A martingale review of some fluctuation theory for spectrally negative Lévy processes*

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Abstract

We give a review of elementary fluctuation theory for spectrally negative Lévy processes using for the most part martingale theory. The methodology is based on techniques found in Kyprianou and Palmowski (2003) which deals with similar issues for a general class of Markov additive processes.

1 Introduction

Two and one sided exit problems for spectrally negative Lévy processes have been the object of several studies over the last 40 years. Significant contributions have come from Zolotarev (1964), Emery (1973), Bingham (1975) and Bertoin (1996a, 1996b, 1997). The principal tools of analysis of these authors are the Wiener-Hopf factorization and Itô's excursion theory.

In recent years, the study of Lévy processes has enjoyed rejuvenation. This has resulted in many applied fields such as the theory of mathematical finance, risk and queues adopting more complicated models which involve an underlying Lévy process. The aim of this text is to give a reasonably self contained approach to some elementary fluctuation theory which avoids the use of specialist theorems. Specifically we avoid the use of the Wiener-Hopf factorization and Itô's excursion theory and rely mainly on martingale arguments together with the Strong Markov property. None of the results we present are new but for the most part, the proofs approach the results from a new angle following Kyprianou and Palmowski (2003).

2 Spectrally negative Lévy processes

Let us start by briefly reviewing what is meant by a spectrally negative Lévy processes. The reader is referred to Bertoin (1996) and Sato (1999) for a complete discussion.

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Suppose that $(\Omega, \mathcal{F}, \mathbb{F}, P)$ is a filtered probability space with filtration $\mathbb{F} = \{\mathcal{F}_t : t \geq 0\}$ satisfying the usual conditions of right continuity and augmentation. In this text, we take as our definition of a Lévy process for $(\Omega, \mathcal{F}, \mathbb{F}, P)$, the strong Markov, $\mathbb{F}$-adapted process $X = \{X_t : t \geq 0\}$ with right continuous paths having the properties that $P(X_0 = 0) = 1$ and for each $0 \leq s \leq t$, the increment $X_t - X_s$ is independent of $\mathcal{F}_s$ and has the same distribution as X_{t-s}. In this sense, it is often said that a Lévy processes has stationary independent increments.

On account of the fact that the process has independent increments, it is not too difficult to show that

$$E\left(e^{i\theta X_t}\right) = e^{-t\Psi(\theta)}$$

where $\Psi(\theta) = \log E\left(\exp\{i\theta X_1\}\right)$. The Lévy-Khinchine formula gives the general form of the function $\Psi(\theta)$. That is,

$$\Psi(\theta) = i\mu\theta - \frac{\sigma^2}{2}\theta^2 + \int_{(-\infty,\infty)} (e^{i\theta x} - 1 - i\theta \mathbf{1}_{|x|<1})\Pi(dx) \tag{1}$$

for every $\theta \in \mathbb{R}$ where $\mu \in \mathbb{R}$, $\sigma > 0$ and Π is a measure on $\mathbb{R} \setminus \{0\}$ such that $\int (1 \wedge x^2)\Pi(dx) < \infty$.

Finally, we say that X is spectrally negative if the measure Π is supported only on $(-\infty, 0)$. We exclude from the discussion however the case of a downward subordinator, that is a spectrally negative Lévy process with monotone decreasing paths. Included in the discussion however are downward subordinators with positive drift (such as one might represent an insurance risk process, dam and storage models or the dual of a virtual waiting time process in an $M/G/1$ queue) and a Browinan motion with drift. Also included are processes (such as α-stable processes for $\alpha \in (1, 2)$) taking the form

$$\sum_{s \leq t} \Delta_s - t\int_{-\infty}^{0} x\Pi(dx), \ t \geq 0$$

where Δ_t us a Poisson point process on $[0, \infty) \times (-\infty, 0)$ with characteristic measure Π. (The process of compensted sums has to be defined in an appropriate limiting sense). Indeed by adding independent copies of any of the above (spectrally negative) processes together one still has a spectrally negative Lévy process.

