E. Copulas, Correlation and Extremal Dependence

1. Describing Dependence with Copulas
2. Survey of Useful Copula Families
3. Simulation of Copulas
4. Understanding the Limitations of Correlation
5. Tail dependence and other Alternative Dependence Measures
6. Fitting Copulas to Data

E1. Modelling Dependence with Copulas

Dependence and independence

The variables $X_1, \ldots, X_d$ are *independent* if and only if

$$F(x_1, \ldots, x_d) = F_1(x_1) \cdots F_d(x_d),$$

for all $(x_1, \ldots, x_d) \in \mathbb{R}^d$. This is denoted $X_1 \perp \cdots \perp X_d$.

Otherwise, the variables are *dependent* and it is of interest to model and *understand* their relation.

A definition

A copula is a multivariate distribution function $C : [0,1]^d \to [0,1]$ with standard uniform margins (or a distribution with such a df).

In the bivariate case this means that C is a function such that

- $C(0,v) = C(u,0) = 0$, for all $u,v \in [0,1]$.
- $C(1,v) = v$ and $C(u,1) = u$, for all $u,v \in [0,1]$.
- Let $u_1 < u_2$ and $v_1 < v_2$ be arbitrary points on $[0,1]$. Then

$$C(u_1,v_1) - C(u_1,v_2) - C(u_2,v_1) + C(u_2,v_2) \geq 0.$$

Key observation

If C is a copula, then

$$H(x, y) = C(F(x), G(y)), \quad x, y \in \mathbb{R}.$$

is a distribution function with margins F and G.

For instance,

$$\lim_{y \to \infty} H(x, y) = C(F(x), 1) = F(x),$$

since $C(u, 1) = u$ for all $u \in [0, 1]$.

Copula model

A copula model for (X, Y) means that

$$H(x, y) = C(F(x), G(y)), \quad x, y \in \mathbb{R}$$

with

$$C \in \mathcal{C}, \quad F \in \mathcal{F}, \quad G \in \mathcal{G}$$

for some specific parametric classes of distributions like the Poisson, Gaussian, Student t, ...

Advantage of this approach

- Varying F and G in the expression

$$H(x,y) = C(F(x), G(y)), \quad x, y \in \mathbb{R}$$

leads to distributions with arbitrary margins.

- Varying C in the expression

$$H(x,y) = C(F(x), G(y)), \quad x, y \in \mathbb{R}$$

leads to different nature of the dependence between X and Y. In particular, when $C(u,v) = uv$, X and Y are *independent*!

The Clayton model

The *Clayton copula* is a copula given by

$$C_\beta^{Cl}(u,v) = \left(u^{-\beta} + v^{-\beta} - 1\right)^{-1/\beta}, \quad u,v \in [0,1]$$

for $\beta > 0$.

The *Clayton copula model* for (X,Y) obtains when

$$H(x,y) = C_\beta^{Cl}(F(x), G(y)), \quad x,y \in \mathbb{R}$$

In particular, F and G *can be anything*! Especially, they can be of a different form, as opposed to classical bivariate distributions.

Sklar's Theorem

Let F be a joint distribution function with margins $F_1, \ldots, F_d$. There exists a copula such that for all $x_1, \ldots, x_d$ in $[-\infty, \infty]$

$$F(x_1, \ldots, x_d) = C(F_1(x_1), \ldots, F_d(x_d)).$$

If the margins are continuous then C is unique; otherwise C is uniquely determined on Ran$F_1 \times$ Ran$F_2 \ldots \times$ RanF_d.

And *conversely*, if C is a copula and $F_1, \ldots, F_d$ are univariate distribution functions, then F defined above is a multivariate df with margins $F_1, \ldots, F_d$.

On uniform distributions

Lemma 1: probability transform

Let X be a random variable with *continuous* distribution function F. Then $F(X) \sim U(0,1)$ (standard uniform).

$$P(F(X) \leq u) = P(X \leq F^{-1}(u)) = F(F^{-1}(u)) = u, \quad \forall u \in (0,1).$$

Lemma 2: quantile transform

Let U be uniform and F the distribution function of *any* rv X. Then $F^{-1}(U) \overset{\mathrm{d}}{=} X$ so that $P(F^{-1}(U) \leq x) = F(x)$.

These facts are the key to all statistical simulation and essential in dealing with copulas.

Idea of proof in continuous case

Henceforth, *unless explicitly stated*, vectors $\mathbf{X}$ will be assumed to have *continuous* marginal distributions. In this case:

$$\begin{aligned} F(x_1,\ldots,x_d) &= P\left(X_1 \le x_1,\ldots,X_d \le x_d\right) \\ &= P\left(F_1(X_1) \le F_1(x_1),\ldots,F_d(X_d) \le F_d(x_d)\right) \\ &= C(F_1(x_1),\ldots,F_d(x_d)). \end{aligned}$$

The unique copula C can be calculated from $F, F_1, \ldots, F_d$ using

$$C(u_1,\ldots,u_d) = F\left(F_1^{-1}(u_1),\ldots,F_d^{-1}(u_d)\right).$$

Copulas and dependence structures

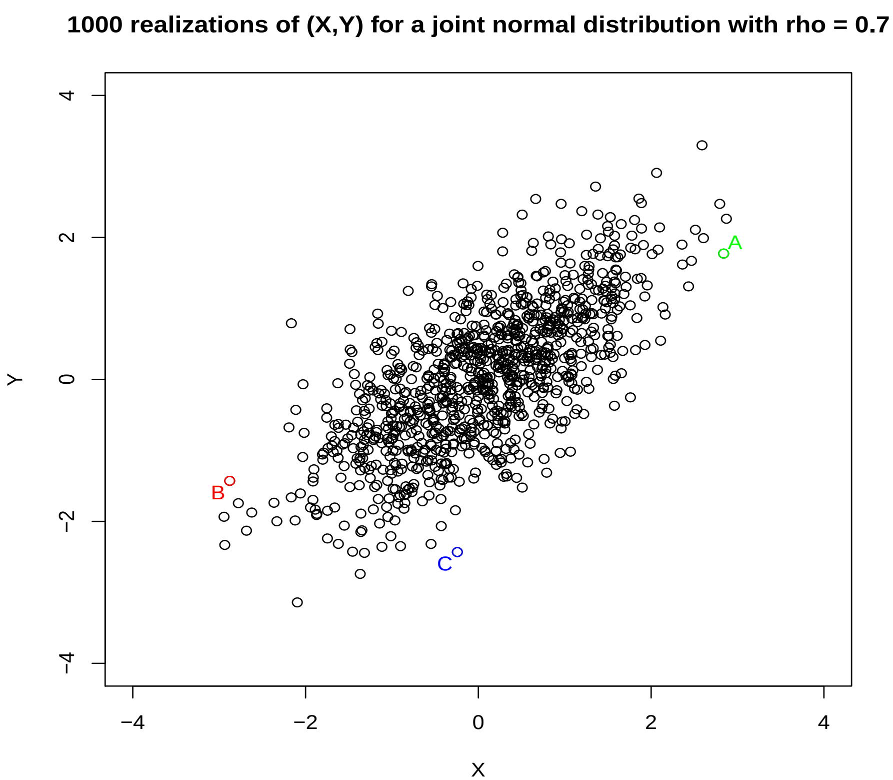
Sklar's theorem shows how a unique copula C fully describes the dependence of $\mathbf{X}$. This motivates a further definition.

