The Wiener chaos expansion for the Cox–Ingersoll–Ross model

M. R. Grasselli * and T. R. Hurd*
Dept. of Mathematics and Statistics
McMaster University
Hamilton ON L8S 4K1
Canada

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Abstract

In this paper we recast the Cox–Ingersoll–Ross model of interest rates into the chaotic representation recently introduced by Hughston and Rafailidis. Beginning with the "squared Gaussian representation" of the CIR model, we find a simple expression for the fundamental random variable X_∞. By use of techniques from the theory of infinite dimensional Gaussian integration, we derive an explicit formula for the nth term of the Wiener chaos expansion of the CIR model, for $n = 0, 1, 2, \ldots$. We then derive a new expression for the price of a zero coupon bond which reveals a connection between Gaussian measures and Ricatti differential equations.

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1 Introduction

In the present paper we shall study the best known example of a term structure with positive interest rates, namely the CIR model [1], in the context of the "chaotic approach" to interest rate dynamics introduced recently by Hughston and Rafailidis [4]. By an interest rate model we mean the specification of a spot rate process r_t and of a market price of risk process λ_t both under the "natural" or physical measure in the economy P . In the chaotic approach, the random nature of the model is assumed to be given by a probability space $(\Omega, \mathcal{F}, P)$ equipped with a Brownian filtration $(\mathcal{F}_t)_{0\leq t\leq\infty}$. The essence of the approach by Hughston and Rafailidis is the specification of the most general term structure with positive interest rates in terms of a single unconstrained random variable they denote X_∞. They then apply a Wiener chaos expansion to X_∞, and interpret the resulting terms as building blocks for models of increasing complexity.

In one version of the CIR model, r_t, λ_t are governed by the equations

$$dr_t = a(b - r_t)dt + c\sqrt{r_t}d\tilde{W}_t, \qquad \|\lambda_t\|^2 = \bar{\lambda}^2 r_t \tag{1}$$

for some positive constants a, b, c, λ with $4ab > c^2$, where $\tilde{W}_t$ is a standard one dimensional P–Brownian motion and λ_t is a vector valued adapted process. By embedding this model inside a general class of squared Gaussian models, we will be led to a natural choice for the basic square integrable random variable X_∞ associated to it. As we shall then demonstrate, the resulting stochastic process $X_t = E_t[X_\infty]$ admits an explicit chaos expansion, one which in general includes terms of every chaotic order.

A central idea in the present paper is the link between the Wiener chaos expansion and the theory of Gaussian functional integration, an essential tool invented to study the mathematical structure of quantum field theory. In fact, the mathematics underlying our example is a consequence of certain basic results in that theory, and can be found in for example [3].

The organization of the paper is as follows. In section 2 we review the essential ingredients for the construction of positive interest rates models in both Flesaker–Hughston and the state price density approaches, and then compare these approaches to the recently introduced chaotic representation of Hughston and Rafailidis [4]. We end the section by describing the structure of the Wiener–Itô chaos expansion and show how it can be expressed in terms of a certain generating functional acting on the space $L^2(\mathbb{R}^N)$.

In section 3, we describe the squared Gaussian formulation of the CIR model and show the spot rate process can be explicitly computed. Based on this representation, we state the form of the random variable X_∞, and give a proof that it lies in L^2. In section 4, we state the exponential quadratic formula which is the main technical tool in this paper. It is a formula for the generating functional of random variables of the form $X = e^{-Y}$ for Y lying in a general class of elements in the second chaos space $\mathcal{H}_2$. In section 5, we compute the generating function for the random variables X_t in the CIR model and, as the main result of the paper, derive their chaos expansion. In section 6, we show that the usual CIR bond pricing formula has a natural derivation within the chaotic framework.

Three appendices focus on the theory of Gaussian functional integration and its relation to the Wiener chaos expansion. Appendix A explores the white noise calculus. Appendix B states and provides a proof of the generating functional theorem. Appendix C provides a proof of the exponential quadratic formula.

2 Positive Interest Rates

2.1 The Flesaker–Hughston approach

Rather than focus on the spot rate process, one can model the system of bond prices directly. Let $P_{tT}, 0 \leq t \leq T$ denote the price at time t for a zero coupon bond which pays one unit of currency at its maturity T. Clearly $P_{tt} = 1$ for all $0 \leq t < \infty$ and furthermore, positivity of the interest rate is equivalent to having

$$P_{ts} \leq P_{tu}, \tag{2}$$

for all $0 \leq t \leq u \leq s$.

Flesaker and Hughston [2, eq. (8)] introduced zero coupon bond prices in the form

$$P_{tT} = \frac{\int_T^\infty h_s M_{ts} ds}{\int_t^\infty h_s M_{ts} ds}, \quad \text{for } 0 \leq t \leq T < \infty, \tag{3}$$

where M_{ts} is a family of strictly positive continuous martingales satisfying $M_{0s} = 1$ and $h_T = -\frac{\partial P_{0T}}{\partial T}$ is a positive deterministic function obtained from the initial term structure. Then the positivity condition (2), as well as $P_{tt} = 1$, holds for all $0 \leq t \leq T < \infty$.

Since the function h is entirely determined by the initial market data P_{0T}, the only exogenous ingredients in the model are the martingales M_{ts}, which are deemed to "embody the random behaviour of the interest rates". When it comes to specify concrete examples, the authors introduce the process [2, eq. (17)]

$$V_t = \int_t^\infty h_s M_{ts} ds, \tag{4}$$

which is easily seen to be a strictly positive supermartingale. They then observe that, due to the martingale property for M_{ts}, the bond prices can be rewritten as

$$P_{tT} = \frac{E_t[V_T]}{V_t}. \tag{5}$$

Absence of arbitrage is taken as an assumption in [2]. If one just assumes that the system of bond prices is given by (3), then further technical conditions should be satisfied by the family of martingales M_{ts} in order to guarantee that it is free from arbitrage.

2.2 State price density and the potential approach

In order to generalize the Flesaker–Hughston framework, Rutkowski [11] and Rogers [10] independently proposed to elevate equation (5) to the starting point of the modelling and to concentrate on the positive adapted continuous process V_t, called the *state price density*.

Positivity of the interest rates is then equivalent to V_t being a supermartingale. In order to match the initial term structure, this supermartingale needs to be chosen so that $E[V_T] = P_{0T}$. If we further impose that $P_{0T} \to 0$ as $T \to \infty$, then V_t satisfies all the properties of what is known in probability theory as a *potential* (namely, a positive supermartingale with expected value going to zero at infinity).