For spectrally negative Lévy processes it is possible to talk of the Laplace exponent $\psi(\lambda)$ defined by

$$E\left(e^{\lambda X_t}\right) = e^{\psi(\lambda)t} \tag{2}$$

in other words, $\psi(\lambda) = -\Psi(-i\lambda)$. Since Π has negative support, we can safely say that $\psi(\lambda)$ exists at least for all $\lambda \geq 0$. Further, it is easy to check that ψ is strictly convex and tends to infinity as λ tends to infinity. This allows us to define,

$$\Phi(q) = \sup\{\lambda \geq 0 : \psi(\lambda) \geq q\},$$

the largest root of the equation $\psi(\lambda) = q$. Further we can identify $\psi'(0^+) = E(X_1) \in [-\infty, \infty)$ which, as we shall see in the next section, determines the long term behaviour of the process.

Suppose now the probabilities $\{P_x : x \in \mathbb{R}\}$ correspond to the conditional version of P where $X_0 = x$ is given. We simply write $P_0 = P$. The equality (2) allows for a Girsanov-type change of measure to be defined, namely via

$$\left.\frac{dP_x^c}{dP_x}\right|_{\mathcal{F}_t} = \frac{\mathcal{E}_t(c)}{\mathcal{E}_0(c)}$$

for any $c \geq 0$ where $\mathcal{E}_t(c) = \exp\{cX_t - \psi(c)t\}$ is the exponential martingale. Note that the fact that $\mathcal{E}_t(c)$ is a martingale follows from the fact that X has stationary independent increments together with (2). It is easy to check that under this change of measure, X remains within the class of spectrally negative processes and the Laplace exponent of X under P_x^c is given by

$$\psi_c(\theta) = \psi(\theta + c) - \psi(c)$$

for $\theta \geq -c$.

3 Exit problems

Let us now turn to the one and two sided exit problems for spectrally negative Lévy processes. The exit problems essentially consist of characterizing the Laplace transforms of τ_a^+, τ_0^- and $\tau_a^+ \wedge \tau_0^-$ where

$$\tau_0^- = \inf\{t \geq 0 : X_t \leq 0\} \text{ and } \tau_a^+ = \inf\{t \geq 0 : X_t \geq a\}$$

for any $a > 0$. Note that X will hit the point a when crossing upwards from below as it can only move continuously upwards. On the other hand, it may either hit 0 or jump over zero when crossing 0 from above depending on the components of the process.

It has turned out (cf. Zolotarev (1964), Emery (1973), Bingham (1975) and Bertoin (1996a, 1996b, 1997)) that one and two sided exit problems of spectrally negative functions can be characterized by the exponential function together with the following two families, $\{W^{(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ and $\{Z^{(q)}(x) : q \geq 0, x \in \mathbb{R}\}$ known as the scale functions which we define below.

Definition 1 *For each $q \geq 0$, $W^{(q)}(x)$ is defined to be identically zero on $(-\infty, 0]$ and its restriction to $(0, \infty)$ is characterized by the following Laplace transform*

$$\int_0^\infty e^{-\beta x} W^{(q)}(x)dx = \frac{1}{\psi(\beta) - q}, \text{ for } \beta > \Phi(q).$$

Further, the function $Z^{(q)}(x)$ is given by the formula

$$Z^{(q)}(x) = 1 + q\int_0^x W^{(q)}(y)dy.$$

Theorem 2 (One sided exit above) *For any $x \leq a$ and $q \geq 0$,*

$$E_x\left(e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\infty)}\right) = e^{-\Phi(q)(a-x)}.$$

Theorem 3 (One sided exit below) *For any $x \in \mathbb{R}$ and $q \geq 0$,*

$$E_x\left(e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\infty)}\right) = Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x)$$

where we understand $q/\Phi(q)$ in the limiting sense for $q = 0$, so that

$$P(\tau_0^+ < \infty) = \begin{cases} 1 - \psi'(0^+)W(x) & \text{if } \psi'(0^+) > 0 \\ 1 & \text{if } \psi'(0^+) \leq 0 \end{cases}.$$