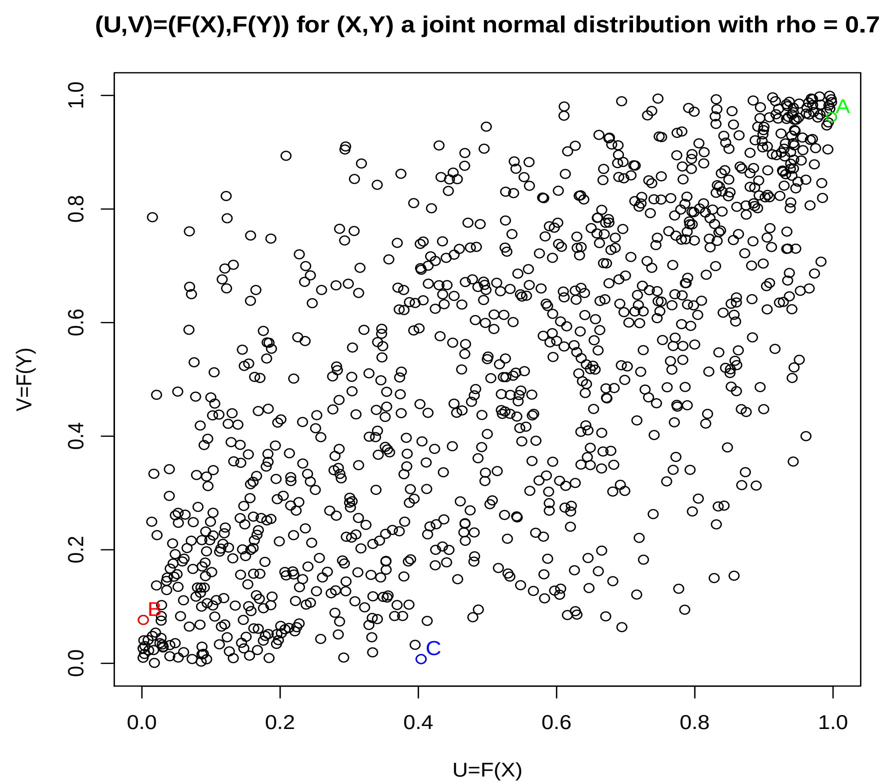
Definition: Copula of $\mathbf{X}$
The copula of $(X_1, \ldots, X_d)$ (or F) is the df C of $(F_1(X_1), \ldots, F_d(X_d))$.

We sometimes refer to C as the *dependence structure* of F.

Invariance
C is invariant under *strictly increasing* transformations of the marginals.
If $T_1, \ldots, T_d$ are strictly increasing, then $(T_1(X_1), \ldots, T_d(X_d))$ has the same copula as $(X_1, \ldots, X_d)$.

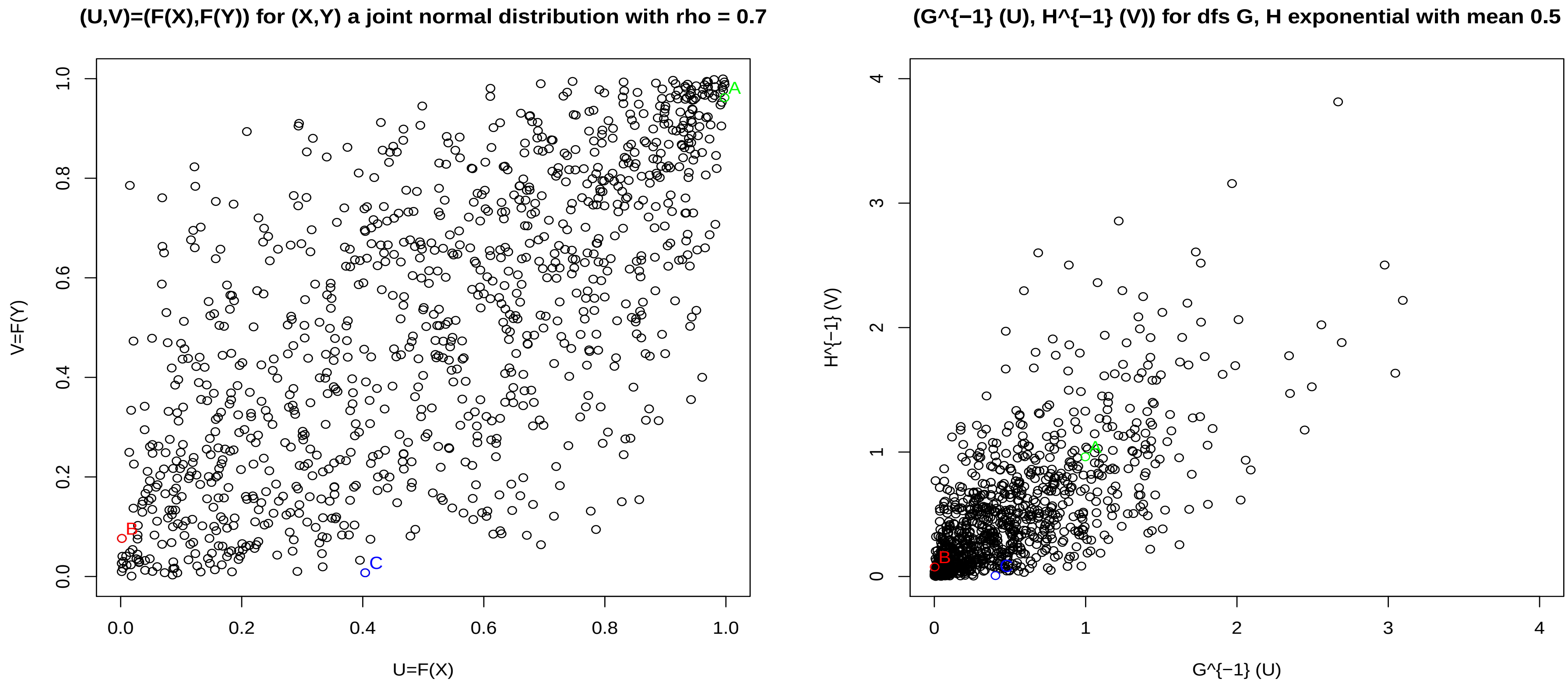
Copulas and dependence structures





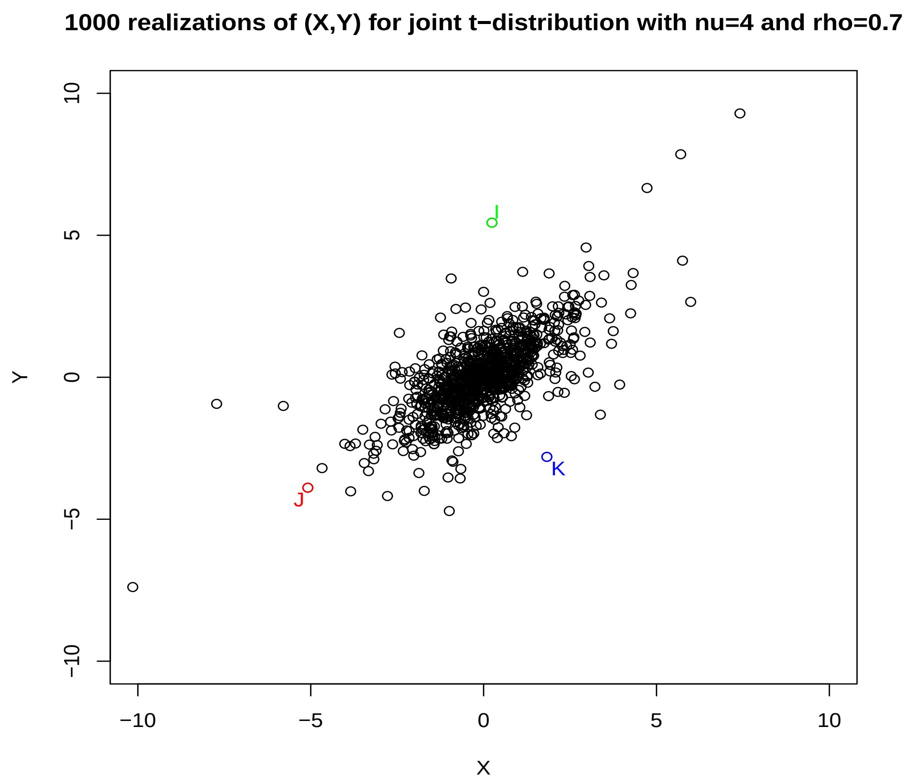
Simulation of a Gaussian copula: The plot on the left is of 1000 samples from $(X, Y)^\top \sim N_2(\mathbf{0}, \Sigma)$, where $\Sigma = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$. We mark three points A, B and C. Applying the cdf F of the (univariate) standard normal distribution to each pair of points gives the plot on the right; the Gaussian copula. Note how the points A, B and C are transformed.