It follows from the Doob–Meyer decomposition that any continuous potential satisfying

$$E\left(\sup_{0\le t\le\infty} V_t^2\right) < \infty \tag{6}$$

can be written as

$$V_t = E_t[A_\infty] - A_t, \tag{7}$$

for a unique (up to indistinguishibility) adapted continuous increasing process A_t with $E(A_\infty^2) < \infty$. Therefore, the model is completely specified by the process A_t, which can be freely chosen apart from the constraint that

$$E\left[\frac{\partial A_T}{\partial T}\right] = -\frac{\partial P_{0T}}{\partial T}. \tag{8}$$

While it is immediate that any Flesaker–Hughston model can be put into a potential form by setting $V_t = \int_t^\infty h_s M_{ts} ds$, the converse result is less direct and was first established by Jin and Glasserman [6, lemma 1].

2.3 Related quantities and absence of arbitrage

Having established that the Flesaker–Hughston and the potential approach are equivalent ways of modelling positive interest rates through the prices of zero coupon bonds, we can now relate them with other standard financial objects. A particularly straightforward path is to follow [11, proposition 1], according to which, given a strictly positive supermartingale V_t, there exist a unique strictly positive (local) martingale Λ_t such that the process $B_t = \Lambda_t / V_t$ is strictly increasing. We identify B_t with a riskless money market account initialized at $B_0 = 1$ and write it as

$$B_t = \exp\left(\int_0^t r_s ds\right), \tag{9}$$

for an adapted process $r_s > 0$, the short rate process.

A sufficient condition for an arbitrage free bond price structure in the potential approach is to require that the local martingale Λ_t be in fact a martingale, since it can then be used as the density for an equivalent martingale measure. It is an interesting open question in the theory to isolate what conditions on the potential V_t would suffice for that.

The formulation up to this point is quite general, in the sense that is does not make use of any particular structure of the underlying filtration (other than the usual conditions). Let us now assume that $\mathcal{F}_t$ is actually generated by an N-dimensional Brownian motion W_t. The market price of risk then arises as the adapted vector valued process λ_t such that

$$d\Lambda_t = -\lambda_t \Lambda_t dW_t. \tag{10}$$

It is also immediate to see that the state price density process is the solution to

$$dV_t = -r_t V_t dt - \lambda_t V_t dW_t, \tag{11}$$

so that the specification of the process V_t is enough to produce both the short rate r_t and the market price of risk λ_t.

It had already been remarked by Flesaker and Hughston [2], that in the Brownian filtration with finite time horizon any positive interest rate model in their formulation corresponds to a model in the HJM family with positive instantaneous forward rates f_{tT}. The converse result, that is, that any interest rate model in HJM form with positive instantaneous forward rate can be written as a Flesaker–Hughston model, was also obtained by Jin and Glasserman [6, theorem 5]. In order to prove this result, they found a rather technical necessary and sufficient condition for positivity in terms of the volatility structure of HJM models, confirming that the HJM formulation is not the most natural one to investigate positive interest rates.

2.4 The Chaotic Approach

We have seen in the Flesaker–Hughston construction that the fundamental ingredients to model the random behaviour of the interest rates are the martingales M_{ts}. In the potential approach, the corresponding role is played by the increasing process A_t in the decomposition $V_t = E_t[A_\infty] - A_t$.

In [4], Hughston and Rafailidis introduced an elegant construction of general positive interest rate models based on a Brownian filtration using even simpler fundamentals. Assume that the state price density V_t is a potential satisfying

$$E\left[\int_0^\infty r_s V_s ds\right] < \infty \tag{12}$$

By integrating (11) on the interval (t, T), taking conditional expectations at time t and the limit $T \to \infty$, one finds that

$$V_t = E_t\left[\int_t^\infty r_s V_s ds\right] \tag{13}$$

Now let σ_t be a vector valued process such that

$$\|\sigma_t\|^2 = r_t V_t, \tag{14}$$

and define the square integrable random variable

$$X_\infty = \int_0^\infty \sigma_s dW_s. \tag{15}$$

It then follows from the Ito isometry that

$$V_t = E_t[X_\infty^2] - E_t[X_\infty]^2, \tag{16}$$

which is called the conditional variance representation of the state price density V_t. To obtain the connection between this representation and the Flesaker–Hughston framework, observe that a direct comparison between (13) and (4) gives that

$$h_s M_{ts} = E_t\left[\|\sigma_s\|^2\right]. \tag{17}$$

Similarly, by comparing the conditional variance representation (16) with the decomposition (7), we see that

$$E_t[X_\infty^2] - X_t^2 = E_t[A_\infty] - A_t,$$

where $X_t = E_t[X_\infty]$. It follows from the uniqueness of the Doob-Meyer decomposition that

$$A_t = [X, X]_t,$$

that is, the quadratic variation of the process X_t.

Conversely, given a zero-mean random variable $X_\infty \in L^2(\Omega, \mathcal{F}, P)$, the representation (16) defines a potential V_t, which can then be used as a state price density to obtain a system of bond prices. The issue of absence of arbitrage can then be adressed in terms of necessary and sufficient conditons on X_∞, and is by and large an open question at this point. The construction in [4] flows in the opposite direction, in the sense that the authors first enumerate a series of axioms to be satisfied by an arbitrage-free interest rate model and then obtain a square integrable random variable corresponding to it.

2.5 Wiener chaos

As Hughston and Rafailidis also observed, the L^2 condition on X_∞ is necessary and sufficient for X_∞ to have the type of orthogonal decomposition known as a Wiener chaos expansion. They interpret the different orders of this decomposition as basic building blocks for models of increasing complexity.

Let W_t be an N–dimensional Brownian motion on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\in\mathbb{R}_+}, P)$. For $h \in L^2(\Delta), \Delta \doteq \mathbb{R}_+ \times \{1, \ldots, N\}$, let $W(h)$ denote the Gaussian random variable $\int h(\tau)dW_\tau$. Here we streamline the handling of vector indices by a compact notation $\tau = (s, \mu) \in \Delta \doteq \mathbb{R}_+ \times \{1, \dots, N\}$ and express integrals as

$$\begin{aligned} \int_\Delta f(\tau) d\tau &\doteq \sum_\mu \int_0^\infty f(s, \mu) ds \\ \int_\Delta f(\tau) dW_\tau &\doteq \sum_\mu \int_0^\infty f(s, \mu) dW_s^\mu \end{aligned} \tag{18}$$

For each $n \geq 0$, let

$$H_n(x) = (-1)^n e^{x^2/2} \frac{d^n}{dx^n} e^{-x^2/2} \tag{19}$$

be the nth Hermite polynomial. The spaces

$$\begin{aligned} \mathcal{H}^{(n)} &\doteq \operatorname{span}\{H_n(W(h)) | h \in L^2(\Delta)\}, \quad n \geq 1, \\ \mathcal{H}^{(0)} &\doteq \mathbb{C} \end{aligned}$$

form an orthogonal decomposition of the space $L^2(\Omega, \mathcal{F}_\infty, P)$ of square integrable random variables:

$$L^2(\Omega, \mathcal{F}_\infty, P) = \oplus_{n=0}^\infty \mathcal{H}^{(n)}$$

Each $\mathcal{H}^{(n)}$ can be understood completely via the *isometries* $J_n : L^2(\Delta_n) \to \mathcal{H}^{(n)}$ given by

$$f_n \mapsto J_n(f_n) = \int_{\Delta_n} f_n(\tau_1, \dots, \tau_n) dW_{\tau_1} \dots dW_{\tau_n} \tag{20}$$

where $\Delta_n \doteq \{(\tau_1, \dots, \tau_n) | \tau_i = (s_i, \mu_i) \in \Delta, 0 \leq s_1 \leq s_2 \leq \dots \leq s_n < \infty\}$.

