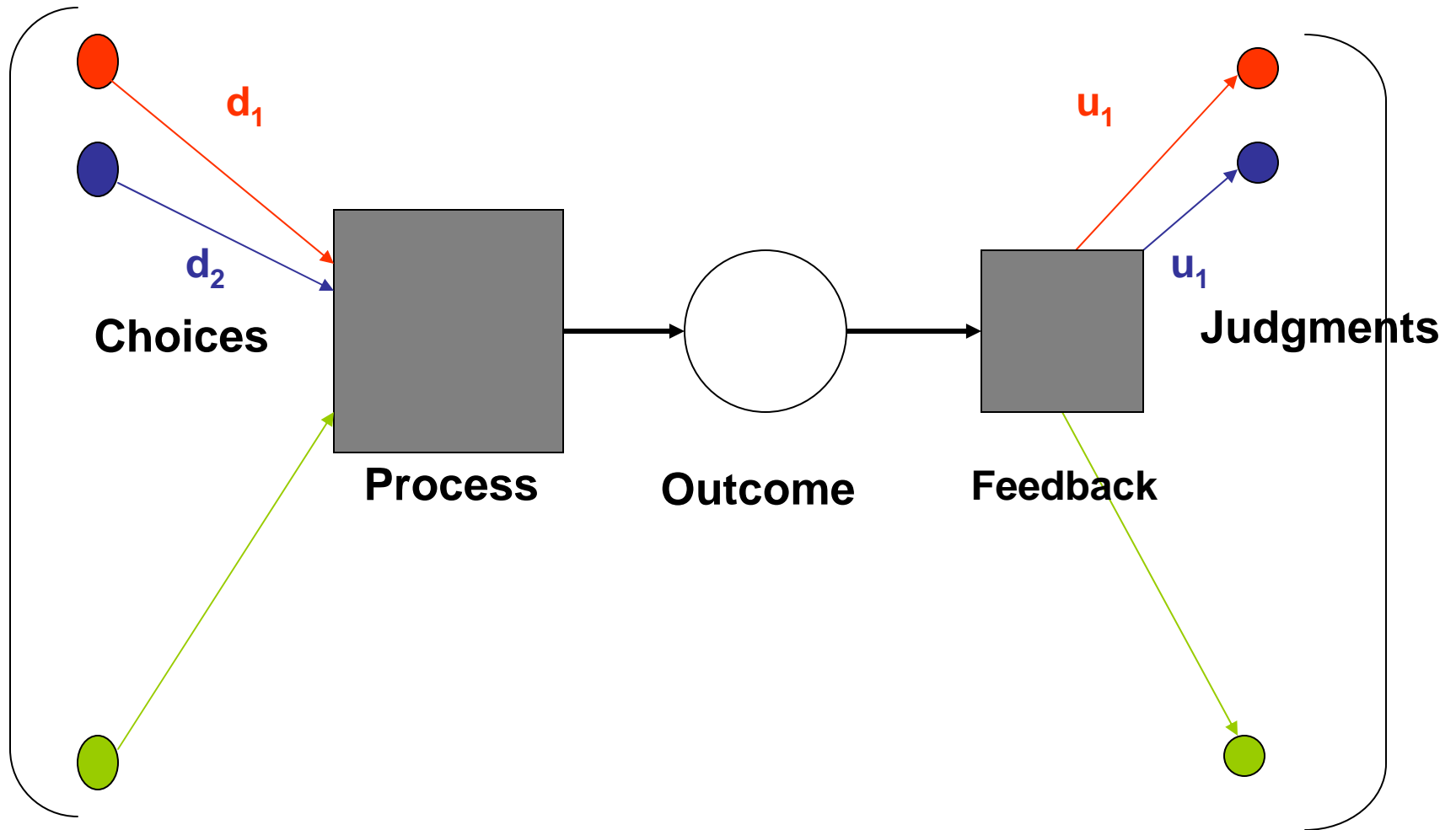
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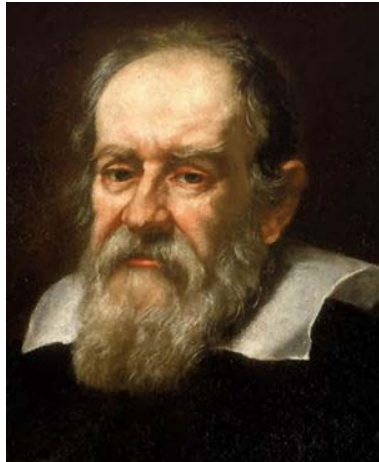
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