Quantitative Risk Management

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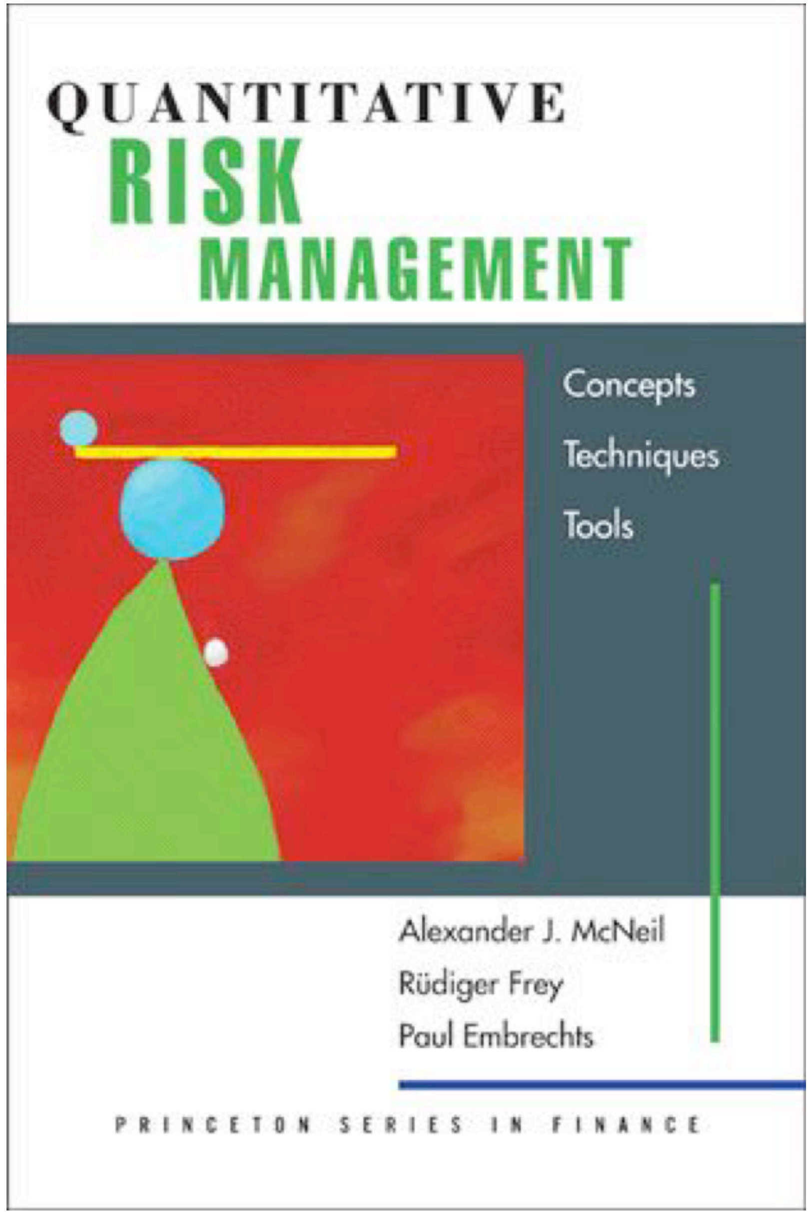
Course webpage:
`www.math.ethz.ch/~donnelly/qrm2010/qrm2010.html`


QUANTITATIVE
RISK
MANAGEMENT
Concepts
Techniques
Tools
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PRINCETON SERIES IN FINANCE

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A1. Risks, Losses and Risk Factors

We concentrate on the following sources of risk.

- **Market Risk** - risk associated with fluctuations in value of traded assets.

- **Credit Risk** - risk associated with uncertainty that debtors will honour their financial obligations

- **Operational Risk** - risk associated with possibility of human error, IT failure, dishonesty, natural disaster etc.

This is a non-exhaustive list; other sources of risk such as *liquidity risk* possible.

Modeling Financial Risks

To model risk we use language of *probability theory*. Risks are represented by *random variables* mapping unforeseen future states of the world into values representing *profits and losses*.

The risks which interest us are *aggregate* risks. In general we consider a *portfolio* which might be

- a collection of *stocks and bonds*;
- a book of *derivatives*;
- a collection of risky *loans*;
- a financial institution's *overall position* in risky assets.