It is convenient and no loss of generality to extend the domain of f_n from the simplex Δ_n to the entire orthant $\Delta^{\times n} = \Delta \times \dots \times \Delta$ as a symmetric function, i.e so that

$$f_n(\tau_1, \dots, \tau_n) = f_n(\tau_{\pi(1)}, \dots, \tau_{\pi(n)})$$

for all permutations π on $\{1, \dots, n\}$.

With these ingredients, one is then led to the result that any $X \in L^2(\Omega, \mathcal{F}_\infty, P)$ can be represented as a *Wiener chaos expansion*

$$X = \sum_{n=0}^\infty J_n(f_n) \tag{21}$$

where the deterministic functions $f_n \in L^2(\Delta_n)$ are uniquely determined by the random variable X.

A special example arises by noting that for $h \in L^2(\Delta)$

$$n! J_n(h^{\otimes n}) = \|h\|^n H_n(W(h)/\|h\|) \tag{22}$$

and furthermore

$$\exp[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau] = \sum_{n=0}^{\infty} \frac{\|h\|^n}{n!} H_n(W(h)/\|h\|) \tag{23}$$

In the notation of quantum field theory (see Appendix A), this example defines the Wick ordered exponential and Wick powers

$$\begin{aligned} :\exp[W(h)]: \ &\doteq \ \exp[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau] \\ :W(h)^n: \ &\doteq \ n! J_n(h^{\otimes n}) \end{aligned} \tag{24}$$

Generating functionals provide one systematic approach to developing explicit formulas for the terms of the chaos expansion in specific examples.

Theorem 2.1 *For any random variable $X \in L^2(\Omega, \mathcal{F}_\infty, P)$, the generating functional $Z_X(h) : L^2(\Delta) \to \mathbb{C}$ defined by*

$$Z_X(h) \doteq E\Big[X \exp[W(h) - \frac{1}{2}\int |h(\tau)|^2 d\tau]\Big] \tag{25}$$

is an entire analytic functional of $h \in L^2(\Delta)$ and hence has an absolutely convergent expansion

$$Z_X(h) = \sum_{n\geq 0} F_X^{(n)}(h) \tag{26}$$

where

$$F_X^{(n)}(h) = \int_{\Delta_n} f_X^{(n)}(\tau_1, \dots, \tau_n) h(\tau_1) \dots h(\tau_n) d\tau_1 \dots d\tau_n \tag{27}$$

Here, $f_X^{(n)}(\tau_1, \dots, \tau_n)$ is the nth Frechet derivative of Z_X at $h = 0$. Finally, the Wiener–Itô chaos expansion of X is

$$X = \sum_{n\geq 0} \int_{\Delta_n} f_X^{(n)}(\tau_1, \dots, \tau_n) dW_{\tau_1} \dots dW_{\tau_n} \tag{28}$$

Proof: See appendix B.

□

For conditional expectations, we have

$$Z_{E_t[X]}(h) = Z_X(\mathbb{1}_t h), \quad t \in \mathbb{R}_+ \tag{29}$$

where $\mathbb{1}_t$ denotes the characteristic function of $[0, t]$.

3 Squared Gaussian models

A number of authors [7],[5], [10] have observed that the CIR model [1] with an integer constraint $N \doteq \frac{4ab}{c^2} \in \mathbb{N}_+ \setminus \{0, 1\}$ lies in the class of so–called squared Gaussian models. By introducing an $\mathbb{R}^N$–valued Ornstein–Uhlenbeck process R_t, governed by the stochastic differential equation

$$dR_t = -\frac{a}{2} R_t \; dt + \frac{c}{2} dW_t \tag{30}$$

where W_t is N–dimensional Brownian motion, the Itô formula together with Lévy's criterion for Brownian motion shows that the square $r_t = \|R_t\|^2$ satisfies (1) where $\tilde{W}_t = \int_0^t \|R_t\|^{-1} R_t \cdot dW_t$ is itself a one–dimensional Brownian motion. We focus on a general family of interest models which includes this example, the so-called extended CIR model and more. Note that we work always in the physical measure and thus to specify the term structure model one needs to determine the market price of risk vector λ as well as the spot rate r.

Definition 3.1 *A pair (r_t, λ_t) of $(\Omega, \mathcal{F}, \mathcal{F}_t, P)$ processes is called an N–dimensional squared Gaussian model of interest rates ($N \geq 2$) if there is an $\mathbb{R}^N$–valued Ornstein–Uhlenbeck process such that $r_t = \|R_t\|^2$ and $\lambda_t = \bar{\lambda} R_t$. R_t satisfies*

$$dR_t = \alpha(t)(\bar{R}(t) - R_t)dt + \gamma(t)dW_t, \quad R|_{t=0} = R_0 \tag{31}$$

where α, γ, λ are symmetric matrix valued and $\bar{R}$ vector valued deterministic measurable functions on $\mathbb{R}_+$. W is standard N–dimensional Brownian motion. In addition we impose boundedness conditions:

- *there is some constant $M > 0$ such that $\alpha(t) \geq M$ and $|\bar{\lambda}(t)|^{-2} \geq M$ for all t*

For the remainder of the paper, we shall suppress vector indices by adopting a matrix multiplication convention, including $\dagger$ for transpose, in which for example $R_t^\dagger \gamma(t) dW_t$ denotes $\sum_{\mu,\nu=1}^N R_t^\mu \gamma^{\mu\nu}(t) dW_t^\nu$. In a later section, we shall also extend the convention to operator multiplication which combines both sums over vector indices as well as integrals over time.

The exact solution of (31) is easily seen to be

$$R_t = \tilde{R}(t) + \int K(t, t_1)(\gamma dW)_{t_1} \tag{32}$$

where

$$\tilde{R}(t) = K(t, 0) R_0 + \int K(t, t_1)\alpha(t_1)\bar{R}(t_1) dt_1 \tag{33}$$

and $K(t, s), t \geq s$ is the matrix valued solution of

$$\begin{cases} dK/dt(t, s) = -\alpha(t) K(t, s) & 0 \leq s \leq t \\ K(t, t) = I & 0 \leq t \end{cases} \tag{34}$$

which generates the Ornstein–Uhlenbeck semigroup.