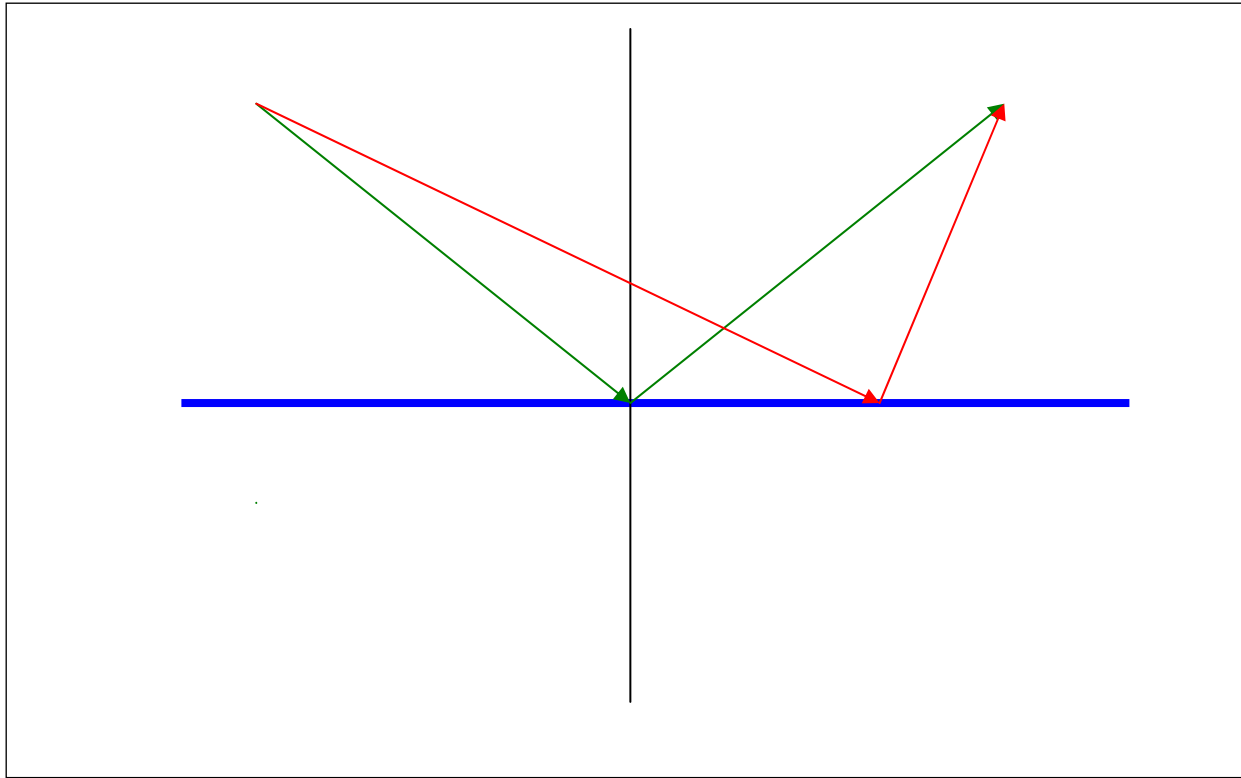
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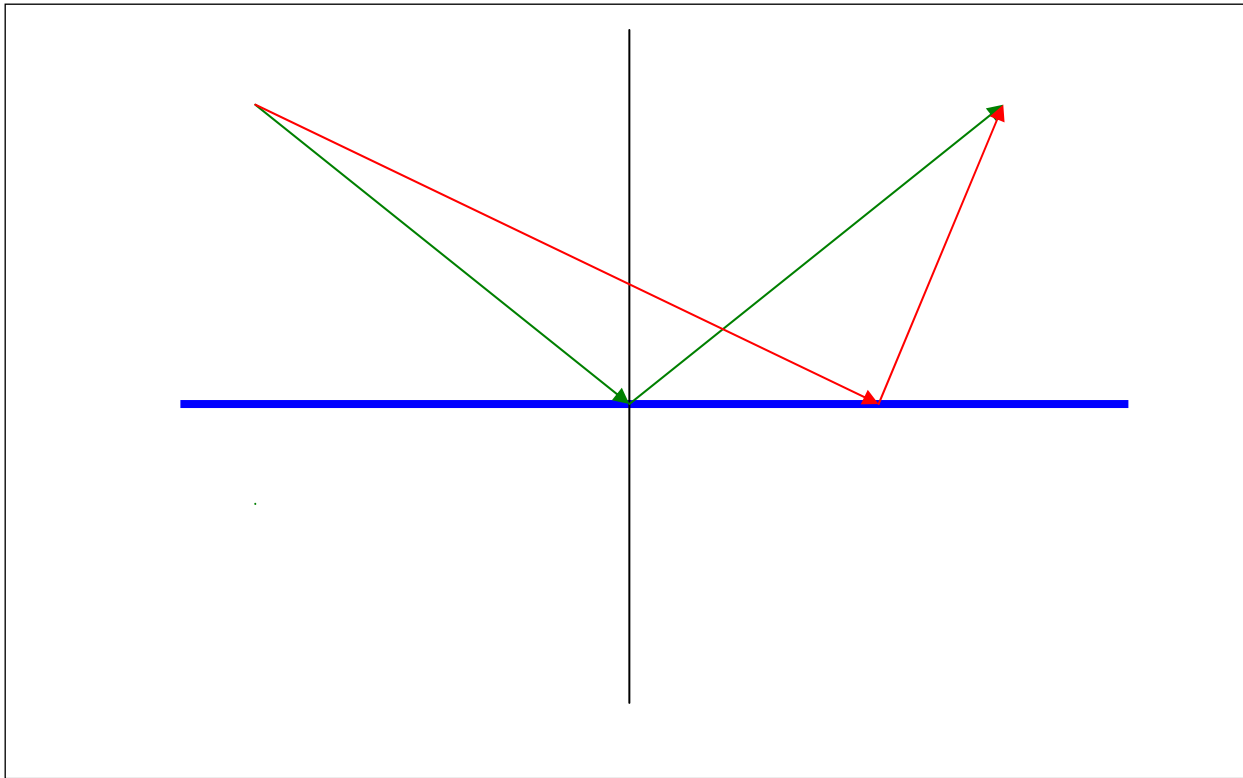


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