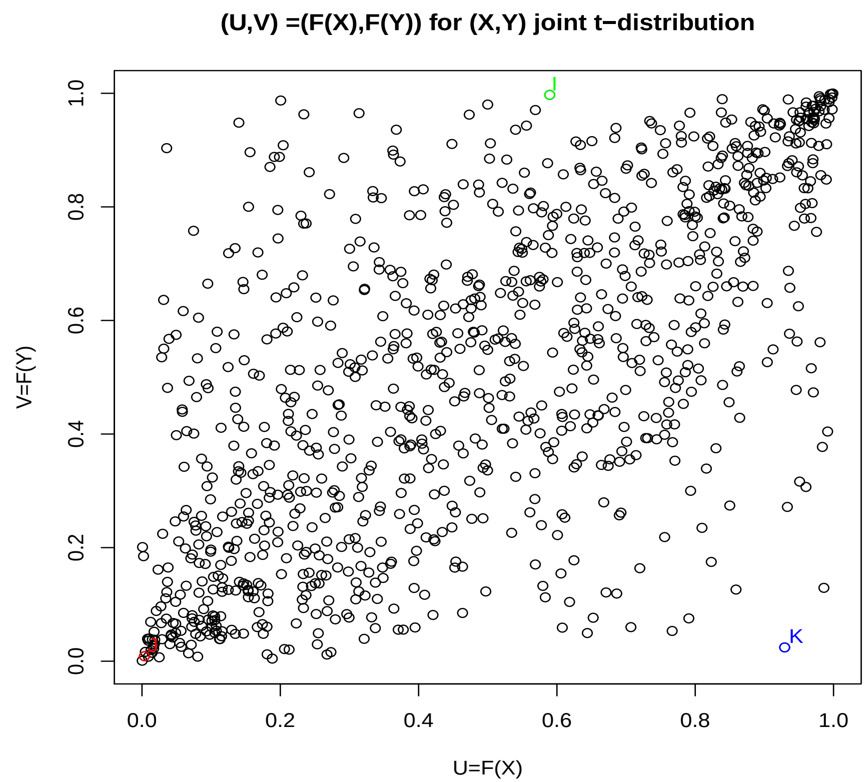
Copulas and dependence structures



Simulation of a Gaussian copula model: The plot on the left is the same Gaussian copula plot as on the previous slide. To simulate from a copula model with marginal dfs which are exponential with mean 0.5 and joined by a Gaussian copula, we apply the inverse of the exponential dfs G and H, where $G(x) = H(x) = 1 - e^{-2x}$, to each pair of points on the left plot. This results in the plot on the right. Again, we mark the same three points A, B and C to allow you to see the transformation.

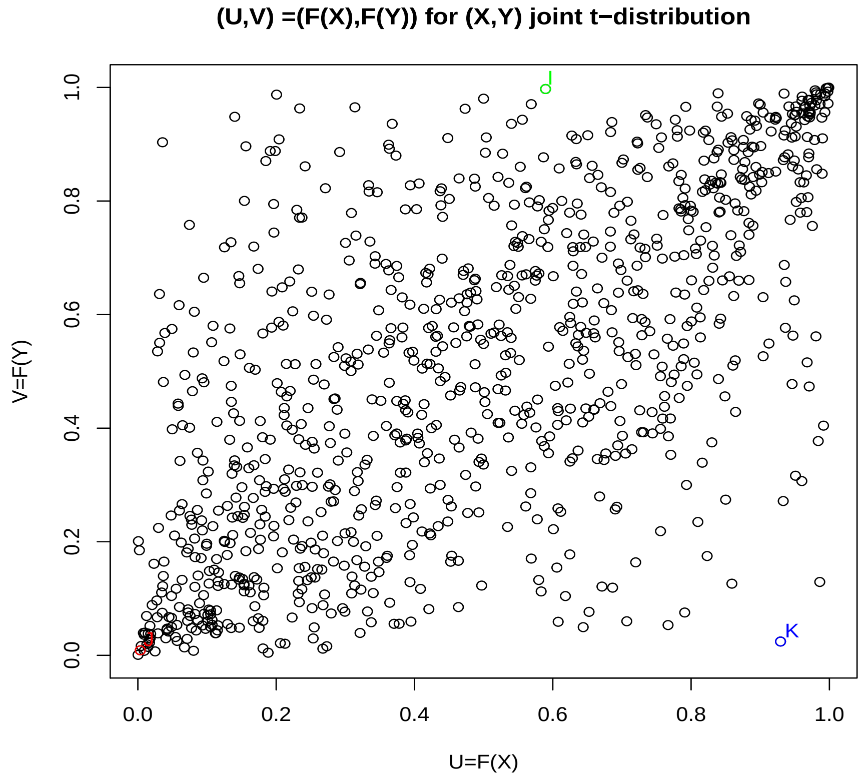
Copulas and dependence structures

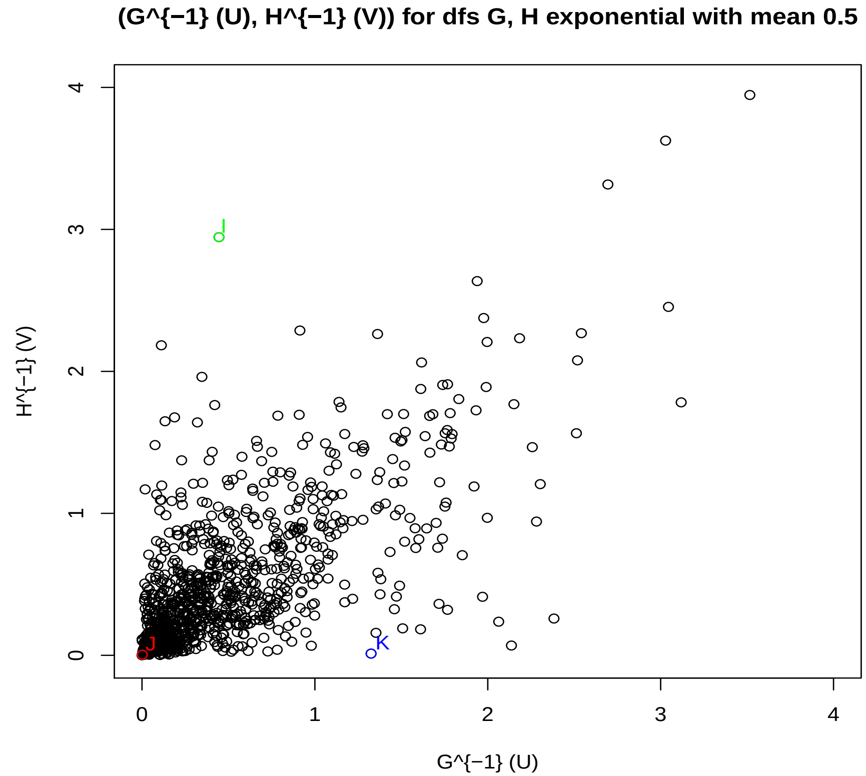




Simulation of a t copula: The plot on the left is of 1000 samples from $(X, Y)^\top \sim t_2(4, \mathbf{0}, \Sigma)$, where $\Sigma = \begin{pmatrix} 1 & 0.7 \\ 0.7 & 1 \end{pmatrix}$. We mark three points I, J and K. Applying the cdf F of the (univariate) standard t distribution to each pair of points gives the plot on the right; the t copula. Note how the points I, J and K are transformed.

Copulas and dependence structures





Simulation of a t copula model: The plot on the left is the same t copula plot as on the previous slide. To simulate from a copula model with marginal dfs which are exponential with mean 0.5 and joined by a t copula, we apply the inverse of the exponential dfs G and H, where $G(x) = H(x) = 1 - e^{-2x}$, to each pair of points on the left plot. This results in the plot on the right. Again, we mark the same three points I, J and K to allow you to see the transformation.