Theorem 4 (Two sided exit) *For any $x \leq a$ and $q \geq 0$,*
(i) $E_x\left(e^{-q\tau_a^+}\mathbf{1}_{(\tau_0^->\tau_a^+)}\right) = W^{(q)}(x)/W^{(q)}(a)$
(ii) $E_x\left(e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^-<\tau_a^+)}\right) = Z^{(q)}(x) - Z^{(q)}(a)W^{(q)}(x)/W^{(q)}(a).$

Remark 5 By changing measure using the exponential martingale, one may extract identities from the above expressions giving the joint Laplace transform of the time to overshoot and overshoot itself.

The proof of the first Theorem we give is not new and follows as an easy consequence of Doob's optional stopping theorem applied to the exponential martingale. The proof of the remaining two theorems are a direct consequence of a special martingale which we shall discuss in Section 5. The proofs of Theorems 3 and 4 are given in Sections 6, 7 and 8. The structure of this text (in particular the proofs of Theorems 3 and 4) is based on new results and methodology for a general class of Markov additive processes given in Kyprianou and Palmowski (2003).

4 Proof of Theorem 2

It follows almost directly from Doob's Optional Stopping Theorem that

$$E_x\left(\frac{\mathcal{E}_{t\wedge\tau_a^+}(\Phi(q))}{\mathcal{E}_0(\Phi(q))}\right) = E_x\left(e^{\Phi(q)(X_{t\wedge\tau_a^+}-x)-q(t\wedge\tau_a^+)}\right) = 1.$$

Clearly

$$e^{\Phi(q)(X_{t\wedge\tau_a^+}-x)-q(t\wedge\tau_a^+)} \leq e^{\Phi(q)(a-x)}$$

and hence by dominated convergence and the fact that $X_{\tau_a^+} = a$ on $\tau_a^+ < \infty$ we have,

$$E_x\left(e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\infty)}\right) = e^{-\Phi(q)(a-x)}.$$

5 The Kella-Whitt martingale

As already mentioned in the introduction, we shall base our proofs for the most part on martingale arguments. A martingale which plays a fundamental role in our calculations is what we shall call the Kella-Whitt martingale, introduced in Kella and Whitt (1992). For completeness we shall introduce this martingale in the following theorem.

Theorem 6 *Let* $\overline{X}_t = \sup_{0\le u\le t} X_u$*, and* $Z_t = \overline{X}_t - X_t$ *then for* $\alpha \ge 0$

$$M_t := \psi(\alpha)\int_0^t e^{-\alpha Z_s}ds + 1 - e^{-\alpha Z_t} - \alpha\overline{X}_t,\ t\ge 0 \tag{3}$$

is a martingale.

Proof. Let $\mathcal{E}_t(\alpha) = \exp\{\alpha X_t - \psi(\alpha)t\}$ and note that

$$\begin{aligned} d\mathcal{E}_t(\alpha) &= \mathcal{E}_{t-}(\alpha)(\alpha dX_t - \psi(\alpha)dt) + \frac{1}{2}\alpha^2\mathcal{E}_{t-}(\alpha)d[X,X]_t^c \\ &\quad +\{\triangle\mathcal{E}_t(\alpha) - \alpha\mathcal{E}_{t-}(\alpha)\triangle X_t\}. \end{aligned}$$