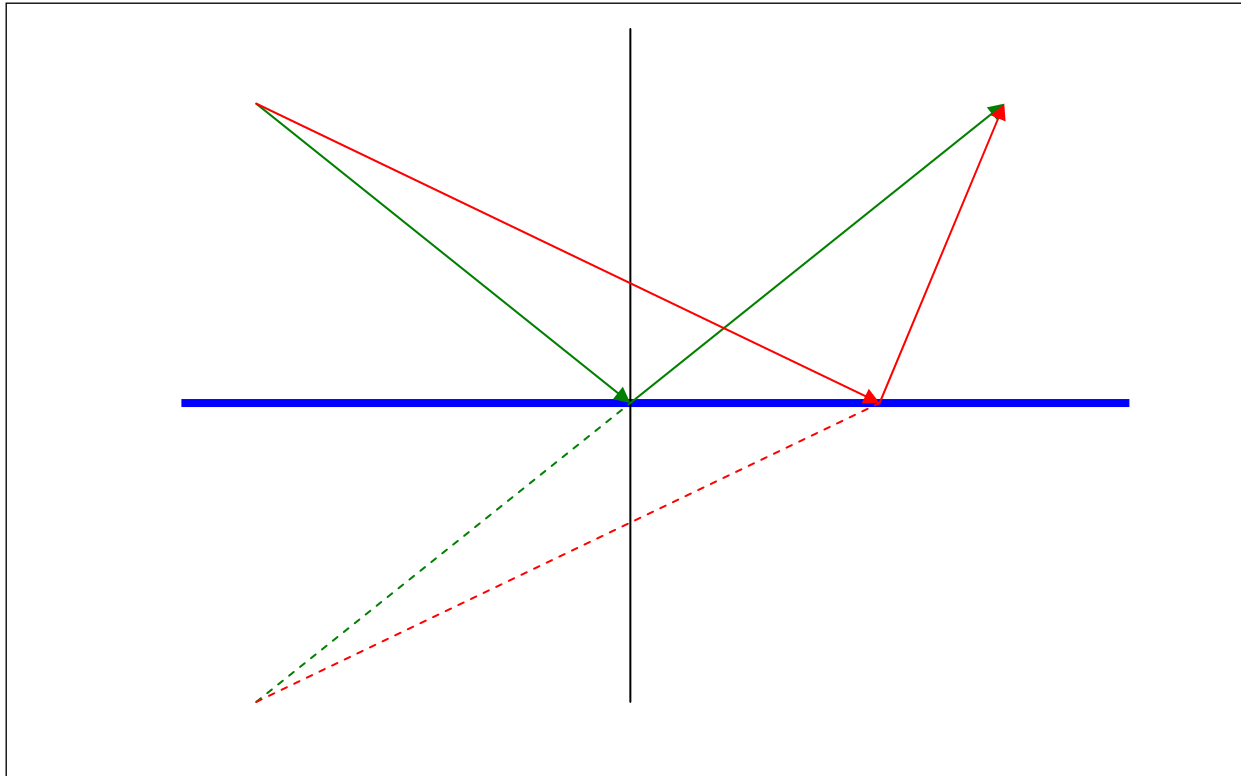
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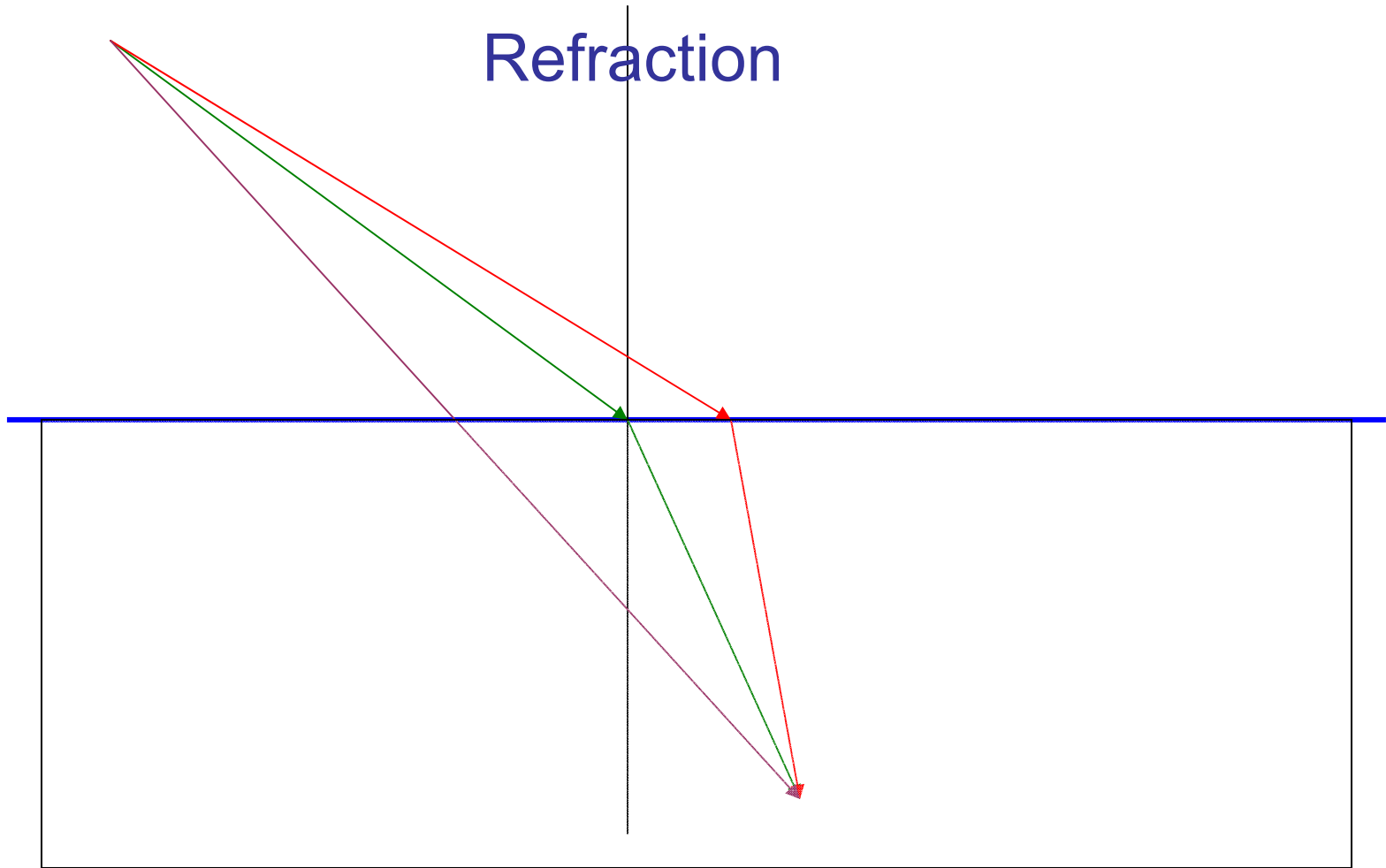