Examples of copulas

- Independence

  $X_1, \ldots, X_d$ are mutually independent $\iff$ their copula C satisfies $C(u_1, \ldots, u_d) = \prod_{i=1}^{d} u_i$.

- *Comonotonicity* - perfect dependence

  $X_i \overset{\text{a.s.}}{=} T_i(X_1)$, T_i strictly increasing, $i = 2, \ldots, d$, $\iff$ C satisfies

  $C(u_1, \ldots, u_d) = \min\{u_1, \ldots, u_d\}$.

- *Countermonotonicity* - perfect negative dependence (d=2)

  $X_2 \overset{\text{a.s.}}{=} T(X_1)$, T strictly decreasing, $\iff$ C satisfies $C(u, v) = \max\{u + v - 1, 0\}$.

Fréchet bounds

Any copula is bounded pointwise, viz.

$$\max\left\{\sum_{i=1}^{d} u_i + 1 - d, 0\right\} \leq C(\mathbf{u}) \leq \min\left\{u_1, \ldots, u_d\right\}.$$

Remark: right hand side is df of $\overbrace{(U, \ldots, U)}^{\text{d times}}$, where $U \sim U(0, 1)$.

E2. Parametric Copula Families

Two most common possibilities are:

- Copulas *implicit* in well-known parametric distributions
  Recall $C(u_1, \ldots, u_d) = F\left(F_1^{-1}(u_1), \ldots, F_d^{-1}(u_d)\right)$.
- *Closed-form* parametric copula families.

Gaussian Copula: an implicit copula

Let $\mathbf{X}$ be standard multivariate normal with correlation matrix P.

$$\begin{aligned} C_P^{\text{Ga}}(u_1, \ldots, u_d) &= P\left(\Phi(X_1) \leq u_1, \ldots, \Phi(X_d) \leq u_d\right) \\ &= P\left(X_1 \leq \Phi^{-1}(u_1), \ldots, X_d \leq \Phi^{-1}(u_d)\right) \end{aligned}$$

where Φ is df of standard normal.

$P = I$ gives independence; as $P \to J$ we get comonotonicity.

Gaussian Copula in the bivariate case

The density of a bivariate normal distribution with correlation ρ and standard normal margins is

$$h_\rho(x,y) = \frac{1}{2\pi\sqrt{1-\rho^2}} \exp\left\{-\frac{x^2+y^2-2\rho xy}{2(1-\rho^2)}\right\}.$$

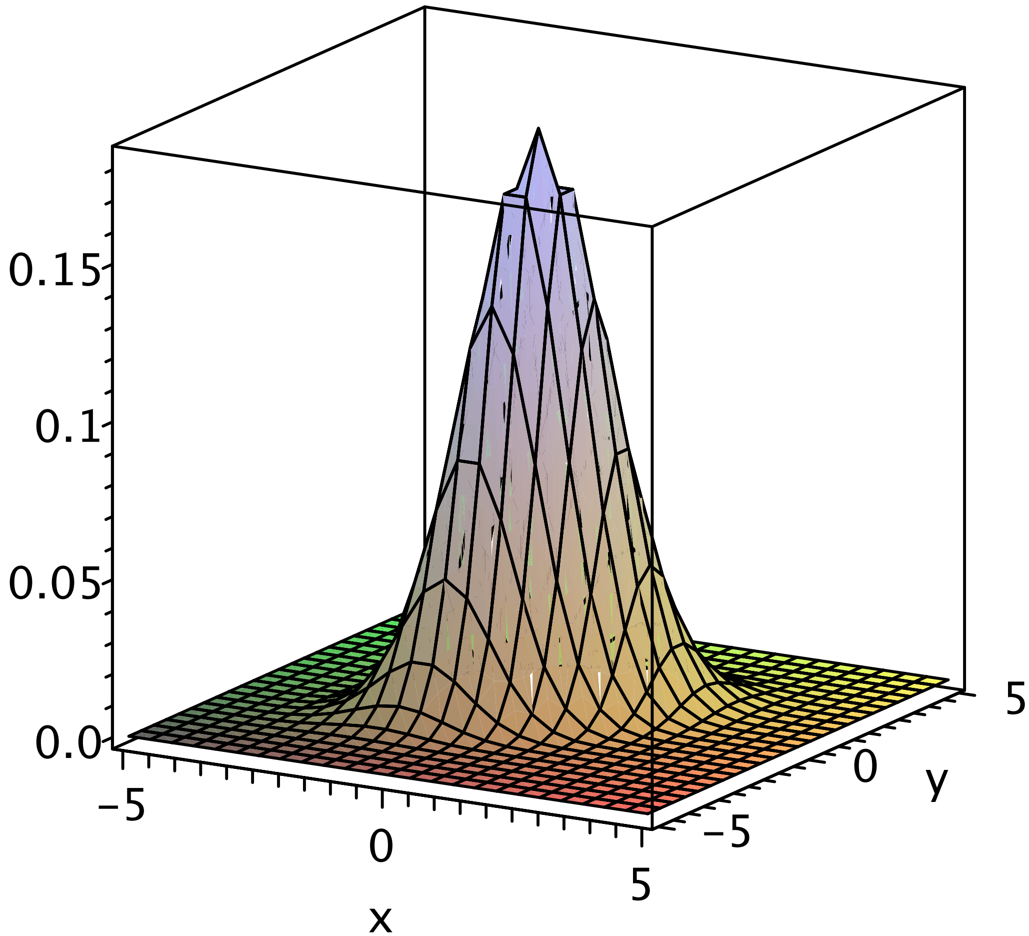
Although the Gaussian copula is not explicit, its density is given by

$$c_\rho(u,v) = \frac{h_\rho\left(\Phi^{-1}(u),\Phi^{-1}(v)\right)}{\varphi\left(\Phi^{-1}(u)\right)\varphi\left(\Phi^{-1}(v)\right)},$$

where

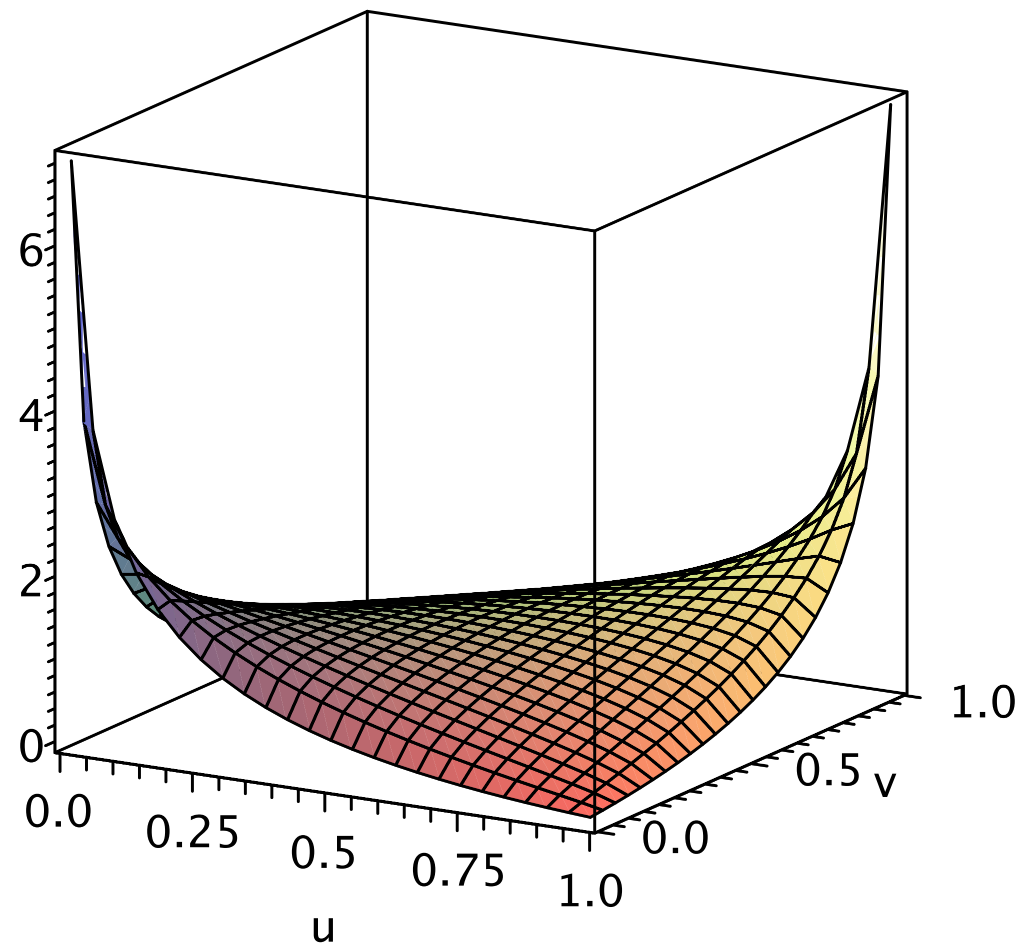
$$\varphi(t) = \Phi'(t) = \frac{1}{\sqrt{2\pi}}\,e^{-t^2/2}, \quad t \in \mathbb{R}.$$