Portfolio Values and Losses

Consider a portfolio and let V_t denote its *value* at time t; we assume this random variable is *observable* at time t.

Suppose we look at risk from perspective of time t and we consider the time period $[t, t+1]$. The value V_{t+1} at the end of the time period is unknown to us.

The distribution of $(V_{t+1} - V_t)$ is known as the profit-and-loss or *P&L distribution*. We denote the *loss* by $L_{t+1} = -(V_{t+1} - V_t)$. By this convention, losses will be positive numbers and profits negative.

We refer to the distribution of L_{t+1} as the *loss distribution*.

Introducing Risk Factors

The Value V_t of the portfolio/position will be modeled as a function of time and a set of d underlying risk factors. We write

$$V_t = f(t, \mathbf{Z}_t) \tag{1}$$

where $\mathbf{Z}_t = (Z_{t,1}, \ldots, Z_{t,d})'$ is the risk factor *vector*. This representation of portfolio value is known as a *mapping*. Examples of typical risk factors:

- (logarithmic) prices of financial assets
- yields
- (logarithmic) exchange rates

Risk Factor Changes

We define the time series of risk factor changes by

$$\mathbf{X}_t := \mathbf{Z}_t - \mathbf{Z}_{t-1}.$$

Typically, *historical* risk factor *time series* are available and it is of interest to relate the changes in these underlying risk factors to the changes in portfolio value.

We have

$$\begin{aligned} L_{t+1} &= -(V_{t+1} - V_t) \\ &= -\left(f(t+1, \mathbf{Z}_{t+1}) - f(t, \mathbf{Z}_t)\right) \\ &= -(f(t+1, \mathbf{Z}_t + \mathbf{X}_{t+1}) - f(t, \mathbf{Z}_t)) \end{aligned} \quad (2)$$

The Loss Operator

Since the risk factor values $\mathbf{Z}_t$ are known at time t the loss L_{t+1} is determined by the risk factor changes $\mathbf{X}_{t+1}$.

Given realization $\mathbf{z}_t$ of $\mathbf{Z}_t$, the loss operator at time t is defined as

$$l_{[t]}(\mathbf{x}) := -(f(t+1, \mathbf{z}_t + \mathbf{x}) - f(t, \mathbf{z}_t)), \tag{3}$$

so that

$$L_{t+1} = l_{[t]}(\mathbf{X}_{t+1}).$$

From the perspective of time t the loss distribution of L_{t+1} is determined by the multivariate distribution of $\mathbf{X}_{t+1}$.

But which distribution exactly? *Conditional* distribution of L_{t+1} given history up to and including time t or *unconditional* distribution under assumption that $(\mathbf{X}_t)$ form stationary time series?

A2. Example: Portfolio of Stocks

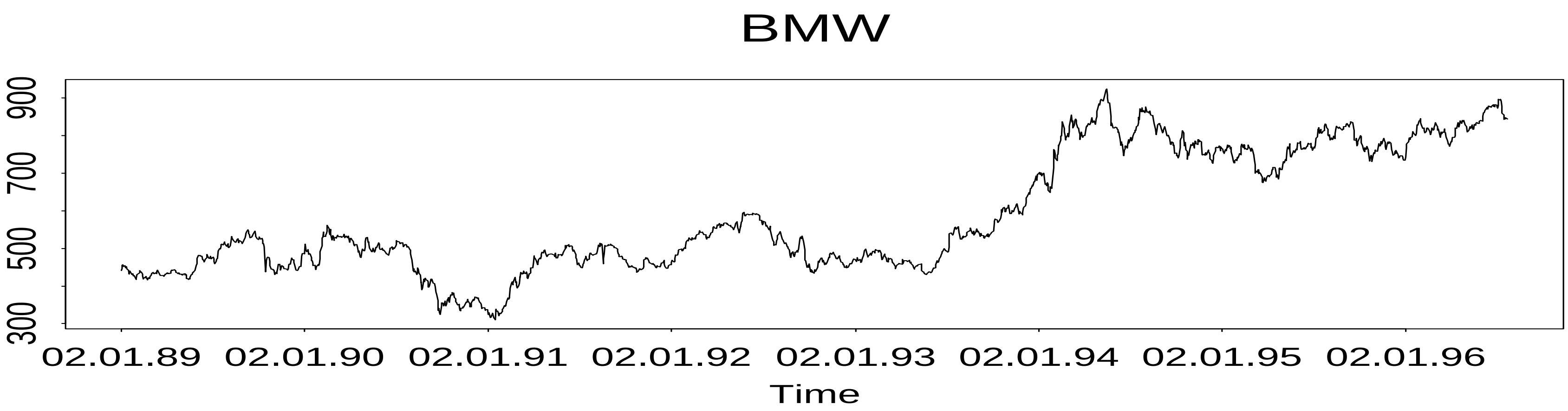
Consider d stocks; let α_i denote number of shares in stock i at time t and let $S_{t,i}$ denote price.