Note also that

$$\begin{aligned} dM_t &= \psi(\alpha)e^{-\alpha Z_{t-}}dt + \alpha e^{-\alpha Z_{t-}}dZ_t - \frac{1}{2}\alpha^2 e^{-\alpha Z_{t-}}d[X,X]_t^c \\ &\quad -\{\triangle e^{-\alpha Z_t} + \alpha\triangle Z_t\} - \alpha d\overline{X}_t \\ &= e^{-\alpha\overline{X}_t+\psi(\alpha)t}[\psi(\alpha)\mathcal{E}_{t-}(\alpha)dt + \alpha\mathcal{E}_{t-}(\alpha)\left(d\overline{X}_t - dX_t\right) \\ &\qquad -\frac{1}{2}\alpha^2\mathcal{E}_{t-}(\alpha)d[X,X]_t^c \\ &\qquad -\mathcal{E}_{t-}(\alpha)\{e^{\alpha\triangle X_t} - 1 - \alpha\triangle X_t\} \\ &\qquad -\alpha e^{\alpha\overline{X}_t-\psi(\alpha)t}d\overline{X}_t] \\ &= e^{-\alpha\overline{X}_t+\psi(\alpha)t}\{-d\mathcal{E}_t(\alpha) + \alpha(\mathcal{E}_t(\alpha) - e^{\alpha\overline{X}_t-\psi(\alpha)t})d\overline{X}_t\}. \end{aligned}$$

Since $\overline{X}_t = X_t$ if and only if $\overline{X}_t$ increases, we may write

$$\begin{aligned} dM_t &= e^{-\alpha\overline{X}_t+\psi(\alpha)t}\{-d\mathcal{E}_t(\alpha) + \alpha(\mathcal{E}_t(\alpha) - e^{\alpha\overline{X}_t-\psi(\alpha)t})\mathbf{1}_{(\overline{X}_t=X_t)}d\overline{X}_t\} \\ &= -e^{-\alpha\overline{X}_t+\psi(\alpha)t}d\mathcal{E}_t(\alpha) \end{aligned}$$

showing that M_t is a local martingale since $\mathcal{E}_t(\alpha)$ is a martingale. To prove that M is a martingale, it suffices to show that for each $t>0$,

$$E\left(\sup_{s\le t}|M_s|\right) < \infty.$$

To this end note that since the events $\{\overline{X}_{\mathbf{e}_q} > x\}$ and $\{\tau_x^+ < \mathbf{e}_q\}$ are almost surely equivalent where $\mathbf{e}_q$ is an exponential distribution independent of X and with intensity $q>0$, it follows that

$$P\left(\overline{X}_{\mathbf{e}_q} > x\right) = E\left(e^{-q\tau_x^+}\mathbf{1}_{(\tau_x^+<\infty)}\right) = e^{-\Phi(q)x}$$

showing that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. It follows that

$$E\left(\overline{X}_{\mathbf{e}_q}\right) = \int_0^\infty q e^{-qt} E\left(\overline{X}_t\right) dt = \frac{1}{\Phi(q)} < \infty$$

and hence since $\overline{X}_t$ is an increasing process $E\left(\overline{X}_t\right) < \infty$ for all t. Now note by the positivity of the process Z and again since $\overline{X}$ increases,

$$E\left(\sup_{s\le t} |M_s|\right) \le \psi(\alpha) t + 2 + \alpha E\left(\overline{X}_t\right) < \infty$$

for each finite $t > 0$. ■

An application involving this martingale, brings us to an identity which is effectively the Wiener-Hopf factorization in disguise. Alternatively one may say that the Wiener-Hopf factorization for spectrally negative Lévy processes brings one to the same conclusion.

Theorem 7 *Let* $\underline{X}_t = \inf_{0\le u\le t} X_t$ *and suppose that* $\mathbf{e}_q$ *is an exponentially distributed random variable with parameter* $q > 0$ *independent of the process* X. *Then for* $\alpha > 0$,

$$E\left(e^{\alpha \underline{X}_{\mathbf{e}_q}}\right) = \frac{q(\alpha - \Phi(q))}{\Phi(q)(\psi(\alpha) - q)}. \tag{4}$$

Proof. We begin by noting some facts which will be used in conjunction with the martingale (3). Recall that $\mathbf{e}_q$ is an exponentially distributed random variable with parameter $q > 0$ independent of the process X.