Density of the bivariate normal distribution



Margins are standard normal; correlation is 50%.

Density of the bivariate Gaussian copula



Correlation parameter is 50%.

Elliptical or Normal Mixture Copulas

The Gaussian copula is an elliptical copula. Using a similar approach we can extract copulas from other multivariate normal mixture distributions.

Examples

- The t copula $C^{\mathrm{t}}_{\nu,P}$
- The generalised hyperbolic copula

The elliptical copulas are rich in parameters - parameter for every pair of variables; easy to simulate.

Archimedean Copulas $d = 2$

These have simple closed forms and are useful for calculations. However, higher dimensional extensions are not rich in parameters.

- Gumbel Copula

$$C_\beta^{Gu}(u,v) = \exp\left(-\left((-\log u)^\beta + (-\log v)^\beta\right)^{1/\beta}\right).$$

$\beta \geq 1$: $\beta = 1$ gives independence; $\beta \to \infty$ gives comonotonicity.

- Clayton Copula

$$C_\beta^{Cl}(u,v) = \left(u^{-\beta} + v^{-\beta} - 1\right)^{-1/\beta}.$$

$\beta > 0$: $\beta \to 0$ gives independence ; $\beta \to \infty$ gives comonotonicity.

- Frank Copula

$$C_\beta^{Fr}(u,v) = -\frac{1}{\beta}\log\left(1 + \frac{(e^{-\beta u}-1)(e^{-\beta v}-1)}{e^{-\beta}-1}\right).$$

$\beta \neq 0$: $\beta \to -\infty$ gives countermonotonicity; $\beta \to 0$ gives independence; $\beta \to \infty$ gives comonotonicity.

Archimedean Copulas in Higher Dimensions

All our Archimedean copulas have the form

$$C(u,v) = \psi(\psi^{-1}(u) + \psi^{-1}(v)),$$

where $\psi : [0,\infty) \mapsto [0,1]$ is strictly decreasing and convex with $\psi(0) = 1$ and $\lim_{t\to\infty} \psi(t) = 0$.

The simplest higher dimensional extension is

$$C(u_1,\ldots,u_d) = \psi(\psi^{-1}(u_1) + \cdots + \psi^{-1}(u_d)).$$

Example: Gumbel copula: $\psi(t) = \exp((-t)^{1/\beta})$:

$$C_\beta^{\mathsf{Gu}}(u_1,\ldots,u_d) = \exp\left(- \left((-\log u_1)^\beta + \cdots + (-\log u_d)^\beta \right)^{1/\beta} \right).$$

These copulas are *exchangeable* (invariant under permutations).

Other Copula Families

- *Extreme Value Copulas*: arise naturally in multivariate extreme value theory; satisfy $C^t(u_1, \dots, u_d) = C(u_1^t, \dots, u_d^t),\ \forall t > 0$, Example: Gumbel.

- *Extremal Copulas*: are copulas of vectors whose components are all either pairwise comonotonic or countermontonic; rank correlation matrix consists of 1's and -1's.
  Let J be subset of $\{1, \dots, d\}$. General form of extremal copula:

$$C(u_1, \dots, u_d) = \max\left\{\min_{i \in J} u_i + \min_{j \in J^c} u_j - 1, 0\right\}.$$

  Example: comonotonicity copula.

E3. Simulating Copulas

Normal Mixture (Elliptical) Copulas

Simulating Gaussian copula C_P^{Ga}

- Simulate $\mathbf{X} \sim N_d(\mathbf{0}, P)$
- Set $\mathbf{U} = \left(\Phi\left(X_1\right), \ldots, \Phi\left(X_d\right)\right)'$ (probability transformation)

Simulating t copula $C_{\nu,P}^{\text{t}}$

- Simulate $\mathbf{X} \sim t_d(\nu, \mathbf{0}, P)$
- $\mathbf{U} = \left(t_\nu\left(X_1\right), \ldots, t_\nu\left(X_d\right)\right)'$ (probability transformation)
  t_ν is df of univariate t distribution.

Meta–Gaussian and Meta–t Distributions

If $(U_1, \ldots, U_d) \sim C_P^{\mathsf{Ga}}$ and F_i are univariate dfs other than univariate normal then

$$\left(F_1^{\leftarrow}\left(U_1\right), \ldots, F_d^{\leftarrow}\left(U_d\right)\right)$$

has a *meta–Gaussian* distribution. Thus it is easy to simulate vectors with the Gaussian copula and arbitrary margins.

In a similar way we can construct and simulate from *meta* t_ν *distributions*. These are distributions with copula $C_{\nu,P}^{\mathsf{t}}$ and margins other than univariate t_ν.

Simulating Archimedean Copulas

For the most useful of the Archimedean copulas (such as Clayton and Gumbel) techniques exist to simulate the exchangeable versions in arbitrary dimensions. The theory on which this is based may be found in Marshall and Olkin (1988).

Algorithm for d-dimensional Clayton copula C_{β}^{Cl}

- Simulate a *gamma* variate X with parameter $\alpha = 1/\beta$.
  This has density $f(x) = x^{\alpha-1}e^{-x}/\Gamma(\alpha)$.

- Simulate d independent standard uniforms $U_1, \ldots, U_d$.

- Return $\left(\left(1 - \frac{\log U_1}{X}\right)^{-1/\beta}, \ldots, \left(1 - \frac{\log U_d}{X}\right)^{-1/\beta}\right)$.

E4. Understanding Limitations of Correlation

Drawbacks of Linear Correlation

Denote the linear correlation of two random variables X_1 and X_2 by $\rho(X_1, X_2)$. We should be aware of the following.

- Linear correlation only gives a scalar summary of (linear) dependence and $\mathrm{var}(X_1), \mathrm{var}(X_2)$ must exist.
- X_1, X_2 independent $\Rightarrow \rho(X_1, X_2) = 0$.
  But $\rho(X_1, X_2) = 0 \not\Rightarrow X_1, X_2$ independent.
  Example: spherical bivariate t-distribution with ν d.f.
- Linear correlation is not invariant with respect to strictly increasing transformations T of X_1, X_2, i.e. generally