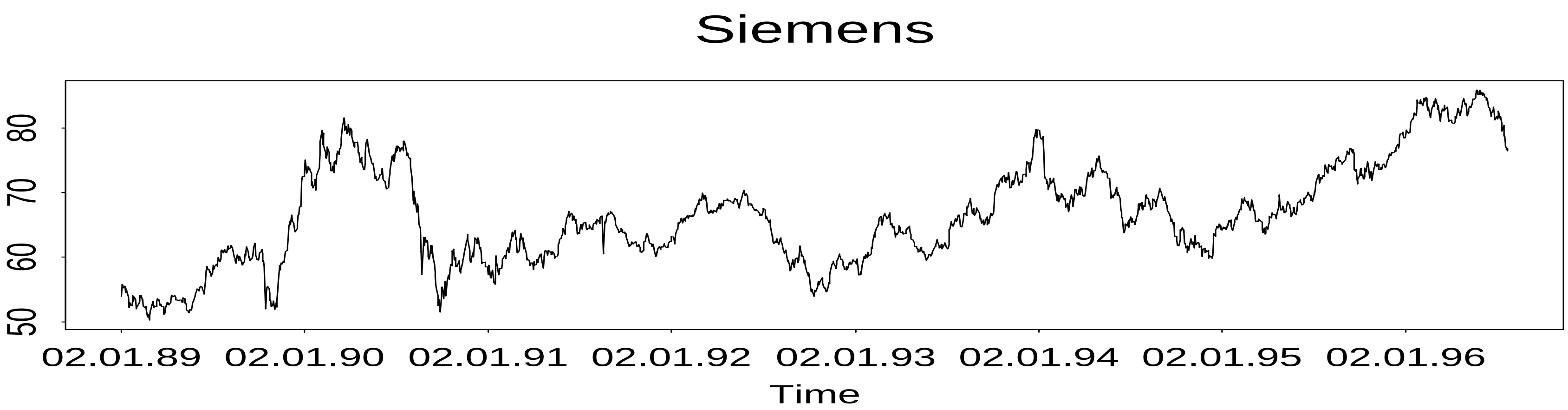
The risk factors: following standard convention we take logarithmic prices as risk factors $Z_{t,i} = \log S_{t,i}, 1 \leq i \leq d$.

The risk factor changes: in this case these are $X_{t+1,i} = \log S_{t+1,i} - \log S_{t,i}$, which correspond to the so-called *log-returns* of the stock.

The Mapping (1)