By (11), the state price density process is

$$V_t = \exp\Big[-\int_0^t \Big(R_s^\dagger (1 + \bar{\lambda}^2/2) R_s ds + R_s^\dagger \bar{\lambda} dW_s\Big)\Big] \tag{35}$$

We thus have a natural candidate for the random variable X_∞:

$$X_\infty = \int_0^\infty \sigma_t^\dagger \; dW_t \tag{36}$$

where σ_t is the $\mathbb{R}^N$–valued process

$$\sigma_t \doteq \exp\Big[-\int_0^t \Big(R_s^\dagger (1/2 + \bar{\lambda}^2/4) R_s ds + \frac{1}{2} R_s^\dagger \bar{\lambda} dW_s\Big)\Big] R_t \tag{37}$$

is the natural solution of $\sigma_t^\dagger \sigma_t = r_t V_t$.

Before proceeding to analyse X_∞ in detail we show that X_∞ is square integrable.

Proposition 3.2 $\lim_{T\to\infty} E[X_T^2] = 1$

Proof: By the Itô isometry and Fubini's theorem for the Itô integral,

$$E[X_T^2] \quad = \quad E\left[\int_0^T \sigma_t^\dagger \sigma_t dt\right] \tag{38}$$

From $\sigma_t^\dagger \sigma_t dt = r_t V_t dt = -dV_t - V_t \lambda^\dagger dW_t$ it follows that $E[X_T^2] = E\left[1 - V_T\right]$. For any $0 < \epsilon < 2M$, $V_T = e^{-Y_1 - Y_2}$ where $Y_1 = \int_0^T R_s^\dagger \big(1 - \frac{\epsilon \bar\lambda^2}{2}\big) R_s ds$ is positive and

$$Y_2 = \frac{1}{1+\epsilon} \int_0^T \left[\frac{1}{2} R_t^\dagger (1+\epsilon)^2 \bar\lambda^2 R_t dt + (1+\epsilon) R_t^\dagger \bar\lambda^\dagger dW_t\right]$$

By the Hölder inequality

$$E[V_T] \le \left(E[e^{(1+1/\epsilon)Y_1}]\right)^{\epsilon/(1+\epsilon)} \left(E[e^{-(1+\epsilon)Y_2}]\right)^{1/(1+\epsilon)}$$

with the second factor equal to 1. Now Y_1 is a positive random variable for which a direct computation shows

$$\mathrm{mean}(Y_1) \quad = \quad C_1 NT\,(1 + \mathcal{O}(T)) \tag{39}$$

$$\mathrm{var}(Y_1) \quad = \quad C_2 NT\,(1 + \mathcal{O}(T)) \tag{40}$$

for positive constants C_1, C_2. An easy application of Chebyshev's inequality

$$\mathrm{Prob}\left(Y_1 \le \frac{C_1 NT}{2}\right) \le \mathcal{O}(\frac{1}{NT}) \tag{41}$$

then implies that $\lim_{T\to\infty} E[V_T] = 0$.

$\square$

4 Exponentiated second chaos

The chaos expansion we seek for the CIR model will be derived from a closed formula for expectations of e^{-Y} for elements

$$Y = A + \int_\Delta B(\tau_1) dW_{\tau_1} + \int_{\Delta_2} C(\tau_1,\tau_2) dW_{\tau_1} dW_{\tau_2} \tag{42}$$

in a certain subset $\mathcal{H}_{\leq 2}^{(+)} \subset \mathcal{H}_{\leq 2} \doteq \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. The formula we present is well known in the theory of gaussian functional integration [3, Chapter 9]. In probability theory, this result gives the Laplace transform of a general class of quadratic functionals of Brownian motion. Many special cases of this result have been studied in probability theory, see for example [12, Chapter 2] and the references contained therein.

If in (42) we define $C(\tau_1, \tau_2) = C(\tau_2, \tau_1)$ when $\tau_1 > \tau_2$, then C is the kernel of a symmetric integral operator on $L^2(\Delta)$:

$$[Cf](\tau) = \int_0^\infty C(\tau, \tau_1) f(\tau_1) d\tau_1 \tag{43}$$

Recall that Hilbert-Schmidt operators are finite norm operators under the norm:

$$\|C\|_{HS}^2 = \int_{\Delta^2} |C(\tau_1, \tau_2)|^2 d\tau_1 d\tau_2$$

We say that $Y \in \mathcal{H}_{\leq 2}$ is in $\mathcal{H}_{\leq 2}^{(+)}$ if C is the kernel of a symmetric Hilbert–Schmidt operator on $L^2(\Delta)$ with non–negative spectrum.

Proposition 4.1 *Let $Y \in \mathcal{H}_{\leq 2}^{(+)}$. Then*

$$\begin{aligned} E[e^{-Y}] &= [\det_2(1+C)]^{-1/2} \\ &\quad \exp\left[-A + \frac{1}{2}\int_{\Delta^2} B(\tau_1)B(\tau_2)(1+C)^{-1}(\tau_1, \tau_2) d\tau_1 d\tau_2\right] \end{aligned} \tag{44}$$

Remark 4.2 *The Carleman–Fredholm determinant is defined as the extension of the formula*

$$\det_2(1+C) = \det(1+C)\exp[-\mathrm{Tr}(C)] \tag{45}$$

from finite rank operators to bounded Hilbert–Schmidt operators; the kernel $(1+C)^{-1}(\tau_1, \tau_2)$ is also the natural extension from the finite rank case.

Proof: See Appendix C.

Using this proposition, it is possible to deduce the chaos expansion of the random variables $e^{-Y}, Y \in \mathcal{H}_{\leq 2}^{(+)}$, a result known in quantum field theory as Wick's Theorem:

Corollary 4.3 *The random variable* $X = e^{-Y}, Y = \int_{\Delta_2} C(\tau_1, \tau_2) dW_{\tau_1} dW_{\tau_2} \in \mathcal{H}^{(+)}_{\leq 2}$ *has Wiener chaos coefficient functions*

$$f_n(\tau_1, \ldots, \tau_n) = \begin{cases} K \sum_{G \in \mathcal{G}_n} \prod_{g \in G} [C(1+C)^{-1}](\tau_{g_1}, \tau_{g_2}) & n \text{ even} \\ 0 & n \text{ odd} \end{cases}$$

where $K = [\det_2(1 + \mathrm{C})]^{-1/2}$ *and for* n *even,* $\mathcal{G}_n$ *is the set of* Feynman graphs *on the* n *marked points* $\{\tau_1, \ldots, \tau_n\}$*. Each Feynman graph* G *is a disjoint union of unordered pairs* $g = (\tau_{g_1}, \tau_{g_2})$ *with* $\cup_{g \in G}\ g = \{\tau_1, \ldots, \tau_n\}$*.*

Proof: We apply Proposition 4.1 with $A = \int h(\tau)^2 d\tau, B(\tau) = h(\tau)$ which yields

$$Z_X(h) = \det{}_2(1+C)^{-1/2} \exp\left[-\frac{1}{2} \int_{\Delta^2} h(\tau_1)h(\tau_2)[\delta(\tau_1, \tau_2) - (1+C)^{-1}(\tau_1, \tau_2)] d\tau_1 d\tau_2\right] \tag{47}$$

The result now comes by evaluating the nth Frêchet derivative at $h = 0$, or equivalently by expanding the exponential and symmetrizing over the points $\tau_1, \ldots, \tau_n$ in the $n/2$th term.