First note that by an application of Fubini's theorem,

$$E\int_0^{\mathbf{e}_q} e^{-\alpha Z_s} ds = \int_0^\infty e^{-qs} E\left(e^{-\alpha Z_s}\right) ds = \frac{1}{q} E\left(e^{-\alpha Z_{\mathbf{e}_q}}\right).$$

Next we recall a well known result, known as the Duality Lemma, which can best be verified with a diagram. That is by defining the process $\{\widetilde{X}_s = X_{(t-s)^-} - X_t : 0 \le s \le t\}$ as the time reversed Lévy process from the fixed moment, t, the law of $\widetilde{X}$ and $-X$ are the same. In particular, this means that

$$-\inf_{0\le s\le t} X_s \stackrel{d}{=} \sup_{0\le s\le t} \widetilde{X}_s = \overline{X}_t - X_t.$$

Using the last two observations we have from (3) that $E\left(M_{\mathbf{e}_q}\right) = 0$ and hence

$$\frac{\psi(\alpha) - q}{q} E\left(e^{\alpha \underline{X}_{\mathbf{e}_q}}\right) = \alpha E\left(\overline{X}_{\mathbf{e}_q}\right) - 1.$$

Recall from the proof of Theorem 6 that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. It now follows that

$$\frac{\psi(\alpha) - q}{q} E\left(e^{\alpha \underline{X}_{\mathbf{e}_q}}\right) = \frac{\alpha - \Phi(q)}{\Phi(q)} \tag{5}$$

and the theorem is now proved. ■

Remark 8 Recall that $\overline{X}_{\mathbf{e}_q}$ is exponentially distributed with parameter $\Phi(q)$. It thus follows that for $\alpha < \Phi(q)$

$$E\left(e^{\alpha \overline{X}_{\mathbf{e}_q}}\right) = \frac{\Phi(q)}{\Phi(q) - \alpha} \tag{6}$$

and hence (5) reads

$$E\left(e^{\alpha \overline{X}_{\mathbf{e}_q}}\right) E\left(e^{\alpha \underline{X}_{\mathbf{e}_q}}\right) = \frac{q}{q - \psi(\alpha)} = E\left(e^{\alpha X_{\mathbf{e}_q}}\right).$$

which is essentially the Wiener-Hopf factorization.

In the previous section it was remarked that $\psi'(0^+)$ characterizes the asymptotic behaviour of X. We may now use the results of the previous Remark and Theorem to elaborate on this point. We do so in the form of a Lemma.

Lemma 9 *We have*
(i) $\limsup_{t\uparrow\infty} X_t = \infty$ *if and only if* $\psi'(0^+) \geq 0$,
(ii) $\liminf_{t\uparrow\infty} X_t = -\infty$ *if and only if* $\psi'(0^+) \leq 0$.

Proof. On account of the strict convexity ψ it follows that $\Phi(0) > 0$ if and only if $\psi'(0^+) < 0$ and hence

$$\lim_{q\downarrow 0} \frac{q}{\Phi(q)} = \begin{cases} 0 & \text{if } \psi'(0^+) < 0 \\ \psi'(0^+) & \text{if } \psi'(0^+) \geq 0 \end{cases}.$$

By taking q to zero in the identity (4) we now have that

$$E\left(e^{\alpha \underline{X}_\infty}\right) = \begin{cases} 0 & \text{if } \psi'(0^+) < 0 \\ \psi'(0^+)\alpha/\psi(\alpha) & \text{if } \psi'(0^+) \geq 0 \end{cases}.$$

Next, recall from (6) that for $\beta > 0$

$$E\left(e^{-\beta \overline{X}_{\mathbf{e}_q}}\right) = \frac{\Phi(q)}{\Phi(q) + \beta}$$

and hence by taking the limit of both sides as q tends to zero,

$$E\left(e^{-\alpha \overline{X}_\infty}\right) = \begin{cases} (\alpha/\Phi(0) + 1)^{-1} & \text{if } \psi'(0^+) < 0 \\ 0 & \text{if } \psi'(0^+) \geq 0 \end{cases}. \tag{7}$$