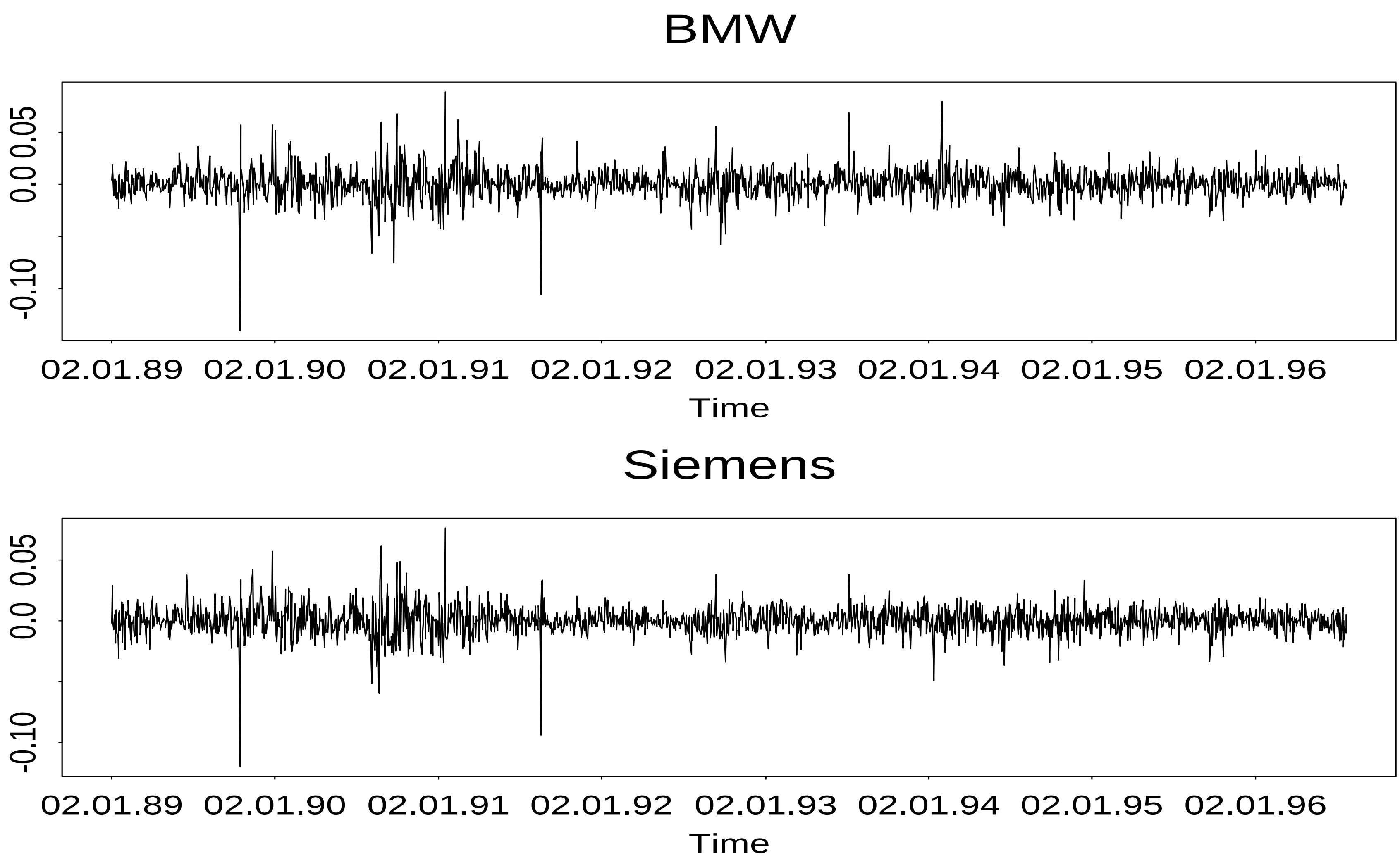
$$V_t = \sum_{i=1}^{d} \alpha_i S_{t,i} = \sum_{i=1}^{d} \alpha_i e^{Z_{t,i}}. \tag{4}$$





BMW and Siemens Data: 1972 days to 23.07.96.

Respective prices on evening 23.07.96: 844.00 and 76.9. Consider portfolio in ratio 1:10 on that evening.



BMW and Siemens Log Return Data: 1972 days to 23.07.96.

Example Continued

The Loss (2)

$$L_{t+1} = -\left(\sum_{i=1}^{d} \alpha_i e^{Z_{t+1,i}} - \sum_{i=1}^{d} \alpha_i e^{Z_{t,i}}\right)$$

$$= -V_t \sum_{i=1}^{d} \omega_{t,i} \left(e^{X_{t+1,i}} - 1\right) \quad (5)$$

where $\omega_{t,i} = \alpha_i S_{t,i}/V_t$ is relative weight of stock i at time t.

The loss operator (3)

$$l_{[t]}(\mathbf{x}) = -V_t \sum_{i=1}^{d} \omega_{t,i} \left(e^{x_i} - 1\right),$$

Numeric Example: $l_{[t]}(\mathbf{x}) = -\left(844(e^{x_1} - 1) + 769(e^{x_2} - 1)\right)$

A3. Conditional or Unconditional Loss Distribution?

This issue is related to the time series properties of $(\mathbf{X}_t)_{t \in \mathbb{N}}$, the series of risk factor changes. If we assume that $\mathbf{X}_t, \mathbf{X}_{t-1}, \ldots$ are *iid* random vectors, the issue does not arise. But, if we assume that they form a strictly *stationary* multivariate *time series* then we must differentiate between conditional and unconditional.

Many standard accounts of risk management fail to make the distinction between the two.

