Credit portfolio modeling

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Credit Risk Methodology

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Basic Model

Basic random variable, if only defaults are considered:

$$L = \sum_{i=1}^{m} l_i \mathbf{1}_{D_i}$$

$$\begin{aligned} m &= \text{\# of counterparties} \\ D_i &= \text{Default of counterparty} \\ l_i &= \text{Loss in the event of default} \\ &= \text{Exposure * Loss Given Default} \end{aligned}$$

The exposure is viewed as a loan equivalent exposure, if the transaction with a counterparty is a traded product (Average Expected Exposure)

Risk?

What is the risk in terms of the random variable loss?

Portfolio Theory: $\sigma(L)$ the standard deviation.

$$\begin{aligned}
\sigma^2(L) &= \sum_{i,j=1}^{m} \mathsf{cov}(l_i \mathbf{1}_{D_i}, l_j \mathbf{1}_{D_j}) \\
&= \sum_{i,j=1, i \neq j}^{m} l_i l_j \mathsf{cov}(\mathbf{1}_{D_i}, \mathbf{1}_{D_j}) \\
&\qquad + \sum_{j=1}^{m} l_j^2 P[D_j](1 - P[D_j])
\end{aligned}$$

The last equation is valid under the assumption that $l_i, i = 1, .., m$ are non-random.

The covariance of the two Bernoulli-distributed random variable $\mathbf{1}_{D_i}, \mathbf{1}_{D_j}$ equals

$$P[D_i \cap D_j] - P[D_i]P[D_j].$$

The determination of the probability of joint defaults of each pair of counterparties is important.

Joint Default Probabilities

- How can joint defaults be observed?
- Single Defaults are derived from relative frequency of defaults in uniform segments. All members in a segment exhibit the same risk characteristics (Rating, Country, Industry, Size, Customer type, etc.).If i is a member of segment A then

$$P[D_i] \quad = \quad \frac{\text{\# Defaulted members in Segment A}}{\text{All members in Segment A}}$$

- Joint Defaults?

$$P[\text{CP in Seg A defaults}, \text{CP in Seg B defaults}]$$
$$= \frac{\#?}{\#???}$$

For joint defaults a model is needed!

Multivariate Merton model

$(A_t^{(i)})_{t\geq 0, i=1,..,m}$ ="Ability to pay" process of the vector of counterparties.

Remark: If the counterparty is an exchange traded firm, then $A_t^{(i)}$ =Value of firm's assets= $F(E,L,t)$ where

$$\begin{aligned} E &= (E_t)_{t\geq 0} \text{ value of equities} \\ L &= (L_t)_{t\leq 0} \text{ Liability Process .} \end{aligned}$$

The function F may be derived by modeling the equity as a contingent claim on the firm (Call option as in Merton '74) or vice versa.

Default in one year can then be defined by the event

$$D_i = \{A_1^{(i)} < C^{(i)}\},$$

where $C^{(i)}$ ="Default Point"= Function of Liabilities.

Joint Defaults if A is multivariate lognormal

$$
\begin{aligned}
& \mathsf{JDP}_{ij} \\
:= \quad & P\left[A_1^{(i)} < C^{(i)}, A_1^{(j)} < C^{(j)}\right] \\
= \quad & \int\limits_{-\infty}^{N^{-1}(P[D_i])} \int\limits_{-\infty}^{N^{-1}(P[D_j])} \frac{1}{2\pi\sqrt{(1-r_{ij}^2)}} \\
& \exp\left(\frac{1}{2(1-r_{ij})}(x_1^2 + x_2^2 - 2r_{ij}x_1x_2)\right)dx_1dx_2
\end{aligned}
$$

Default correlation

$$
\begin{aligned}
\rho_{ij} \quad & = \quad \mathsf{corr}(\mathbf{1}_{\{A_1^{(i)} < C^{(i)}\}}, \mathbf{1}_{\{A_1^{(j)} < C^{(j)}\}}) \\
& = \quad \frac{\mathsf{JDP}_{ij} - P[D_i]P[D_j]}{\sqrt{P[D_i](1-P[D_i])P[D_j](1-P[D_j])}}
\end{aligned}
$$

Portfolio Standard Deviation

$$
\begin{aligned}
& \sigma(L_p) \\
= & \sum_{i,j=1}^{m} l_i l_j \sqrt{P[D_i](1-P[D_i])P[D_j](1-P[D_j])}\rho_{ij}
\end{aligned}
$$