□

5 The chaotic expansion for squared Gaussian models

We now derive the chaos expansion for the squared Gaussian model defined by (31). For simplicity, in the remainder of the paper we take $\bar{\lambda} = 0$.

In view of (36) it will be enough to find the chaos expansion for σ^{μ}_T, $T < \infty$ which we will derive from a closed formula for the following generating functional $Z(h, k) = E\left[e^{-Y_T}\right]$ with

$$\begin{aligned} Y_T &= \frac{1}{2}\int_0^T R_t^\dagger R_t dt - \int_0^T h^\dagger(t) dW_t \\ &\quad + \frac{1}{2}\int_0^T \|h(t)\|^2 dt - \int_0^T k^\dagger(t) R_t dt \end{aligned} \tag{48}$$

Proposition 5.1 *$Z(h,k)$ is an entire analytic functional of $(h,k) \in L^2([0;T]) \otimes \mathbb{R}^{2M}$. Moreover*

$$\lim_{t \to T^-} \frac{\delta Z(h,0)}{\delta k^\mu(t)}\Big|_{k=0} = Z_{\sigma^\mu_T}(h) \tag{49}$$

where $Z_{\sigma^\mu_T}(h)$ is defined by (25) with $X = \sigma^\mu_T, \mu = 1, \ldots, M$.

Proof: Analyticity in (h,k) follows by repeating the argument given in Appendix B. By the definition of Frêchet differentiation and continuity of the $t \to T^-$ limit, (49) follows.

□

Substitution of (32,(33) into (48) leads to

$$\begin{aligned}
\int_0^T R_t^\dagger R_t dt &= \int_0^T \|\tilde{R}(t)\|^2 dt + 2\int_\Delta [\tilde{R}^\dagger K_T](t)\gamma(t) dW_t \\
&\quad +2\int_{\Delta_2} [\gamma K_T^\dagger K_T \gamma](t_1,t_2) dW_{t_1} dW_{t_2} + \int_0^T \mathrm{tr}[\gamma K_T^\dagger K_T \gamma](t,t) dt \\
\int_0^T k^\dagger(t) R_t dt &= \int_0^T \Big(k^\dagger(t)\tilde{R}(t) dt + [k^\dagger K_T \gamma](t) dW_t\Big)
\end{aligned} \tag{50}$$

where we define $K_T(t_1,t_2) = \mathbb{1}(t_1 \le T) K(t_1,t_2)$ and use an operator multiplication notation. Thus the exponent Y_T appearing in (48) has the form of (42) with

$$\begin{aligned}
C_T &= \gamma K_T^\dagger K_T \gamma \\
B_T &= -h - k^\dagger K_T \gamma + \tilde{R}^\dagger K_T \gamma \\
A_T &= \int_0^T \left[\frac{1}{2}\tilde{R}^\dagger \tilde{R} + \frac{1}{2}\mathrm{tr}(\gamma K_T^\dagger K_T \gamma) + \frac{1}{2} h^\dagger h + k^\dagger \tilde{R}\right] dt
\end{aligned} \tag{51}$$

Now we note that C_T is manifestly a positive semi-definite operator and the Hilbert–Schmidt norm $\|C_T\|^2_{HS} = \mathcal{O}(T)$ and hence the Proposition (44) applies to $E[e^{-Y}]$ leading to the general formula for the generating functional

$Z(h,k)$. Differentiation once with respect to k then yields

$$\begin{aligned} Z_{\sigma_T}(h) &= M_T \exp -\frac{1}{2}\left[\int_0^T (\tilde{R}^\dagger \tilde{R} + h^\dagger h)\right] \\ & \quad \left(-\tilde{R} + K_T\gamma(1+C_T)^{-1}\left[h - \gamma K_T^\dagger \tilde{R}\right]\right)(t) \\ & \exp\left[\int_{\Delta_2} [h-\gamma K_T^\dagger \tilde{R}]^\dagger(t_1)[h-\gamma K_T^\dagger \tilde{R}](t_2)(1+C_T)^{-1}(t_1,t_2)dt_1 dt_2\right] \end{aligned} \tag{52}$$

where

$$M_T = e^{-\frac{1}{2}\mathrm{tr} C_T}(\det{}_2(1+C_T))^{-1/2} = (\det(1+C_T))^{-1/2} \tag{53}$$

These formulas simplify considerably if the function $\tilde{R}$ vanishes, which is true in the simple CIR model when $r_0 = 0$. In this case, one can easily evaluate the nth Frêchet derivative of Z_{σ_T} at $h = 0$ as in the proof of Corollary 4.3 and determine the following partly explicit form for the nth term of the chaos expansion.

Theorem 5.2 *The nth term of the chaos expansion of σ_T for the CIR model with $\bar{\lambda} = 0$ and initial condition $r_0 = 0$ is zero for n even. For n odd, the kernel of the expansion is the function $f_T^{(n)}(\cdot) : \Delta_n \to \mathbb{R}$*

$$f_T(t_1,\dots,t_n) = M_T \sum_{G\in\mathcal{G}_n^*}\prod_{g\in G} L(g) \tag{54}$$

where

$$L(g) = \begin{cases} [C_T(1+C_T)^{-1}](t_{g_1},t_{g_2}) & t\notin g \\ \\ (K_T\gamma(1+C_T)^{-1})(T,t_{g_2}) & t\in g \end{cases} \tag{55}$$

Here, $\mathcal{G}_n^*$ is the set of Feynman graphs*, each Feynman graph G being a partition of $\{t_1,\dots,t_n,T\}$ into pairs $g = (t_{g_1},t_{g_2})$.*