If we cannot assume that risk factor changes form a stationary time series for at least some window of time extending from the present back into intermediate past, then any statistical analysis of loss distribution is difficult.

The Conditional Problem

Let $\mathcal{F}_t$ represent the *history* of the risk factors up to the present.

More formally $\mathcal{F}_t$ is sigma algebra generated by past and present risk factor changes $(\mathbf{X}_s)_{s \leq t}$.

In the conditional problem we are interested in the distribution of $L_{t+1} = l_{[t]}(\mathbf{X}_{t+1})$ *given* $\mathcal{F}_t$, i.e. the conditional (or predictive) loss distribution for the next time interval given the history of risk factor developments up to present.

This problem forces us to model the *dynamics* of the risk factor time series and to be concerned in particular with predicting *volatility*. This seems the most suitable approach to market risk.

The Unconditional Problem

In the unconditional problem we are interested in the distribution of $L_{t+1} = l_{[t]}(\mathbf{X})$ when $\mathbf{X}$ is a *generic* vector of risk factor changes with the same distribution $F_{\mathbf{X}}$ as $\mathbf{X}_t, \mathbf{X}_{t-1}, \ldots$.

When we neglect the modeling of dynamics we inevitably take this view. Particularly when the time interval is large, it may make sense to do this. Unconditional approach also typical in credit risk.

More Formally

Conditional loss distribution: distribution of $l_{[t]}(\cdot)$ under $F_{[\mathbf{X}_{t+1}|\mathcal{F}_t]}$.

Unconditional loss distribution: distribution of $l_{[t]}(\cdot)$ under $F_{\mathbf{X}}$.

A4. Linearization of Loss

Recall the general formula (2) for the loss L_{t+1} in time period $[t, t+1]$. If the mapping f is differentiable we may use the following first order *approximation* for the loss

$$L_{t+1}^{\Delta} = -\left(f_t(t, \mathbf{Z}_t) + \sum_{i=1}^{d} f_{z_i}(t, \mathbf{Z}_t) X_{t+1,i} \right), \tag{6}$$

- – f_{z_i} is partial derivative of mapping with respect to risk factor i
  – f_t is partial derivative of mapping with respect to time

- The term $f_t(t, \mathbf{Z}_t)$ only appears when mapping explicitly features time (derivative portfolios) and is sometimes neglected.

Linearized Loss Operator

Recall the loss operator (3) which applies at time t. We can obviously also define a linearized loss operator

$$l^{\Delta}_{[t]}(\mathbf{x}) = -\left(f_t(t, \mathbf{z}_t) + \sum_{i=1}^{d} f_{z_i}(t, \mathbf{z}_t) x_i \right), \tag{7}$$

where notation is as in previous slide and $\mathbf{z}_t$ is realization of $\mathbf{Z}_t$.

Linearisation is convenient because linear functions of the risk factor changes may be easier to handle analytically. It is crucial to the *variance-covariance method*. The quality of approximation is best if we are measuring risk over a short time horizon and if portfolio value is almost linear in risk factor changes.

Stock Portfolio Example

Here there is no explicit time dependence in the mapping (4). The partial derivatives with respect to risk factors are

$$f_{z_i}(t, \mathbf{z}_t) = \alpha_i e^{z_{t,i}}, \; 1 \leq i \leq d,$$

and hence the linearized loss (6) is

$$L_{t+1}^{\Delta} = -\sum_{i=1}^{d} \alpha_i e^{\mathbf{z}_{t,i}} X_{t+1,i} = -V_t \sum_{i=1}^{d} \omega_{t,i} X_{t+1,i},$$

where $\omega_{t,i} = \alpha_i S_{t,i}/V_t$ is relative weight of stock i at time t. This formula may be compared with (5).

Numeric Example: $l_{[t]}^{\Delta}(\mathbf{x}) = -(844x_1 + 769x_2)$

A5. Example: European Call Option

Consider portfolio consisting of one standard European call on a non-dividend paying *stock* S with *maturity* T and *exercise* price K.