It is now clear that

$$\begin{aligned} P(\limsup_{t\uparrow\infty} X_t &= \infty) = 1 \Leftrightarrow \psi'(0^+) \geq 0 \\ P(\liminf_{t\uparrow\infty} X_t &= -\infty) = 1 \Leftrightarrow \psi'(0^+) \leq 0 \end{aligned}$$

which corresponds to the two cases of the statement of the theorem. ■

6 Proof of Theorem 4 (i)

We prove this theorem in stages. We prove the theorem for the case that $\psi'(0^+) > 0$ and $q = 0$, then for the case that $q > 0$ (no restriction on $\psi'(0^+)$). Finally the case that $\psi'(0^+) \leq 0$ and $q = 0$ is achieved by passing to the limit as q tends to zero.

Assume then that $\psi'(0^+) > 0$ so that $-\underline{X}_\infty$ is almost surely finite. As earlier seen in the proof of Lemma 9, by taking q to zero in 4 it follows that

$$E\left(e^{\alpha \underline{X}_\infty}\right) = \psi'(0) \frac{\alpha}{\psi(\alpha)}.$$

Integration by parts shows that

$$\begin{aligned} E\left(e^{\alpha \underline{X}_\infty}\right) &= \int_0^\infty e^{-\alpha x} P(-\underline{X}_\infty \in dx) \\ &= \alpha \int_0^\infty e^{-\alpha x} P(-\underline{X}_\infty < x) dx \\ &= \alpha \int_0^\infty e^{-\alpha x} P_x(\underline{X}_\infty > 0) dx. \end{aligned}$$

Now define the function

$$W(x) = \frac{1}{\psi'(0^+)} P_x(\underline{X}_\infty > 0)$$

Clearly $W(x) = 0$ for $x \leq 0$ and it has Laplace transform on $(0, \infty)$ given by $1/\psi(\alpha)$. A simple argument using the law of total probability and the Strong Markov Property now yields for $x \in (0, a)$

$$\begin{aligned} P_x(\underline{X}_\infty &> 0) = E_x P(\underline{X}_\infty > 0 | \mathcal{F}_{\tau_a^+}) \\ &= E_x\left(\mathbf{1}_{(\tau_a^+ < \tau_0^-)} P_a(\underline{X}_\infty > 0)\right) + E_x\left(\mathbf{1}_{(\tau_a^+ > \tau_0^-)} P_{X_{\tau_0^-}}(\underline{X}_\infty > 0)\right) \\ &= P_a(\underline{X}_\infty > 0) P_x(\tau_a^+ < \tau_0^-). \end{aligned}$$

That is to say

$$P_x(\tau_a^+ < \tau_0^-) = \frac{W(x)}{W(a)}$$

and clearly the same equality holds even when $x \leq 0$.

Now assume that $q > 0$. In this case, by convexity of ψ, we know that $\Phi(q) > 0$ and hence $\psi'_{\Phi(q)}(0) = \psi'(\Phi(q)) > 0$ (again by convexity). Changing measure using the Girsanov density, we have for $x \in (0, a)$

$$\begin{aligned} E_x\left(e^{-q\tau_a^+} \mathbf{1}_{(\tau_a^+ < \tau_0^-)}\right) &= E_x\left(\frac{\mathcal{E}_{\tau_a^+}(\Phi(q))}{\mathcal{E}_0(\Phi(q))} \mathbf{1}_{(\tau_a^+ < \tau_0^-)}\right) e^{-\Phi(q)(a-x)} \\ &= e^{-\Phi(q)(a-x)} P_x^{\Phi(q)}\left(\tau_a^+ < \tau_0^-\right). \end{aligned}$$