In classical portfolio theory, the risk contribution of a counterparty i is defined by

$$
\beta_i \quad = \quad \frac{\sigma_i}{\sigma(L)} \sum_{j=1}^{m} l_j \sigma_j \rho_{ij},
$$

with $\sigma_i = \sqrt{P[D_i](1-P[D_i])}$. Since

$$
\sum_{i=1}^{m} \beta_i l_i \quad = \quad \sigma(L),
$$

it is a nice allocation rule for risk. Additionally,

$$
\beta_i \quad = \quad \frac{\partial \sigma(L_p)}{\partial l_i}.
$$

Economic Capital

Economic Capital is usually defined to be a quantile of the loss distribution minus the mean of the loss distribution.

$$EC(\alpha) \quad = \quad q_\alpha(L) - E[L].$$

Assuming that the returns on a single asset in the portfolio are joint normal distributed, the portfolio return L is also normal. Then the economic capital is also given by multiples of the standard deviation. These multipliers CM are also called "Capital Multipliers".

$$EC(\alpha) \quad = \quad \sigma(L) * CM(\alpha).$$

Loss Distribution
Uniform Portfolio

$p_i = p, l_i = 1 \ \forall \ i = 1, .., m$ and $r_{ij} = r \ \forall \ i, j = 1, .., m, \ i \neq j$. Zerlege

$$W_t^i = \sqrt{r} B_t^0 + \sqrt{1-r} B_t^i,$$

where $B^j, j = 0, .., m$ are independent Brownian Motions.

Conditioning on the systematic factor B_1^0 yields for the percentage portfolio loss: $P[L = \frac{k}{m}] =$

$$\binom{m}{k} P\left[A_1^1 < C_1, .., A_1^k < C_k,\right.$$
$$\left. A_1^{k+1} > C_{k+1}, .., A_1^m > C_m\right]$$
$$= \binom{m}{k} \int_{-\infty}^{\infty} P\left[A_1^i < C_i, i = 1, .., k,\right.$$
$$\left. A_1^j > C_j, j = k+1, .., m \,\middle|\, B_1^0 = x\right] P[B_1^0 \in dx]$$
$$= \binom{m}{k} \int P\left[B_1^i < \frac{\ln\frac{C_i}{A_0^i} - (\mu_i - \frac{1}{2}\sigma_i^2) - \sigma_i\sqrt{r}x}{\sqrt{(1-r)}\cdot\sigma_i},\right.$$
$$i = 1, .., k,$$
$$\left. B_1^i > \frac{\ln\frac{C_i}{A_0^i} - (\mu_i - \frac{1}{2}\sigma_i^2) - \sigma_i\sqrt{\rho}x}{\sqrt{(1-r)}\sigma_i}, i = k+1, .., m\right]$$
$$P[B_1^0 \in dx]$$
$$= \binom{m}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{1}{\sqrt{1-r}}(c + \sqrt{r}x)\right)^k$$
$$\cdot\left(1 - \Phi\left(-\frac{1}{\sqrt{1-r}}(c + \sqrt{\rho}x)\right)\right)^{m-k} \Phi(dx).$$

In the last equation

$$c = c_i = \frac{1}{\sigma_i}(\ln(A_0^i/C_i) + \mu_i - \frac{1}{2}\sigma_i^2),$$

since $c_i = c = \Phi^{-1}(p)$.

Limiting distribution $m \to \infty$

$$\begin{aligned}
& F_m(\theta) \\
= \quad & P[\text{percentage loss} < \theta] \\
= \quad & \sum_{k=0}^{[m\theta]} P[\text{percentage loss} = k/m] \\
= \quad & \sum_{k=0}^{[m\theta]} \binom{m}{k} \int_{-\infty}^{\infty} \Phi\left(-\frac{1}{\sqrt{1-r}}(c+\sqrt{r}x)\right)^k \\
& \qquad \left(1 - \Phi\left(-\frac{1}{\sqrt{1-r}}(c+\sqrt{r}x)\right)\right)^{m-k} \Phi(dx)
\end{aligned}$$