The Black-Scholes value of this asset at time t is $C^{BS}(t, S_t, r, \sigma)$ where

$$C^{BS}(t, S; r, \sigma) = S\Phi(d_1) - Ke^{-r(T-t)}\Phi(d_2),$$

Φ is standard normal df, r represents risk-free interest rate, σ the volatility of underlying stock, and where

$$d_1 = \frac{\log(S/K) + (r + \sigma^2/2)(T-t)}{\sigma\sqrt{T-t}} \text{ and } d_2 = d_1 - \sigma\sqrt{T-t}.$$

While in BS model, it is assumed that *interest rates* and *volatilities* are constant, in reality they tend to *fluctuate* over time; they should be added to our set of risk factors.

The Issue of Time Scale

Rather than measuring time in units of the time horizon (as we have implicitly done in most of this chapter) it is more common when *derivatives* are involved to *measure time in years* (as in the Black Scholes formula).

If Δ is the length of the time horizon measured in years (i.e. $\Delta = 1/260$ if time horizon is one day) then we have

$$V_t = f(t, \mathbf{Z}_t) = C^{BS}(t\Delta, S_t; r_t, \sigma_t).$$

When linearizing we have to recall that

$$f_t(t, \mathbf{Z}_t) = C_t^{BS}(t\Delta, S_t; r_t, \sigma_t)\Delta.$$

Example Summarised

The risk factors: $\mathbf{Z}_t = (\log S_t, r_t, \sigma_t)'$.

The risk factor changes:
$\mathbf{X}_t = (\log(S_t/S_{t-1}), r_t - r_{t-1}, \sigma_t - \sigma_{t-1})'$.

The mapping (1)

$$V_t = f(t, \mathbf{Z}_t) = C^{BS}(t\Delta, S_t; r_t, \sigma_t),$$

The loss/loss operator could be calculated from (2). For derivative positions it is quite common to calculate linearized loss.

The linearized loss (6)

$$L_{t+1}^{\Delta} = -\left(f_t(t, \mathbf{Z}_t) + \sum_{i=1}^{3} f_{z_i}(t, \mathbf{Z}_t) X_{t+1,i} \right).$$

The Greeks

It is more common to write the linearized loss as

$$L_{t+1}^{\Delta} = -\left(C_t^{BS}\Delta + C_S^{BS} S_t X_{t+1,1} + C_r^{BS} X_{t+1,2} + C_\sigma^{BS} X_{t+1,3}\right),$$

in terms of the derivatives of the BS formula.

- C_S^{BS} is known as the *delta* of the option.
- C_σ^{BS} is the *vega*.
- C_r^{BS} is the *rho*.
- C_t^{BS} is the *theta*.

A6. Risk Measurement

Risk measures are used for the following purposes:

- Determination of *risk capital*. Risk measure gives amount of capital needed as a buffer against (unexpected) future losses to satisfy a regulator.

- *Management tool*. Risk measures are used in internal limit systems.

- *Insurance premia* can be viewed as measure of riskiness of insured claims.

Our interpretation. Risk measure gives amount of capital that needs to be added to a position with loss L, so that the position becomes *acceptable* to an (internal/external) regulator.

Approaches to Risk Measurement

- *Notional-amount approach*. Risk of a portfolio is defined as the (weighted) sum of the notational values of the individual securities.

- *Factor sensitivity measures*. Give the change in portfolio value for a given predetermined change in one of the underlying risk factors.

- *Scenario-based risk measures*. One considers a number of future scenarios and measures the maximum loss of the portfolio under these scenarios.

- *Risk measures based on loss distributions*. Statistical quantities describing the loss distribution of the portfolio.

Risk Measures Based on Loss Distributions

Risk measures attempt to quantify the riskiness of a portfolio. The most popular risk measures like VaR describe the right tail of the loss distribution of L_{t+1} (or the left tail of the P&L).

To address this question we put aside the question of whether to look at conditional or unconditional loss distribution and assume that this has been decided.

Denote the distribution function of the loss $L := L_{t+1}$ by F_L so that $P(L \leq x) = F_L(x)$.