According to our previous calculations for the case that X drifts to infinity, we can now identify

$$E_x\left(e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+<\tau_0^-)}\right)=\frac{W^{(q)}(x)}{W^{(q)}(a)} \tag{8}$$

such that $W^{(q)}(x)=e^{\Phi(q)x}W_{\Phi(q)}(x)$ where $W_{\Phi(q)}(x)$ is identically zero on $(-\infty,0]$ which has Laplace transform $1/\psi_{\Phi(q)}(\alpha)$ on $(0,\infty)$. Taking Laplace transforms of $W^{(q)}(x)$ it appears now that for $\alpha>\Phi(q)$,

$$\begin{aligned}\int_0^\infty e^{-\alpha x}W^{(q)}(x)dx &= \int_0^\infty e^{-(\alpha-\Phi(q))x}W_{\Phi(q)}(x)dx\\ &= \frac{1}{\psi_{\Phi(q)}(\alpha-\Phi(q))}\\ &= \frac{1}{\psi(\alpha)-q}\end{aligned} \tag{9}$$

where in the last equality we have used the fact that for $c>0$, $\psi_c(\theta)=\psi(\theta+c)-\psi(c)$.

As mentioned at the beginning of the proof, the final missing case of X not drifting to infinity (ie $\psi'(0^+)\leq 0$) and $q=0$ is achieved by passing to the limit as $q\downarrow 0$ in (8) and (9). Note one uses dominated convergence when taking the limit in (9) as the function $W^{(p)}(x)$ can be written $e^{\Phi(p)x}P_x^{\Phi(p)}(\underline{X}_\infty>0)\leq e^{\Phi(q)x}$ for all $0\leq p\leq q$.

7 Proof of Theorem 3

Let us break off from the proof of Theorem 4 and quickly prove Theorem 3. Now that it is clear that for $\alpha>\Phi(q)$

$$\int_0^\infty e^{-\alpha x}W^{(q)}(x)dx=\frac{1}{\psi(\alpha)-q}$$

and hence by integrating by parts,

$$\int_0^\infty e^{-\alpha x}W^{(q)}(dx)=\frac{\alpha}{\psi(\alpha)-q}$$

we can interpret (4) as saying that

$$P(-\underline{X}_{\mathbf{e}_q}\in dx)=\frac{q}{\Phi(q)}W^{(q)}(dx)-qW^{(q)}(x)dx$$

and hence with an easy manipulation, for $x > 0$

$$\begin{aligned} P_x(\underline{X}_{\mathbf{e}_q} &< 0) = P_x(\mathbf{e}_q > \tau_0^-) \\ &= E_x\left(e^{-q\tau_0^-}\mathbf{1}_{(\tau_0^- < \infty)}\right) \\ &= 1 + q\int_0^x W^{(q)}(y)dy - \frac{q}{\Phi(q)}W^{(q)}(x) \\ &= Z^{(q)}(x) - \frac{q}{\Phi(q)}W^{(q)}(x). \end{aligned} \tag{10}$$

Note that since $Z^{(q)}(x) = 1$ and $W^{(q)}(x) = 0$ for all $x \in (-\infty, 0]$, the statement is valid for all $x \in \mathbb{R}$. The proof is now complete for the case that $q > 0$.

Recalling that $\lim_{q\downarrow 0} q/\Phi(q)$ is either $\psi'(0^+)$ if the process drifts to infinity, or zero otherwise, the proof is completed by taking the limit in q.