Substitution

$$s = s(x) = \Phi\left(\frac{1}{\sqrt{1-r}} \cdot \left(\Phi^{-1}(p) - \sqrt{r} \cdot x\right)\right)$$

yields

$$\begin{aligned} F_m(\theta) &= \sum_{k=0}^{[m\theta]} \binom{m}{k} \int_0^1 s^k (1-s)^{m-k} \\ &\qquad d\Phi\left(\frac{1}{\sqrt{r}} \cdot (\sqrt{1-r} \cdot \Phi^{-1}(s) - \Phi^{-1}(p))\right). \end{aligned}$$

Because of the law of large numbers

$$\sum_{k=0}^{[m\theta]} \binom{m}{k} s^k (1-s)^{m-k} \to \mathbf{1}_{(0,\theta]}(s), 0 < s < 1$$

we obtain the density of the limiting distribution

$$\begin{aligned} \frac{\sqrt{1-r}}{\sqrt{r}} \exp\Bigg[-\frac{1}{2r} \cdot (\sqrt{1-r} \cdot \Phi^{-1}(s) \\ -\Phi^{-1}(p))^2 + \frac{1}{2}(\Phi^{-1}(s))^2\Bigg]. \end{aligned}$$

Applications

Basket Credit Derivatives / Synthetic CDO's

Basic Concepts: "Sell" the risk of a subportfolio to the investors.

In mathematical terms: The holder of a tranch, that covers the losses between e.g. $\alpha\%$ and $\beta\%$ might have - depending on the contract specification - the (percentage) expected loss ($\sim$ spread)

$$s \quad = \quad \int \frac{(x-\alpha)^+ \wedge (\beta-\alpha)}{(\beta-\alpha)} f_{p,\rho}(x)dx.$$

Since the overall expected loss (in percentage) is p we can obtain ρ.

Extending the above argument to portfolio consisting of many uniform portfolios we might try to derive implied correlations from a set of equations

$$s_i = \int \frac{(x-\alpha_i)^+ \wedge (\alpha_{i+1}-\alpha_i)}{(\alpha_{i+1}-\alpha_i)} f_{p_1,..,p_k,\rho_1,..,\rho_q}(x)dx,$$

$i = 1,..,l$. Here s_i is the spread of tranch i with boundaries α_i, α_{i+1}.

Contributory Economic Capital
(Normal Distribution)

Let us now consider the case

$$L = \sum_{i=1}^{m} l_i X_i,$$

with $(X_1, ..., X_m)$ multivariate normal.

Under this normal assumption also the allocation of the total EC to all the single assets in the portfolio is proportional to β_i. The contributory economic capital is defined in CAPM by

$$\xi(\alpha, i) = CM(\alpha)\beta_i.$$

Then

$$\sum_{i=1}^{m} l_i \xi(\alpha, i) = EC(\alpha)$$

yields a reasonable allocation rule for capital.
Also

$$\xi(\alpha, i) = \frac{\partial EC(\alpha)}{\partial l_i.}$$

Also the marginal EC, $MC(\alpha, i, h)$ defined by

$$MC(\alpha, 1, h) \quad = \quad EC(\alpha, L + h\cdot) - EC(\alpha, L)$$

equals

$$h\xi(\alpha, i).$$

Non-Normal Returns

But L and $L_i = l_i \mathbf{1}_{D_i}$ are highly non-normal. Therefore

$$\frac{\partial q_\alpha(L)}{\partial l_i} \neq \xi(\alpha, i)$$

Moreover it might happen that

$$\xi(\alpha, i) > 1$$

implying that more capital is needed than the possible loss that you are exposed to.

Alternative Capital Definition

In light of the non-normality another Capital Definition should be considered. Economic Capital viewed as a Risk Measures should also satisfy the $Coherency\ Axioms$ formulated by Artzner et al. A prominent example of a coherent risk measure is similar to a kind of lower partial moment

$$C(L) \quad := \quad E[L|L > K],$$

where K is a threshold, used to define "Large Losses", e.g.
$K = q_{\alpha'}(L)$, then $C(L)$ is coherent.
$K =$fraction of equity capital
$K =$ experienced large losses

Properties

- C uses information beyond the threshold (e.g. beyond the quantiles).
- It takes into account how large large losses are.
- Capital defined by C suffices to survive even worse than large losses on average.