$$\rho(T(X_1), T(X_2)) \neq \rho(X_1, X_2).$$

A Fallacy in the Use of Correlation

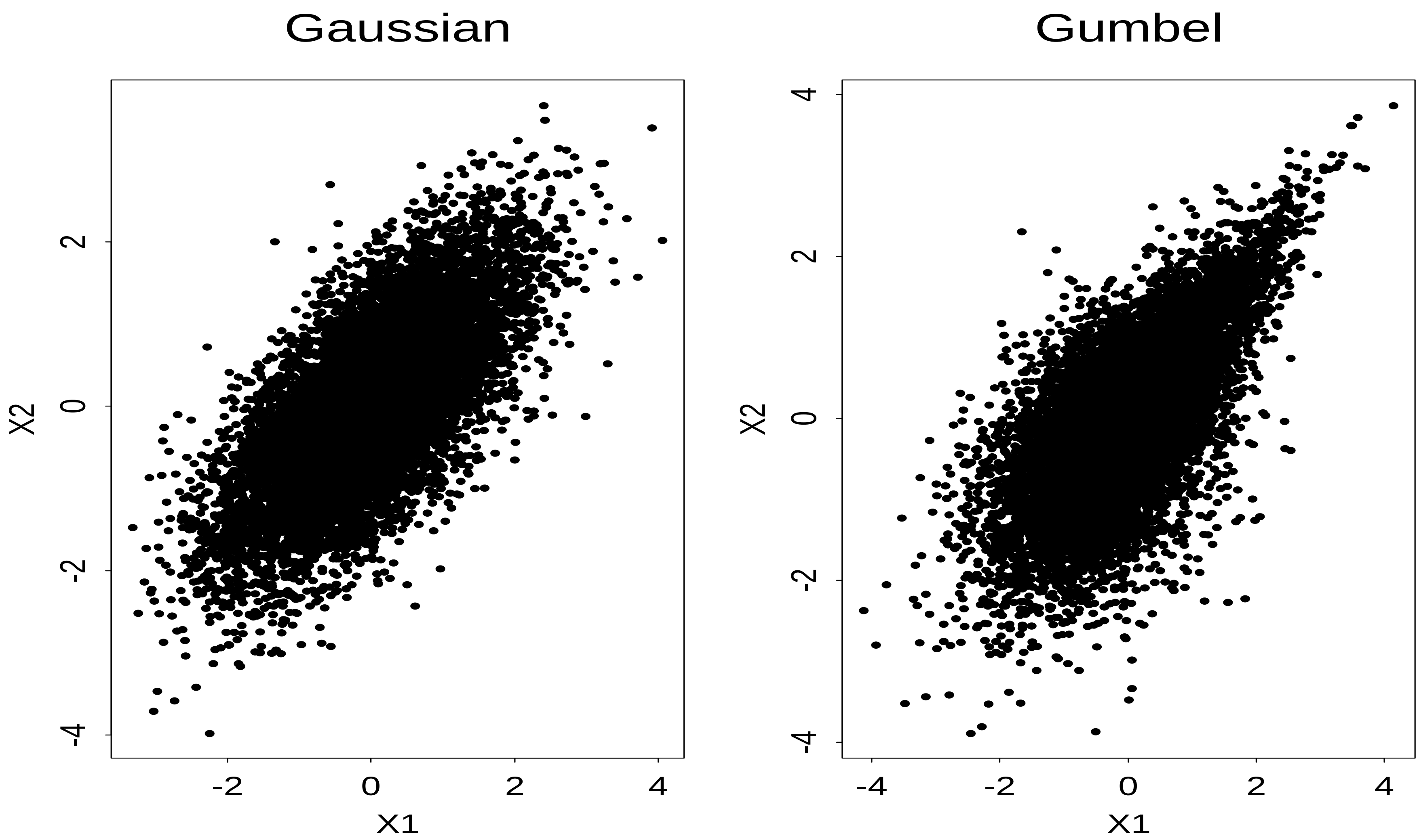
Consider the random vector $(X_1, X_2)'$.

"Marginal distributions and correlation determine the joint distribution".

- True for the class *bivariate* normal distributions or, more generally, for elliptical distributions.

- *Wrong* in general, as the next example shows.

Gaussian and Gumbel Copulas Compared



Margins are standard normal; correlation is 70%.

Fallacy 1 continued

Sometimes Fallacy 1 is hidden in statements like:
"If two random variables X_1 and X_2 are uncorrelated, they may be considered as approximately independent".

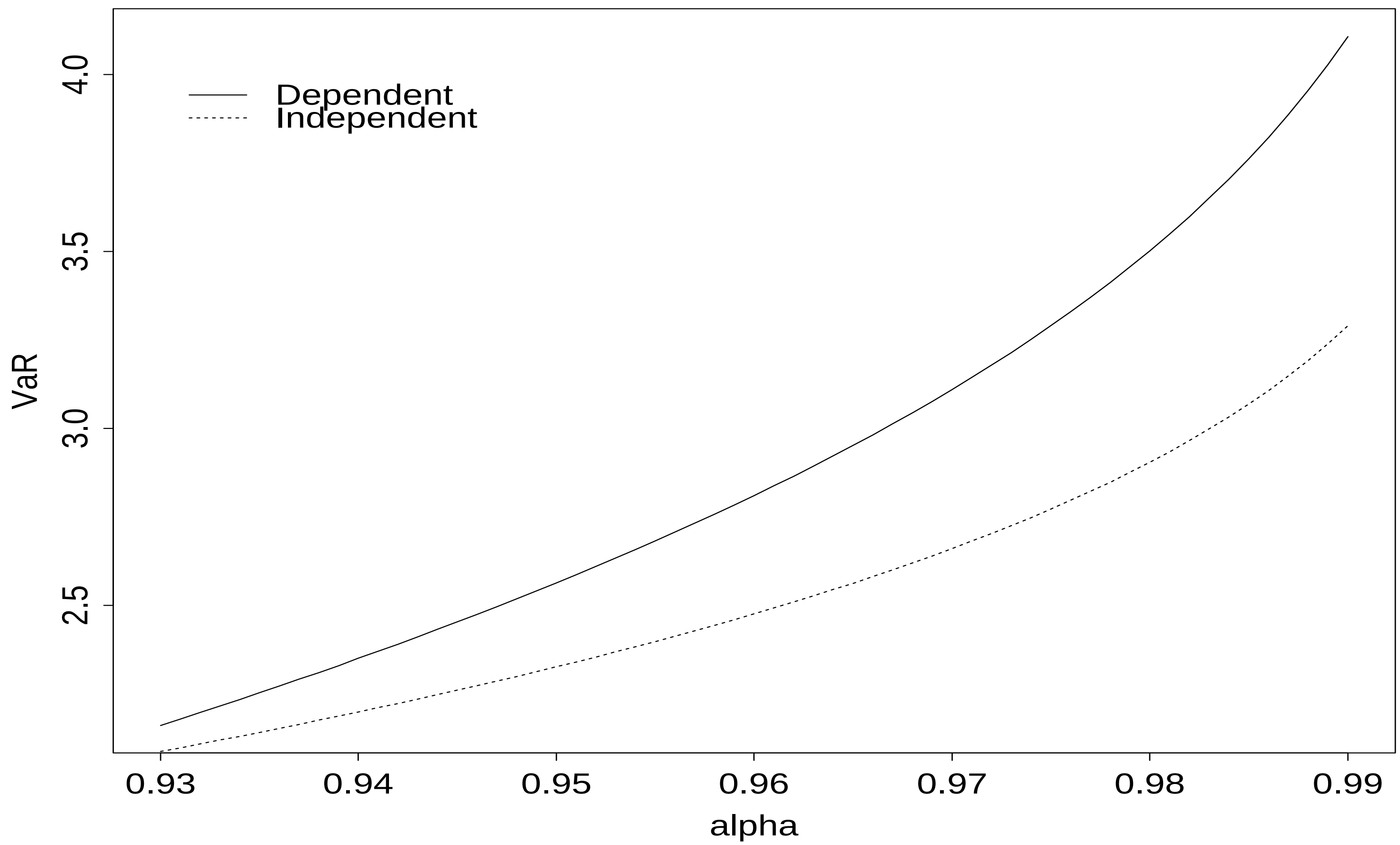
Consider two portfolios of risks. Set

$$X_1 = Z \quad \text{(Profit\&Loss Country A)},$$
$$X_2 = V \cdot Z \quad \text{(Profit\&Loss Country B)},$$

V, Z independent, $Z \sim N(0,1)$,
$P(V = +1) = P(V = -1) = 1/2$.
V switches between perfect positive and negative dependence.

$X_2 \sim N(0,1)$ and $\rho(X_1, X_2) = 0$.
But $(X_1, X_2)'$ is *not* bivariate normal.