The chaos expansion for X_∞ itself is exactly the same, except that the variable T is treated as an additional Itô integration variable. The explicit expansion up to fourth order is:

$$
\begin{aligned}
X_\infty &= \int_{\Delta_2} M_T[K_T\gamma(1+C_T)^{-1}](T,t_1)dW_{t_1}dW_T \\
&+ \int_{\Delta^{(4)}} M_T[K_T\gamma(1+C_T)^{-1}](T,t_3)[C_T(1+C_T)^{-1}](t_1,t_2)dW_{t_1}dW_{t_2}dW_{t_3}dW_T \\
&+ \int_{\Delta^{(4)}} M_T[K_T\gamma(1+C_T)^{-1}](T,t_2)[C_T(1+C_T)^{-1}](t_1,t_3)dW_{t_1}dW_{t_2}dW_{t_3}dW_T \\
&+ \int_{\Delta^{(4)}} M_T[K_T\gamma(1+C_T)^{-1}](T,t_1)[C_T(1+C_T)^{-1}](t_2,t_3)dW_{t_1}dW_{t_2}dW_{t_3}dW_T \\
&\quad \cdots
\end{aligned}
\tag{56}
$$

6 Bond pricing formula

In this section we give a derivation of the price of a zero coupon bond in the CIR model. To keep things as simple as possible we take $\lambda = 0$ so $V_T = \exp[-\int_0^T r_s ds]$. Since any non–dividend paying asset is assumed to satisfy $S_t = E_t[V_t^{-1}V_T S_T], t \leq T$, the price at time t of a zero coupon bond paying 1 at time T is given by

$$
P_{tT} = E_t[V_t^{-1}V_T] \tag{57}
$$

For $t \leq s \leq T$

$$
R_s^\mu = K_T(s,t)R_t^\mu + \frac{c}{2}\int_t^s K_T(s,s_1)dW_{s_1}^\mu \tag{58}
$$

where $K_T(s,t) = e^{-a(s-t)/2}\mathbb{1}(t \leq s \leq T)$. Hence $-\log[V_t^{-1}V_T] = \sum_\mu \int_t^T (R_s^\mu)^2 ds$ can be written

$$
\sum_\mu \left[\frac{1}{2}(R_t^\mu)^2 B_T(t,t) + \frac{cR_t^\mu}{2}\int_t^T B_T(t,s)dW_s^\mu + \frac{c^2}{4}\int_t^T\int_t^{s_2} B_T(s_1,s_2)dW_{s_1}^\mu dW_{s_2}^\mu\right] + \frac{c^2 N}{4}\int_t^T B_T(s,s)ds \tag{59}
$$

where $B_T(s_1,s_2) = 2\int_0^T K_T(s,s_1)K_T(s,s_2)ds$.

Taking the conditional expectation of $V_t^{-1}V_T$ by use of Proposition 4.1 leads to the desired formula

$$
\begin{aligned}
P_{tT} &= \left(\det(1 + \frac{c^2}{4}B_T)\right)^{-N/2} \qquad (60)\\
& \prod_\mu \exp\left[-\frac{1}{2}(R_t^\mu)^2 \left(B_T(1 + \frac{c^2}{4}B_T)^{-1}\right)(t,t)\right] \qquad (61)
\end{aligned}
$$

Thus P_{tT} has the exponential affine form $\exp[-\beta(t,T)r_t - \alpha(t,T)]$ with

$$
\begin{aligned}
\beta(t,T) &= \frac{1}{2}[B_T(1 + \frac{c^2}{4}B_T)^{-1}](t,t)\\
\alpha(t,T) &= \frac{N}{2}\log\bigl(\det(1 + \frac{c^2}{4}B_T)\bigr) \qquad (62)
\end{aligned}
$$

The known formula has the same form, with

$$
\begin{aligned}
\beta(t,T) &= \frac{2(e^{\rho(T-t)} - 1)}{(\rho + a)(e^{\rho(T-t)}) + 2\rho}, \quad \rho^2 = a^2 + 8\gamma^2\\
\alpha(t,T) &= \frac{2ab(3\rho + a)e^{\rho(T-t)}}{(\rho + a)[(\rho + a)(e^{\rho(T-t)}) + 2]\rho} \qquad (63)
\end{aligned}
$$

which can be derived as solutions of the pair of Ricatti ordinary differential equations

$$
\begin{aligned}
\frac{\partial\beta}{\partial t} &= \frac{c^2\beta^2}{2} + a\beta - 1\\
\frac{\partial\alpha}{\partial t} &= -ab\beta \qquad (64)
\end{aligned}
$$

One can demonstrate using power series expansions for that (62) do in fact solve the Ricatti equations and hence agree with the usual formula. This example does point to the rather subtle general relationship between kernels such as $(1 + \frac{c^2}{4}B_T)^{-1}$ and solutions of Ricatti equations deserving of further study.

7 Discussion

We have shown how the CIR model, at least in integer dimensions, can be viewed within the chaos framework of Hughston and Rafailidis as arising from a somewhat special random variable X_∞. This random variable can be understood as derived from exponentiated second chaos random variables $e^{-Y}, Y \in \mathcal{H}^{(+)}_{\leq 2}$. Such exponentiated $\mathcal{H}^{(+)}_{\leq 2}$ variables form a rich and natural family which is likely to include many more candidates for applicable interest rate models. Although their analytic properties are complicated, there do exist approximation schemes which can in principle be the basis for numerical methods.

On the theoretical side, this family is distinguished by its natural invariance properties. Most notably, as will be investigated elsewhere, it is invariant under conditional $\mathcal{F}_t$–expectations: $\log E_t[e^{-Y}] \in \mathcal{H}^{(+)}_{\leq 2}$ whenever $Y \in \mathcal{H}^{(+)}_{\leq 2}$. Note in particular that this implies that these can be used as the Radon–Nikodym derivatives of measure changes which generalize the Girsanov transform, and which can greatly enrich the tools applicable in finance.

This application of the chaos expansion to squared Gaussian models also illustrates a deep connection between methods developed for quantum field theory, and the methods of Malliavin calculus. Many of the very rich analytic properties of this example reflect well known techniques widely used in mathematical physics.

To conclude, even if the representation of the CIR model and its generalizations we present is not simple, it does show the way to the use of a powerful set of analytical techniques which might prove to be very useful in financial modeling.

A White noise calculus

Here we describe the white noise calculus, which can be regarded as a reformulation of the calculus of Wiener measure into concepts familiar to practitioners in quantum field theory as Gaussian functional integration. We follow the discussion of Oksendal [9].