Alternative Contributory Economic Capital

This capital definition yields also a new definition of contributory economic capital

$$C_i(L) \quad := \quad E[\mathbf{1}_{D_i} | L > K].$$

Average contribution of counterparty i to the portfolio loss, when large losses occur.
Theorem:

- $C(L) = \sum_{i=1}^{m} l_i C_i$
- If $K \notin \{\sum_{k=1}^{n} l_{i_k} | \{i_1, .., i_n\} \subset \{1, .., m\}\}$ then

$$C_i \quad = \quad \frac{\partial C(L)}{\partial^+ l_i}$$

- C_i is a coherent risk measure on the probability space generated by the portfolio.

Remarks

- $C_i < 1$

- These figures are a by-product if the loss distribution is generated by a Monte-Carlo-Simulation.

- First statistics of the distribution of L_i given $L > K$. Other statistics like variance could be useful. (Conditional variance is probably not coherent.)

- Definition reflects a causality relation. If counterparty i adds more to the overall loss than counterparty j in bad situations for the bank, also business with i should be more costly (asuming stand alone risk characteristics are the same).

- A function C is a coherent risk measure iff C is a generalized scenario, i.e. there is a set of probability measures $\mathcal{Q}$ such that

$$C(X) = sup\{E^Q[X] | Q \in \mathcal{Q}\}.$$

Since $L, L_i \geq 0$ the capital allocation rule carries over to all coherent risk measures.

- If $L = \sum l_i X_i$ then the capital allocation also works as long as the X_is are positive.

Simulation Study

Portfolio of 40 counterparties with 5 year default probabilities. New capital allocation rule based on shortfall risk changes the order of capital consumption.

Validation of credit risk models

Default probabilities

$$P[D_i] = \text{Default Probabilities}$$

Determination

1. Step Rating

e.g. from 1="AAA", best creditworthiness to 10="C" worst

2. Step Calibration

$$P[D_i] = \frac{\#\text{Defaults in Rating } j(i)}{\#\text{in Rating } j(i)}$$

Challenge: Validation of default probability, usually they are assumed to be independent. But there are *Dependent defaults*. These are modeled through A_t = Ability to Pay Process

$$dA_t^i \quad = \quad \mu_A(i) A_t^i dt + \sigma_A(i) A_t^i dW_t^i, \qquad (1)$$

Brownian Motion W^i and

$$D_i \quad = \quad \{A_1^i < C_i\}, \tag{2}$$

C_i default boundary

$$P[D_i] = \Phi\left((\sigma_A(i))^{-1}\left(\ln(C_i/A_0^i) - \mu_A(i) + \frac{1}{2}\sigma_A^2(i)\right)\right),$$

Φ Standard normal distribution function.

Implementation

Single default probabilities $P[D_i]$ are given.

Multiple defaults

$$
\begin{aligned}
& P[A_1^{i_1} < C_{i_1}, ..., A_1^{i_k} < C_{i_k}] \\
= \quad & P[W_1^{i_1} < \Phi^{-1}(p_{i_1}), ..., W_1^{i_k} < \Phi^{-1}(p_{i_k})]
\end{aligned}
$$

It remains to get the correlations of

$$r = \log A - E[\log A].$$

Can be obtained from Asset time series for some counterparties.

But most counterparties are not listed.

Recent question: Estimation of ρ in uniform portfolios. The observation is a set of dependent "0" and "1". How can we estimate the parameter ρ of the hidden random variable "Asset-Value"?

Validation

Is the EC quota correct? If the confidence level equals 99-98%, EC is only breeched in 1 out of 5000 years. You can't test this statistically!

Possible Approaches

1. Analysis in many subportfolios, i.e. Cross-Sectional Data instead of time series

Problem: Subportfolios are correlated

Try to identify portfolios which are almost uncorrelated

More ideas.

Randomized Subportfolios to make them independent?

2. Parametric Bootstrap.

Generate under H_0 many realisations of the "spatial" distribution of losses.

- Are these realisations "in the neighborhood" to the observed one?

- "Near" in the sense of point process distributions?

Statistical Tests about rejection of H_0, error probabilities

3. Parameter optimization (especially implied correlations), model selection, model validation with (non-parametric) boostrap techniques or resampling of dependent data?

Literature:

Efron/Tibshirani Chapter 17: Cross-Validation and other estimates of prediction error.

Davison/Hinkley Bootstrap Methods and their Application

Chapter 8, Complex Dependence (incl. Spatial processes).