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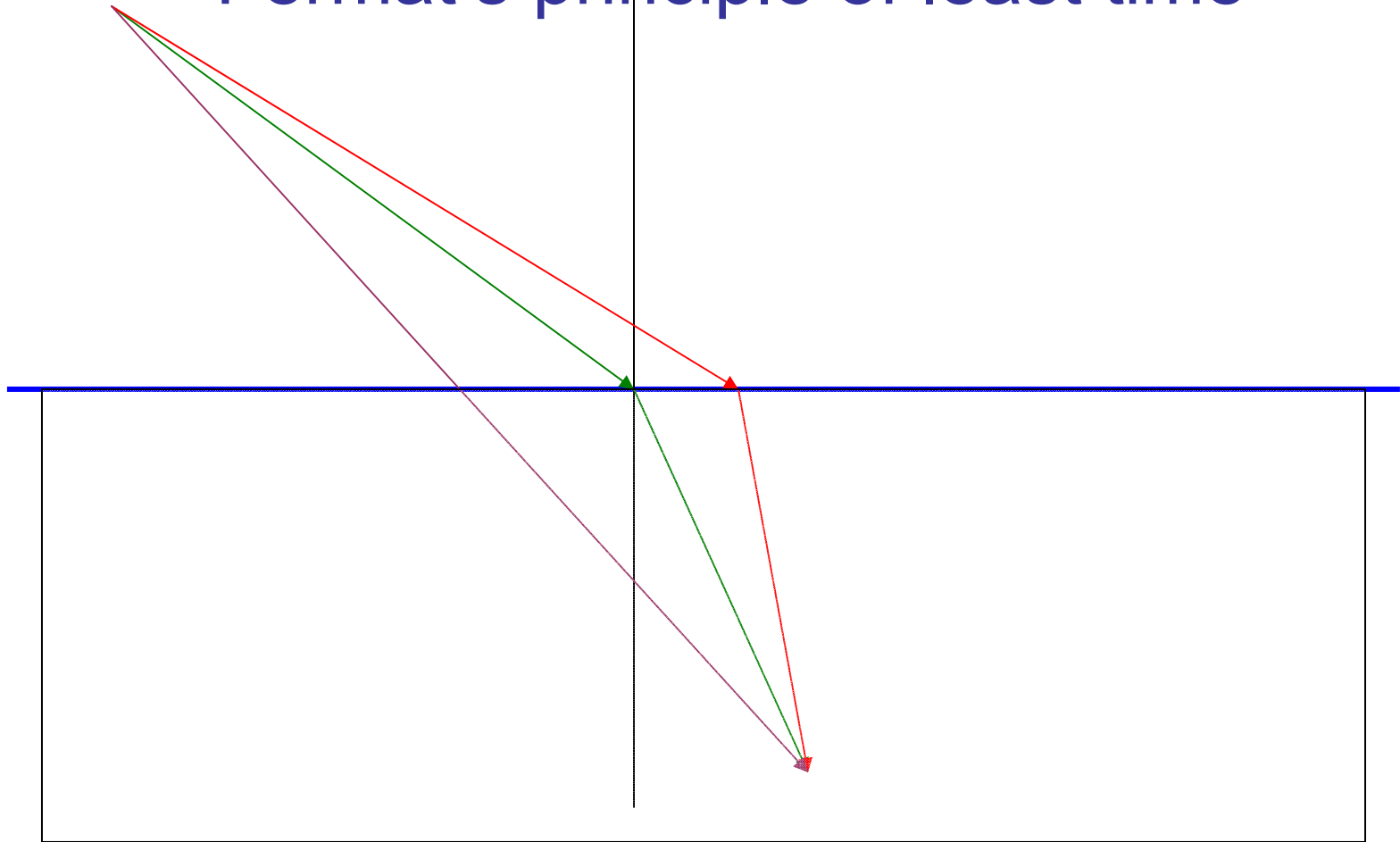
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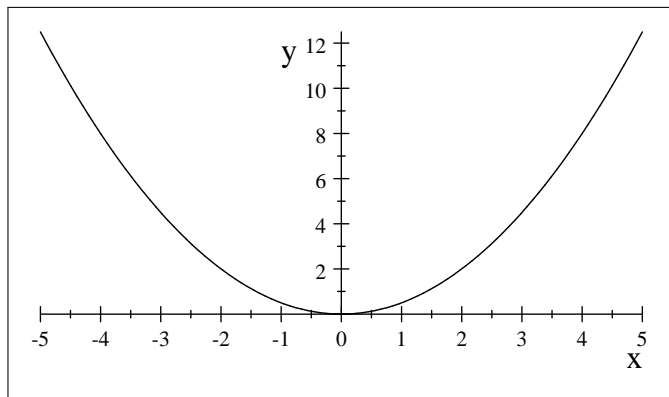