8 Proof of Theorem 4 (ii)

Fix $q > 0$. The Strong Markov Property together with the identity (10) give us that

$$\begin{aligned} & P_x\left(\underline{X}_{\mathbf{e}_q} < 0 | \mathcal{F}_{t\wedge\tau_a^+\wedge\tau_0^-}\right) \\ = \; & e^{-q(t\wedge\tau_a^+\wedge\tau_0^-)}P_{X_{t\wedge\tau_a^+\wedge\tau_0^-}}\left(\underline{X}_{\mathbf{e}_q} < 0 | \mathcal{F}_{t\wedge\tau_a^+\wedge\tau_0^-}\right) \\ = \; & e^{-q(t\wedge\tau_a^+\wedge\tau_0^-)}\left(Z^{(q)}(X_{t\wedge\tau_a^+\wedge\tau_0^-}) - \frac{q}{\Phi(q)}W^{(q)}(X_{t\wedge\tau_a^+\wedge\tau_0^-})\right) \end{aligned}$$

showing that the right hand side is a martingale for $t \geq 0$. Note also that with a similar methodology we have (using that $W^{(q)}(X_{\tau_0^-\wedge\tau_a^+}) = \mathbf{1}_{(\tau_a^+ < \tau_0^-)}W^{(q)}(a)$)

$$\begin{aligned} & P_x\left(e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+ < \tau_0^-)} | \mathcal{F}_{t\wedge\tau_a^+\wedge\tau_0^-}\right) \\ = \; & \mathbf{1}_{(t < \tau_0^-\wedge\tau_a^+)}e^{-qt}P_{X_t}\left(e^{-q_{\tau_a^+}}\mathbf{1}_{(\tau_a^+ < \tau_0^-)}\right) + \mathbf{1}_{(t > \tau_0^-\wedge\tau_a^+)}e^{-q\tau_a^+}\mathbf{1}_{(\tau_a^+ < \tau_0^-)} \\ = \; & \mathbf{1}_{(t < \tau_0^-\wedge\tau_a^+)}e^{-qt}\frac{W^{(q)}(X_t)}{W^{(q)}(a)} + \mathbf{1}_{(t > \tau_0^-\wedge\tau_a^+)}e^{-q(\tau_0^-\wedge\tau_a^+)}\frac{W^{(q)}(X_{\tau_0^-\wedge\tau_a^+})}{W^{(q)}(a)} \\ = \; & e^{-q(t\wedge\tau_a^+\wedge\tau_0^-)}\frac{W^{(q)}(X_{t\wedge\tau_0^-\wedge\tau_a^+})}{W^{(q)}(a)} \end{aligned}$$

showing again that the right hand side is a martingale for $t \geq 0$.

Now it follows by linearity that

$$e^{-q(t\wedge\tau_a^+\wedge\tau_0^-)}\left(Z^{(q)}(X_{t\wedge\tau_a^+\wedge\tau_0^-}) - \frac{Z^{(q)}(a)}{W^{(q)}(a)}W^{(q)}(X_{t\wedge\tau_a^+\wedge\tau_0^-})\right)$$

is also a martingale for $t \geq 0$. In fact it is a uniformly integrable martingale and hence its terminal expectation is equal to its initial expectation. That is to say

$$\begin{aligned} & E_x\left(e^{-q(\tau_a^+\wedge\tau_0^-)}\left(Z^{(q)}(X_{\tau_a^+\wedge\tau_0^-}) - \frac{Z^{(q)}(a)}{W^{(q)}(a)}W^{(q)}(X_{\tau_a^+\wedge\tau_0^-})\right)\right) \\ = & E_x\left(e^{-q\tau_0^-}\mathbf{1}_{(\tau_a^+>\tau_0^-)}\right) \\ = & Z^{(q)}(x) - \frac{Z^{(q)}(a)}{W^{(q)}(a)}W^{(q)}(x) \end{aligned}$$

where as usual we have used the fact that $Z^{(q)}(X_{\tau_a^+\wedge\tau_0^-}) = 1$ and $W^{(q)}(X_{\tau_a^+\wedge\tau_0^-}) = 0$ if $\tau_0^- < \tau_a^+$ and $Z^{(q)}(X_{\tau_a^+\wedge\tau_0^-}) = Z^{(q)}(a)$ and $W^{(q)}(X_{\tau_a^+\wedge\tau_0^-}) = W^{(q)}(a)$ if $\tau_0^- > \tau_a^+$.

For the case that $q = 0$, we again take limits as q tends to zero.

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