Let $\mathcal{S}$ be the Schwartz space of smooth functions on $\mathbb{R}_+$ of rapid decrease, and $\mathcal{S}'$ its topological dual, the space of tempered distributions on $\mathbb{R}_+$. If $\phi \in \mathcal{S}'$ and $f \in \mathcal{S}$, we write

$$\phi(f) \doteq \langle \phi, f \rangle$$

for their canonical pairing. We also use the formal notation

$$\phi(f) = \int_{\mathbb{R}_+} f(s)\phi_s ds$$

to represent the 'smearing' of the distribution ϕ over the test function f. It acquires a rigorous meaning, however, in the cases where ϕ is itself a function on $\mathbb{R}_+$ for which the pointwise product $\phi(s)f(s)$ is integrable for all $f \in \mathcal{S}$.

Define now the functional

$$S\{f\} \doteq e^{-\frac{1}{2}\|f\|^2} = e^{-\frac{1}{2}\langle f,f\rangle_{L^2}}, \tag{65}$$

where $\langle\cdot,\cdot\rangle_{L^2}$ denotes the real-valued inner product in $L^2(\mathbb{R}_+)$. Observe that this functional satisfies

1. (continuity) $f_n \to f$ in $\mathcal{S}$ implies that $S\{f_n\} \to S\{f\}$;
2. (positive definiteness) $\sum_{i,j=1}^N \overline{c_i} c_j S\{f_i - f_j\} \geq 0$ for all $f_i \in \mathcal{S}$ and all $c_i \in \mathbb{C}$;
3. (normalization) $S\{0\} = 1$.

It follows from the Bochner-Minlos theorem that there exists a unique Borel probability measure μ on $\mathcal{S}'$ such that $S\{f\}$ corresponds to a *moment generating functional*, that is, for all $f \in \mathcal{S}$

$$\int_{\mathcal{S}'} e^{i\phi(f)} d\mu(\phi) = S\{f\} = e^{-\|f\|^2/2}. \tag{66}$$

The measure space $(\mathcal{S}', \mathcal{B}, \mu)$, where $\mathcal{B}$ is the Borel σ-algebra of $\mathcal{S}'$, is called the *white noise probability measure*.

In Euclidean quantum field theory, a given Borel measure μ on $\mathcal{S}'$ characterizes the family of 'fields' $\phi \in \mathcal{S}'$ through the properties of the random variables $\phi(f) : \mathcal{S}' \to \mathbb{R}$ obtained for each $f \in \mathcal{S}$. One tries to construct measures μ so that the generating functional $S\{f\}$ satisfy the so-called Osterwalder-Schrader axioms, in order to guarantee that the fields ϕ have certain required physical properties. The family of Euclidean free fields is obtained when

$$S_C\{f\} \doteq e^{-\frac{\langle f, Cf\rangle}{2}} = \int_{\mathcal{S}'} e^{i\phi(f)} d\mu_C(\phi),$$

where C is the integral kernel of a positive, continuous, nondegenerate Euclidean covariant bilinear form C on $\mathcal{S} \times \mathcal{S}$. We see that the special case

$S\{f\} = e^{-\|f\|^2/2}$ is obtained when $C(s,t) = \delta(t-s)$, called the "ultralocal" covariance. For each $\phi \in \mathcal{S}'$, the random variables $\{\phi(f) : f \in \mathcal{S}\} \subset L^2(\mathcal{S}', \mathcal{B}, \mu)\}$ form a Gaussian family with mean zero and covariances

$$E_\mu[\phi(f)\phi(g)] = \int_{\mathbb{R}_+ \times \mathbb{R}_+} f(s)g(t)C(s,t)dsdt = \langle f, g \rangle. \tag{67}$$

The theory of martingales makes its appearance in white noise calculus through the concept of Wick ordered random variables. To introduce them, recall that the Hermite polynomials are defined by

$$H_n(x) = (-1)^n e^{\frac{x^2}{2}} \frac{d^n}{dx^n}(e^{-\frac{x^2}{2}}),$$

from which it is easy to see that they satisfy the recurrence relation

$$\begin{aligned} \left(x - \frac{d}{dx}\right) H_{n-1}(x) &= H_n(x), \quad n = 1, 2, \ldots \\ H_0(x) = 1. \end{aligned}$$

The first couple of Hermite polynomials are then

$$\begin{aligned} H_0(x) &= 1 \\ H_1(x) &= x \\ H_2(x) &= x^2 - 1 \\ H_3(x) &= x^3 - 3x, \ldots \end{aligned}$$

We begin by defining the *Wick ordered* exponential for any $f \in \mathcal{S}$ to be

$$: e^{\phi(f)} := e^{\phi(f) - \|f\|^2/2}. \tag{68}$$

Then we have

Proposition A.1 *For any $f \in \mathcal{S}$,*

$$: e^{\phi(f)} := \sum_{n \geq 0} \frac{\|f\|^n}{n!} H_n\left(\frac{\phi(f)}{\|f\|}\right), \tag{69}$$

where H_n is the nth Hermite polynomial. Moreover, for any $f, g \in \mathcal{S}$,

$$E_\mu\left[H_n\left(\frac{\phi(f)}{\|f\|}\right) H_m\left(\frac{\phi(g)}{\|g\|}\right)\right] = \delta_{nm}\ (n!) \left(\frac{\langle f, g \rangle}{\|f\|\ \|g\|}\right)^n \tag{70}$$

Proof: For any $a, b \in \mathbb{R}$ we have the absolutely convergent expansions

$$\begin{aligned} e^{a-b^2/2} &= e^{-(b-a/b)^2/2+a^2/(2b^2)} \\ &= \sum_{n\geq 0} \frac{b^n}{n!}(-1)^n e^{a^2/(2b^2)} \frac{d^n}{dx^n}\left(e^{-x^2/2}\right)\Big|_{x=a/b} \\ &= \sum_{n\geq 0} \frac{b^n}{n!} H_n(a/b) \end{aligned} \tag{71}$$

where the last line makes use of the defining property of the Hermite polynomials. Using this with $a = \phi(f), b = \|f\|$ gives (69). The orthogonality relation (70) follows by expanding the identity

$$E_\mu[: e^{\phi(f)} :: e^{\phi(g)} :] = e^{\langle f,g\rangle} \tag{72}$$

in powers of f, g and comparing to the expansion derived from (69).