VaR (Quantile) for two different dependence models



$\mathrm{VaR}_\alpha(X_1 + X_2)$ for X_1, X_2 independent and X_1, X_2 dependent.

Fallacy 2

"Given marginal distributions F_1 and F_2 for X_1 and X_2, all linear correlations between -1 and $+1$ can be attained through specification of the joint distribution".

- This is again true for elliptical distributions but not true in general. If F_1 and F_2 are not of the same type, then $\rho(X_1, X_2) < 1$.

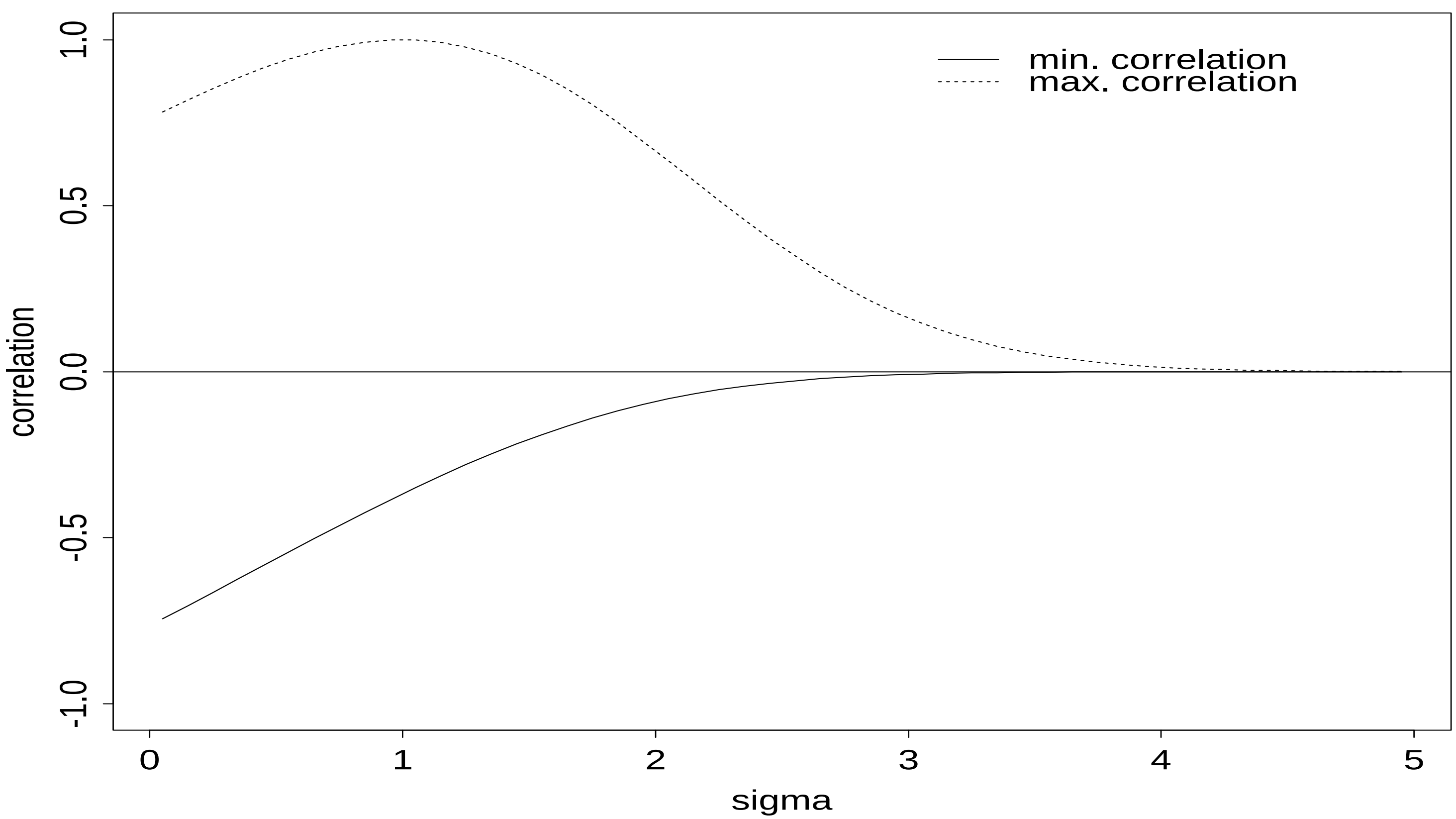
- **Theorem** (Höffding 1940)

  1. The set of possible correlations is a closed interval $[\rho_{\min}, \rho_{\max}]$.

  2. $\rho_{\max}$ is attained iff X_1, X_2 comonotonic. $\rho_{\min}$ is attained iff X_1, X_2 countermonotonic.

Example of Extremal Correlations

Take $X_1 \sim$ Lognormal$(0,1)$, and $X_2 \sim$ Lognormal$(0,\sigma^2)$.
Let σ vary and plot ρ_{min} and ρ_{max} against σ



E5. Alternative Dependence Concepts

Rank Correlation (let C denote copula of X_1 and X_2)
Spearman's rho

$$\begin{aligned}\rho_S(X_1, X_2) &= \rho(F_1(X_1), F_2(X_2)) = \rho(\text{copula}) \\ \rho_S(X_1, X_2) &= 12 \int_0^1 \int_0^1 \{C(u_1, u_2) - u_1 u_2\} du_1 du_2.\end{aligned}$$

Kendall's tau
Take an independent copy of (X_1, X_2) denoted $(\widetilde{X}_1, \widetilde{X}_2)$.

$$\begin{aligned}\rho_\tau(X_1, X_2) &= 2P\Big((X_1 - \widetilde{X}_1)(X_2 - \widetilde{X}_2) > 0\Big) - 1 \\ \rho_\tau(X_1, X_2) &= 4 \int_0^1 \int_0^1 C(u_1, u_2) dC(u_1, u_2) - 1.\end{aligned}$$

Properties of Rank Correlation

(not shared by linear correlation)

True for Spearman's rho (ρ_S) or Kendall's tau (ρ_τ).

- ρ_S depends only on copula of $(X_1, X_2)'$.

- ρ_S is invariant under strictly increasing transformations of the random variables.

- $\rho_S(X_1, X_2) = 1 \iff X_1, X_2$ comonotonic.

- $\rho_S(X_1, X_2) = -1 \iff X_1, X_2$ countermonotonic.

Kendall's Tau in Elliptical Models

Suppose $\mathbf{X} = (X_1, X_2)'$ has *any elliptical distribution*; for example $\mathbf{X} \sim t_2(\nu, \boldsymbol{\mu}, \Sigma)$. Then

$$\rho_\tau(X_1, X_2) = \frac{2}{\pi} \arcsin\left(\rho(X_1, X_2)\right). \tag{1}$$

Remarks:

1. In case of infinite variances we simply interpret $\rho(X_1, X_2)$ as $\Sigma_{1,2}/\sqrt{\Sigma_{1,1}\Sigma_{2,2}}$.
2. Result of course implies that if $\mathbf{Y}$ has copula $C^t_{\nu,P}$ then $\rho_\tau(Y_1, Y_2) = \frac{2}{\pi}\arcsin(P_{1,2})$.
3. An estimator of ρ_τ is given by

$$\widehat{\rho_\tau}(X_1, X_2) = \frac{1}{\binom{n}{2}} \sum_{1 \le i < j \le n} \text{sgn}\left[\left(X_{i,1} - X_{j,1}\right)\left(X_{i,2} - X_{j,2}\right)\right].$$

Tail Dependence or Extremal Dependence

Objective: measure dependence in joint tail of bivariate distribution. When limit exists, coefficient of *upper* tail dependence is

$$\lambda_u(X_1, X_2) = \lim_{q \to 1} P(X_2 > \mathrm{VaR}_q(X_2) \mid X_1 > \mathrm{VaR}_q(X_1)),$$

Analogously the coefficient of *lower* tail dependence is

$$\lambda_\ell(X_1, X_2) = \lim_{q \to 0} P\left(X_2 \leq \mathrm{VaR}_q(X_2) \mid X_1 \leq \mathrm{VaR}_q(X_1)\right).$$

These are functions of the copula given by

$$\begin{aligned} \lambda_u &= \lim_{q \to 1} \frac{\overline{C}(q,q)}{1-q} = \lim_{q \to 1} \frac{1 - 2q + C(q,q)}{1-q}, \\ \lambda_\ell &= \lim_{q \to 0} \frac{C(q,q)}{q}. \end{aligned}$$

Tail Dependence

Clearly $\lambda_u \in [0,1]$ and $\lambda_\ell \in [0,1]$.