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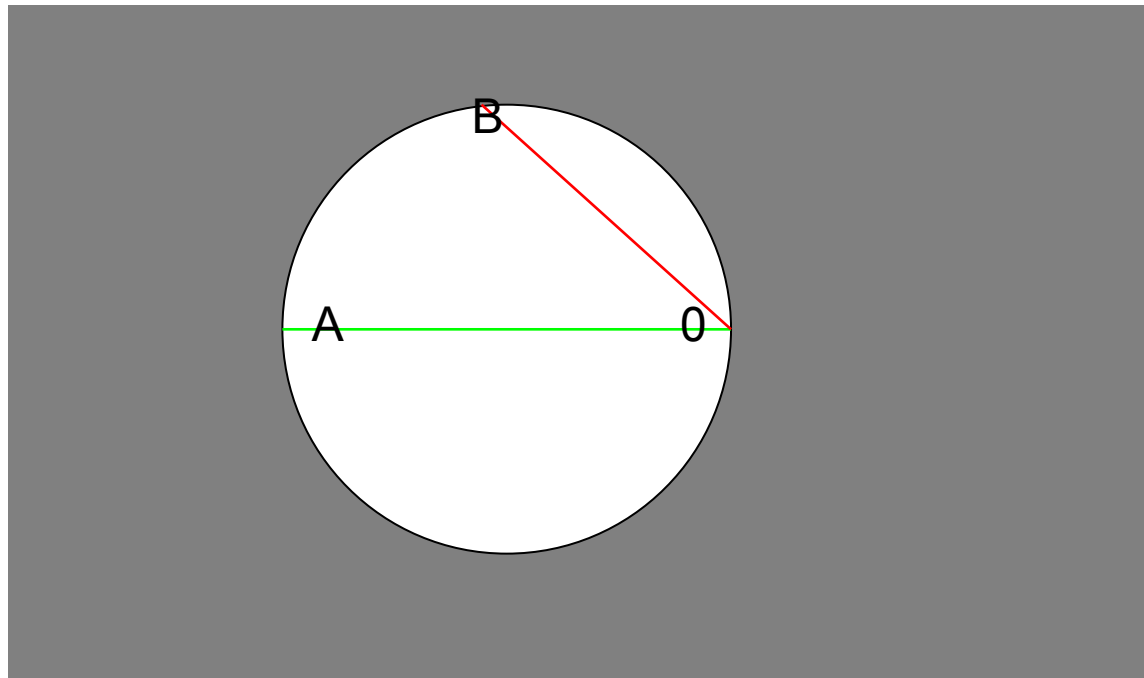
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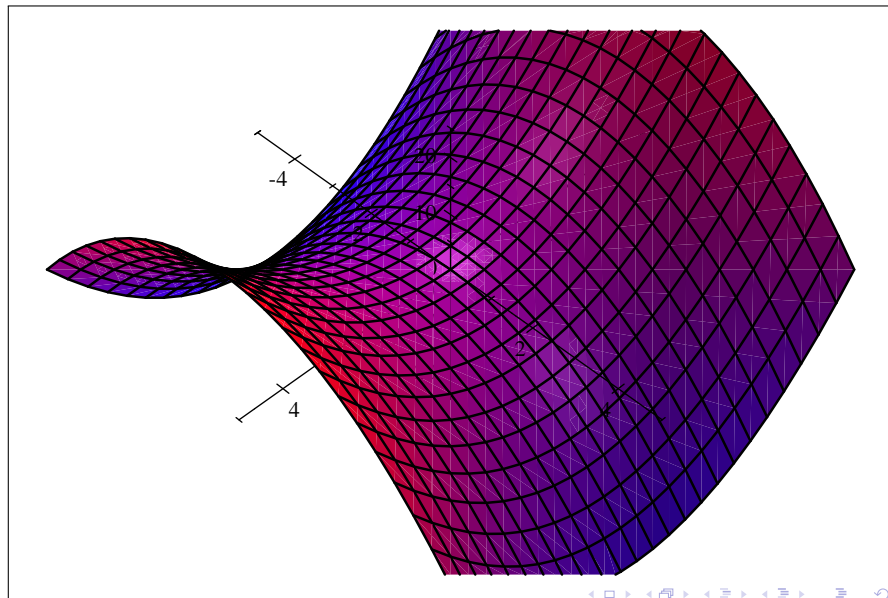