VaR and Expected Shortfall

- Primary risk measure: *Value at Risk* defined as

$$\mathrm{VaR}_\alpha = q_\alpha(F_L) = F_L^{\leftarrow}(\alpha)\,, \tag{8}$$

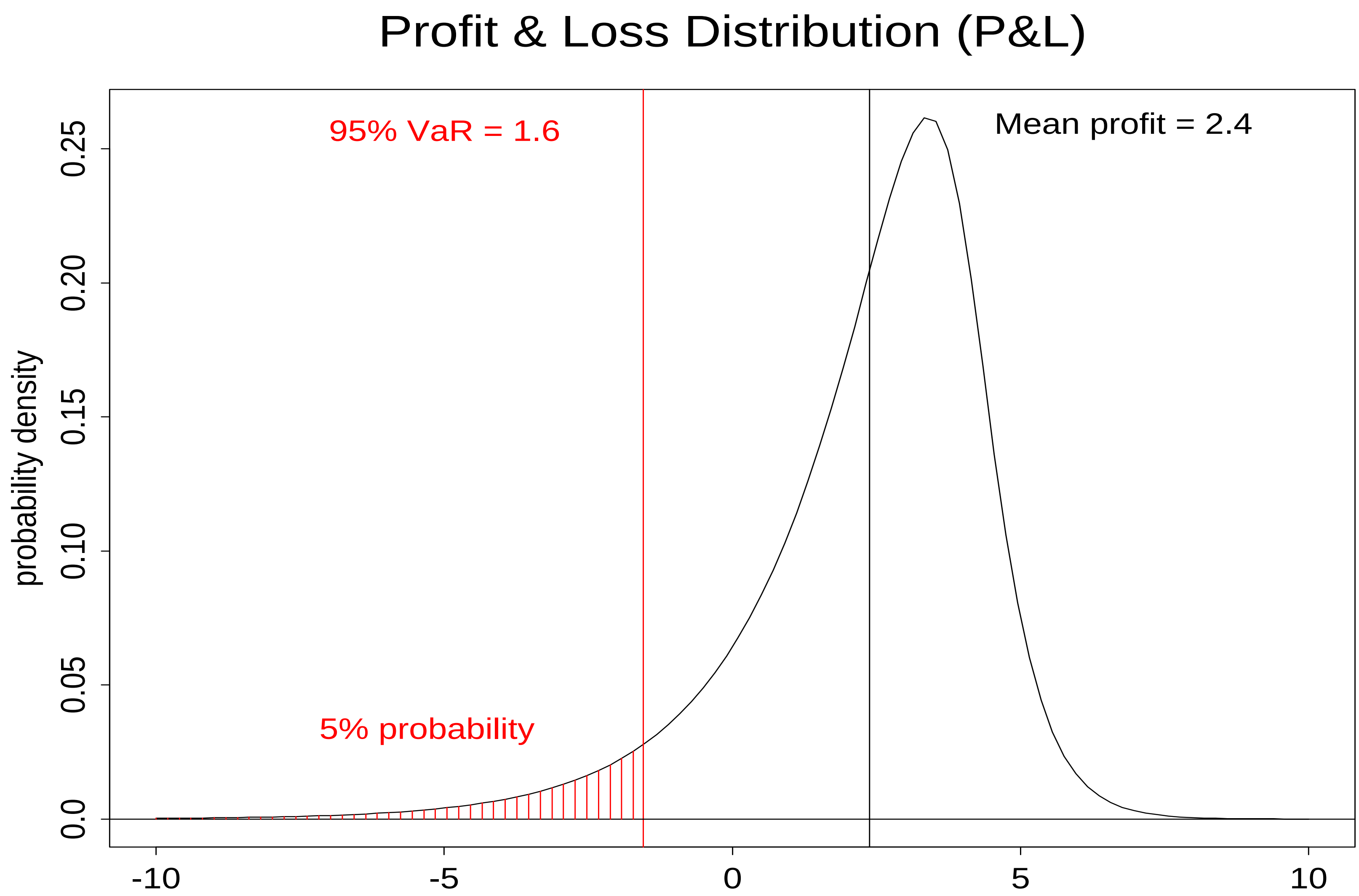
i.e. the α-quantile of F_L.