For elliptical copulas $\lambda_u = \lambda_\ell =: \lambda$. True of all copulas with radial symmetry: $(U_1, U_2) \stackrel{\text{d}}{=} (1 - U_1, 1 - U_2)$.

Terminology:

$\lambda_u \in (0,1]$: upper tail dependence,

$\lambda_u = 0$: asymptotic independence in upper tail,

$\lambda_\ell \in (0,1]$: lower tail dependence,

$\lambda_\ell = 0$: asymptotic independence in lower tail.

Examples of tail dependence

The Gaussian copula is asymptotically independent for $|\rho| < 1$.

The t copula is tail dependent when $\rho > -1$.

$$\lambda = 2\bar{t}_{\nu+1}\left(\sqrt{\nu+1}\sqrt{1-\rho}/\sqrt{1+\rho}\right).$$

The Gumbel copula is upper tail dependent for $\beta > 1$.

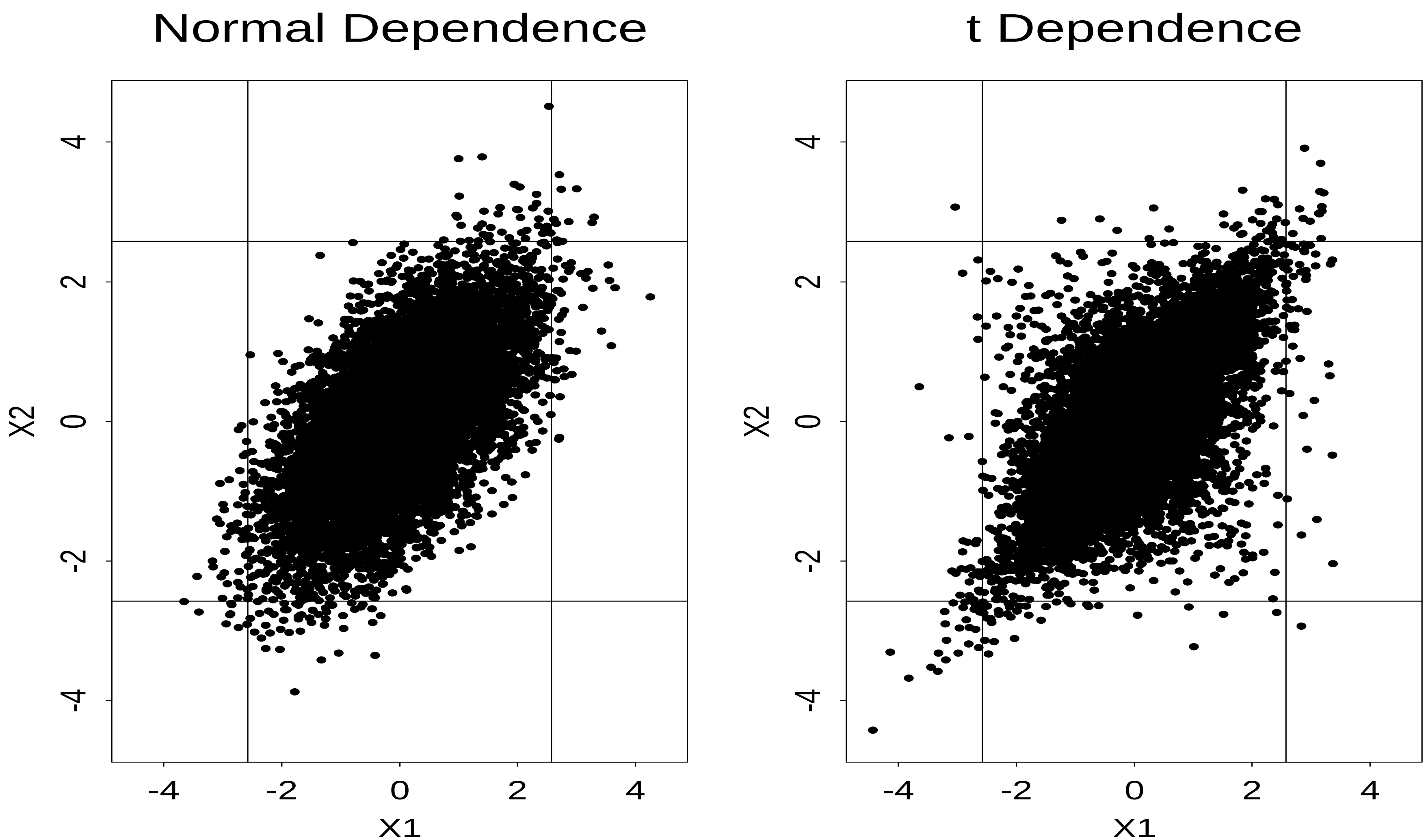
$$\lambda_u = 2 - 2^{1/\beta}.$$

The Clayton copula is lower tail dependent for $\beta > 0$.

$$\lambda_\ell = 2^{-1/\beta}.$$

Recall dependence model in Fallacy 1b: $\lambda_u = \lambda_\ell = 0.5$.

Gaussian and t3 Copulas Compared



Copula parameter $\rho = 0.7$; quantiles lines 0.5% and 99.5%.

Joint Tail Probabilities at Finite Levels

ρ	C	Quantile			
		95%	99%	99.5%	99.9%
0.5	N	1.21×10^{-2}	1.29×10^{-3}	4.96×10^{-4}	5.42×10^{-5}
0.5	t8	1.20	1.65	1.94	3.01
0.5	t4	1.39	2.22	2.79	4.86
0.5	t3	1.50	2.55	3.26	5.83
0.7	N	1.95×10^{-2}	2.67×10^{-3}	1.14×10^{-3}	1.60×10^{-4}
0.7	t8	1.11	1.33	1.46	1.86
0.7	t4	1.21	1.60	1.82	2.52
0.7	t3	1.27	1.74	*2.01*	2.83

For normal copula probability is given.
For t copulas the *factor* by which Gaussian probability must be multiplied is given.

Joint Tail Probabilities, $d \geq 2$

ρ	C	Dimension d			
		2	3	4	5
0.5	N	1.29×10^{-3}	3.66×10^{-4}	1.49×10^{-4}	7.48×10^{-5}
0.5	t8	1.65	2.36	3.09	3.82
0.5	t4	2.22	3.82	5.66	7.68
0.5	t3	2.55	4.72	7.35	10.34
0.7	N	2.67×10^{-3}	1.28×10^{-3}	7.77×10^{-4}	5.35×10^{-4}
0.7	t8	1.33	1.58	1.78	1.95
0.7	t4	1.60	2.10	2.53	2.91
0.7	t3	1.74	2.39	2.97	3.45

We consider only *99% quantile* and case of *equal correlations*.

Financial Interpretation

Consider daily returns on *five financial instruments* and suppose that we believe that all correlations between returns are equal to 50%. However, we are unsure about the best multivariate model for these data.

If returns follow a multivariate Gaussian distribution then the probability that on any day all returns fall below their *1% quantiles* is 7.48×10^{-5}. In the long run such an event will happen once every 13369 trading days on average, that is roughly *once every 51.4 years* (assuming 260 trading days in a year).

On the other hand, if returns follow a multivariate t distribution with four degrees of freedom then such an event will happen 7.68 times more often, that is roughly *once every 6.7 years*.

References

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- [Joe, 1997] (on dependence in general)
- [Nelsen, 2006] (standard reference on bivariate copulas)
- [Genest and Favre, 2007] (theory of copula fitting)
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