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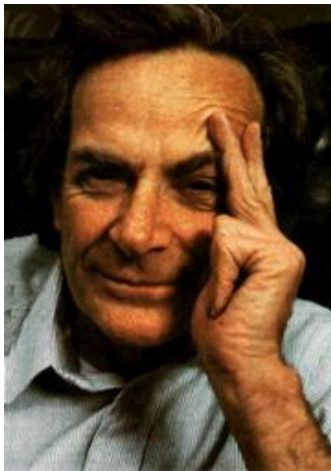


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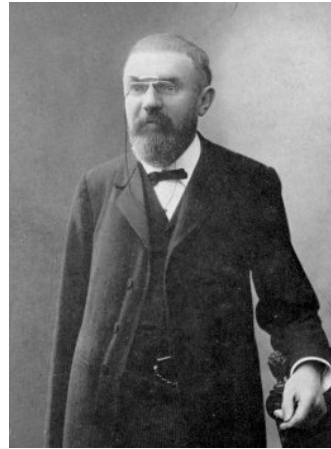
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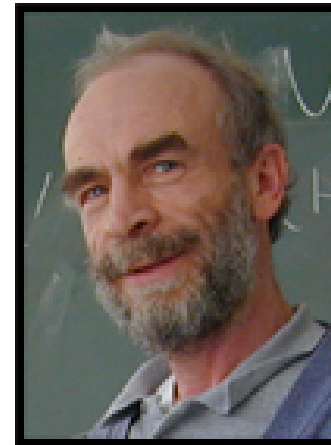
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