- Alternative risk measure: *Expected shortfall* defined as

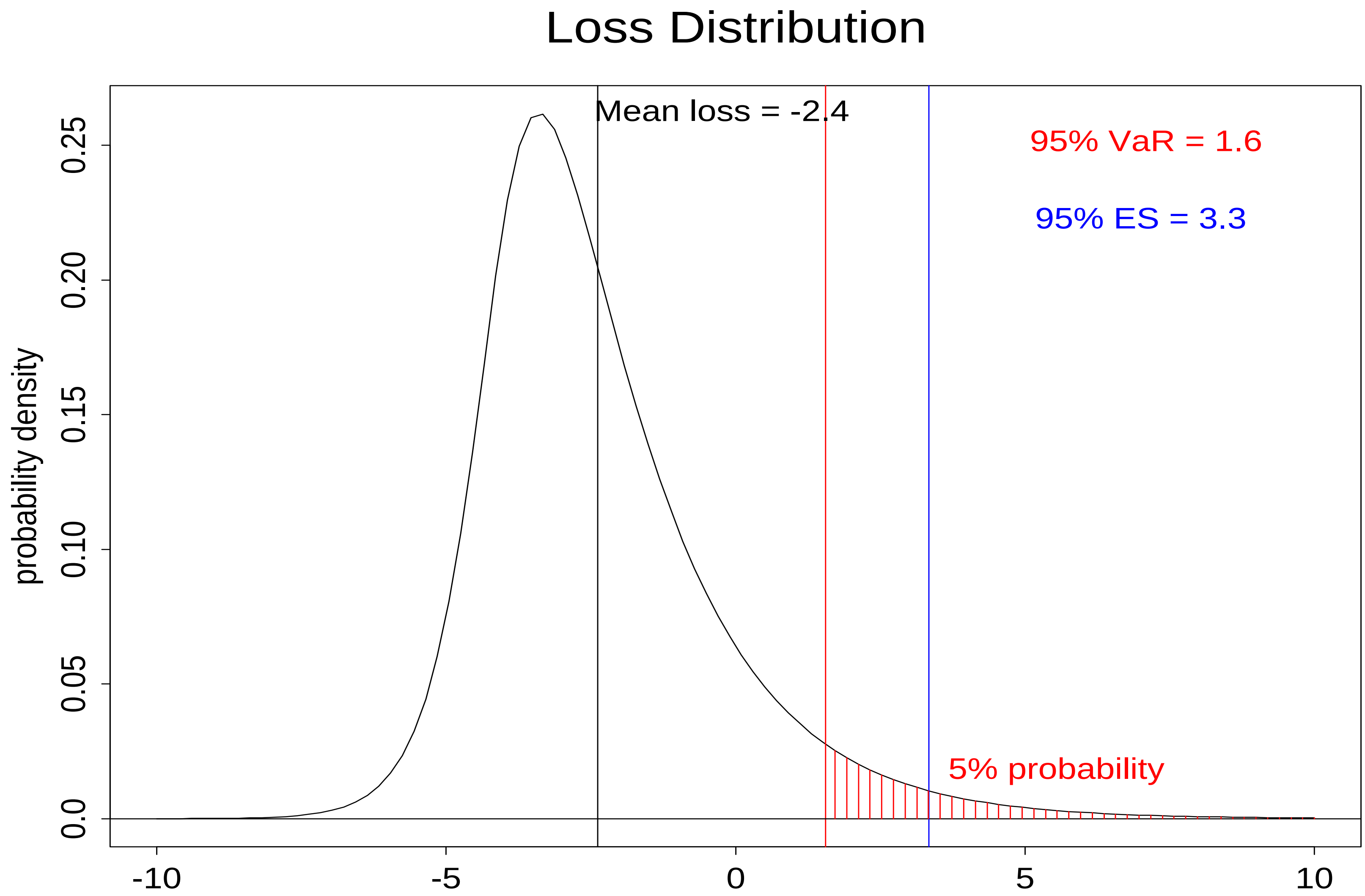
$$\mathsf{ES}_\alpha = E\left(L \;\middle|\; L > \mathrm{VaR}_\alpha\right)\,, \tag{9}$$

i.e. the *average* loss when VaR is exceeded. ES_α gives information about *frequency and size* of large losses.

VaR in Visual Terms



Losses and Profits



VaR - badly defined!

The VaR bible is the book by Philippe Jorion.[Jorion, 2007].

The following "definition" is very common:

"VaR is the $maximum$ expected loss of a portfolio over a given time horizon with a certain confidence level."

It is however mathematically meaningless and potentially misleading. In *no sense* is VaR a maximum loss!
We can lose more, sometimes much more, depending on the *heaviness of the tail* of the loss distribution.

References

On risk management:

- [McNeil et al., 2005] (methods for QRM)
- [Crouhy et al., 2010] (on risk management)
- [Jorion, 2007] (on VaR)

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[Jorion, 2007] Jorion, P. (2007). *Value at Risk: The New Benchmark for Measuring Financial Risk*. McGraw-Hill, New York, 3rd edition.

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