From this we define the *Wick ordered* monomials as the random variables

$$: \phi(f)^n := \|f\|^n H_n\left(\frac{\phi(f)}{\|f\|}\right), \tag{73}$$

so that linearity and convergent power series imply that

$$: e^{\phi(f)} := \sum_{n\geq 0} \frac{: \phi(f)^n :}{n!}.$$

Formally, we express the Wick ordered monomials as

$$: \phi(f)^n := \int_{\mathbb{R}_+^n} f(s_1)\dots f(s_n) \ : \phi_{s_1}\dots\phi_{s_n} : \ ds_1 \dots ds_n$$

and from the orthogonalization (70) we can define the *Wick products*

$$: \phi(f_1)\dots\phi(f_n) : \ = \int_{\mathbb{R}_+^n} f_1(s_1)\dots f_n(s_n) \ : \phi_{s_1}\dots\phi_{s_n} : \ ds_1 \dots ds_n, \tag{74}$$

$\mathcal{H}^{(n)}$ is defined to be the span of $\{: \phi(f)^n : | f \in \mathcal{S}\}$ and consists of precisely the random variables $(n!)^{-1} \int_{\mathbb{R}_+^n} f(s_1, \dots, s_n) : \phi_{s_1}\dots\phi_{s_n} : \ ds_1 \dots ds_n$ where f lies in $\tilde{T}_n$ the L^2 completion of the space of *symmetric* functions in $\mathcal{S}^{\otimes n}$. These subspaces form an orthogonal decomposition of $L^2(\mathcal{S}', \mathcal{B}, \mu)$. In quantum field theory this is known as the Fock space decomposition of the Hilbert space of quantum states into n particle sectors for $n \geq 0$.

The relationship between Wiener measure and the white noise measure is to identify $\phi(f) \doteq \int f(s)\phi_s ds$ with $W(f) \doteq \int f(s)dW_s$ for all $f \in \mathcal{S} \subset L^2(\mathbb{R}_+)$. One is then lead to the formal relation $\phi_s = dW_s/ds$, that is "white noise" is the "derivative" of Brownian motion. This identification extends to all orders in the chaos expansion via:

$$(n!)^{-1} \int_{\mathbb{R}_+^n} f(s_1, \ldots, s_n) :\phi_{s_1} \ldots \phi_{s_n}: \, ds_1 \ldots ds_n = \int_{\Delta_n} \tilde{f}(s_1, \ldots, s_n) dW_{s_1} \ldots dW_{s_n} \tag{75}$$

where the bijection $f \leftrightarrow \tilde{f}$ between $\tilde{T}_n$ and $L^2(\Delta_n)$ is the restriction map and its inverse. Finally, this leads to the identification $L^2(\mathcal{S}', \mathcal{B}, \mu) \equiv L^2(\Omega, \mathcal{F}_\infty, P)$.

One useful consequence of Wick products and the chaos expansion is that conditional expectations can be handled systematically.

Proposition A.2 *Let $f \in L^2(\mathbb{R}_+)$ and $t \in [0, \infty)$. Then*

$$\begin{aligned} E[: e^{\phi(f)} : | \mathcal{F}_t] &= : e^{\phi(f_{(t)})} : \\ E[: (\phi(f))^n : | \mathcal{F}_t] &= : (\phi(f_{(t)}))^n : \end{aligned} \tag{76}$$

where $[f_{(t)}](s) = \mathbb{1}(s \leq t) f(s)$. In other words, the processes $: e^{\phi(f_{(t)})} :, : (\phi(f_{(t)}))^n :$ are martingales.

B Generating functional for chaos coefficients

We derive Theorem 2.1 in the notation of the white noise calculus. To begin, we recall the definition of analyticity for a function between complex Banach spaces.

Definition B.1 *Let $f : B \to C$ where B, C are complex Banach spaces. Then f is analytic at a point $x \in B$ if*

1. *for any finite set $\{x_1, \ldots, x_n\} \subset B$ of points the function $F : \mathcal{C}^n \to \mathcal{C}$*

$$F(\zeta_1, \ldots, \zeta_n) = f(x + \sum_i \zeta_i x_i) \tag{77}$$

   is analytic at $0 \in \mathcal{C}^n$;

2. f is continuous at x.

Proof of Theorem 2.1 One can check directly that the map $h \mapsto : e^{\phi(h)} :$ is entire analytic between $L^2_{\mathcal{C}}(\mathbb{R}_+)$ and $L^2_{\mathcal{C}}(\mathcal{S}', \mathcal{B}, \mu)$. Similarly, $Z_X(h)$ is analytic for any $h \in L^2_{\mathcal{C}}(\mathbb{R}_+)$.

Finally, to verify (28), it is enough to take the expectation of the equation multiplied by $: \phi(g)^n :$ for arbitrary $g \in L^2(\Delta), n \geq 0$:

$$\begin{aligned} E[X : \phi(g)^n :] &= \frac{d^n}{d\lambda^n} Z_X(\lambda g)\Big|_{\lambda=0} \\ & n! \int_{\Delta^{(n)}} f_X^{(n)}(s_1, \dots, s_n) g(s_1) \dots g(s_n) ds_1 \dots ds_n \end{aligned} \tag{78}$$

whereas by (70),

$$E[\sum_{m \geq 0} (m!)^{-1} \int_{\mathbb{R}_+^m} f_X^{(m)}(s_1, \dots, s_m) : \phi(s_1) \dots \phi(s_m) : ds_1 \dots ds_m : \phi(g)^n :]$$

$$= n! \int_{\mathbb{R}_+^m} g(s_1) \dots g(s_n) f_X^{(n)}(s_1, \dots, s_n) ds_1 \dots ds_n \tag{79}$$

□

C The exponential quadratic formula

Here we prove the formula (44) in the context of one–dimensional white noise calculus. The proof extends easily to the multidimensional case.
Proof: (Proposition 4.1)

1. (Step 1) First we note that Y can be approximated in the Hilbert space $\mathcal{H}_{\leq 2}$ by random variables of the form

$$Y = A + \sum_{i=1}^{M} [b_i \phi(g_i) + c_i : \phi(g_i)^2 : /2] \tag{80}$$

with $\{g_i\}$ a finite orthonormal set in $L^2(\Delta)$ and numbers $c_i > 0, b_i$. Then $X_i = \phi(g_i)$ form a collection of independent $N(0,1)$ random variables. For Y of this type, there is a factorization

$$E[e^{-Y}] = e^{-A} (2\pi)^{-M/2} \prod_i \int_{\mathbb{R}} \exp[-b_i x - (1 + c_i) x^2/2 + c_i/2)] dx \tag{81}$$

into one dimensional Gaussian integrals. Each integral gives the factor

$$(2\pi)^{1/2}(1+c_i)^{-1/2}\exp\frac{1}{2}[c_i+b_i^2(1+c_i)^{-1}] \tag{82}$$

leading to a formula for $E[e^{-Y}]$ which agrees with (4.1) for Y of this form.

2. (Step 2) Since the formula is true for Y in a dense subset, it is now enough to prove that the map $Y \mapsto E[e^{-Y}]$ is continuous provided the kernel C satisfies the stated conditions. By the definition of the Carleman–Fredholm determinant

$$det_2(1+C) = \exp[\mathrm{Tr}[\log(1+C)-C] \tag{83}$$

Since there is a constant M so that $\log(1+x)-x \leq Mx^2$ for $x>0$ we see that $det_2(1+C)$ is well-defined and continuous for C non–negative, the entire right hand side of (4.1) is continuous in C.

□

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