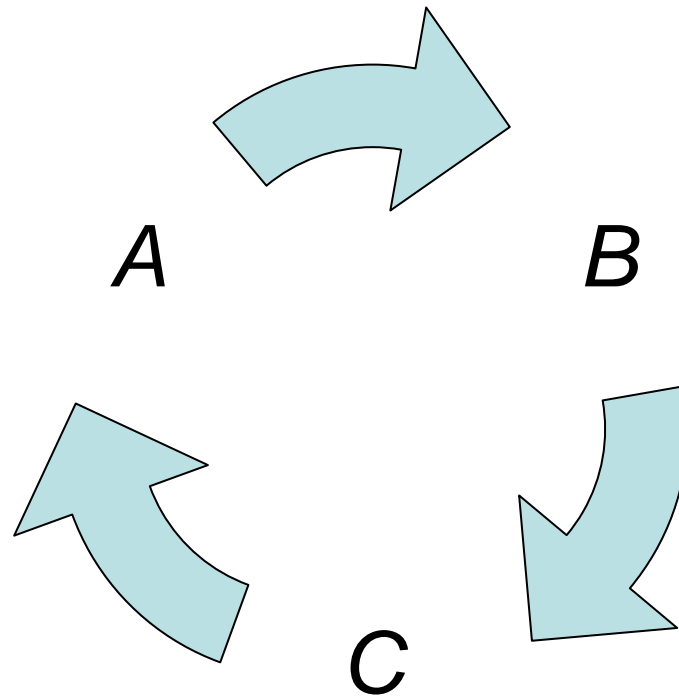
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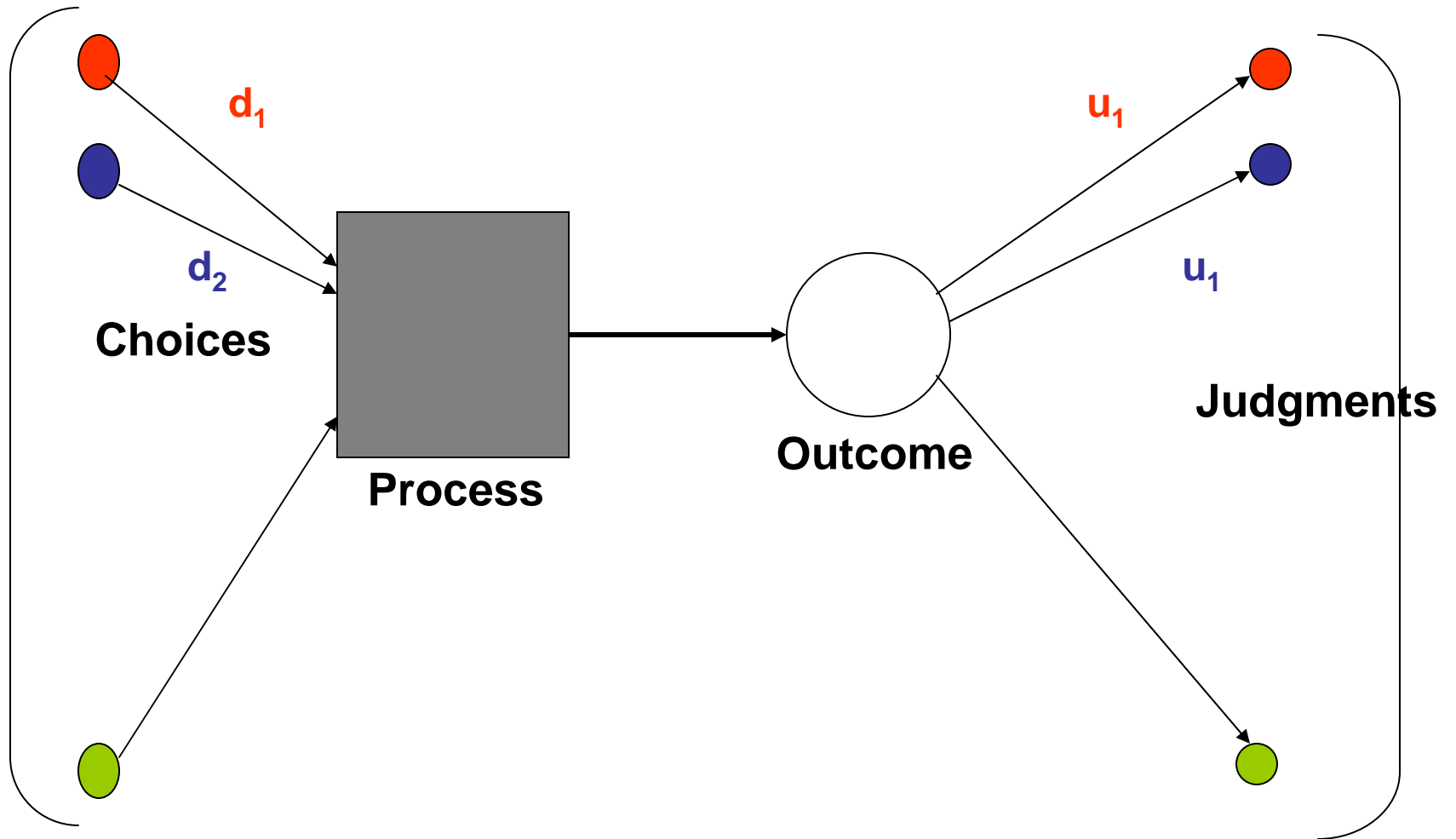
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