Hazard rate for credit risk and hedging defaultable contingent claims

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Abstract

We provide a concise exposition of theoretical results that appear in modelling default time as a random time, we study in details the invariance martingale property and we establish a representation theorem which leads, in a complete market setting, to the hedging portfolio of a defaultable claim.

1 Introduction

We study a financial market, where primary assets S are traded, including a risk-free asset. The filtration generated by the discounted prices is denoted by $\mathbf{F}^S$. The agent has a knowledge on the asset prices modelled by a filtration $\mathbf{F}$. The default occurs at some random time τ, and the agents are advertized when this time occurs.
In the *structural approach*, the default time τ is a stopping time in the filtration $\mathbf{F}^S$ and it is assumed that the agents have all the information contained on prices, i.e., $\mathbf{F} = \mathbf{F}^S$, whereas in the *reduced-form approach*, the default arrives "by surprise". An intermediary case is when τ is a $\mathbf{F}^S$-stopping time and $\mathbf{F} \subset \mathbf{F}^S$, for example when the filtration $\mathbf{F}$ is the trivial one or when $\mathbf{F}$ is the filtration generated by the information of prices observed at discrete times, as in Duffie and Lando [9].

We give a representation theorem in a general setting, leading to the hedging of defaultable contingent claim using defaultable zero-coupon and default-free assets, when the default-free market is complete. We discuss the role of the hypothesis of invariance of martingales and establish that this hypothesis holds when markets are arbitrage free and when the default-free market is complete. We show that under some regularity condition, the default time is the first time where a stochastic barrier is reached, as in Cox process modelling.

2 The model

We study a model of financial market where a riskless asset is traded, as well as risky financial assets. We limit our study, mainly for simplicity of notation, to the case where there are only one risky financial

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asset, whose price at time t is denoted by S_t. The interest rate is a non-negative process r, we denote by $R_t = \exp\left(-\int_0^t r_s ds\right)$ the discounted factor and by $S_t^0 = \exp\left(\int_0^t r_s ds\right)$ the savings account. The filtration generated by the price of the discounted asset $\widetilde{S}_t = S_t/S_t^0$ is denoted $\mathbf{F}^S = (\mathcal{F}_t^S = \sigma(\widetilde{S}_s, s \leq t); t \geq 0)$. We refer to the market where the traded assets are the savings account and the risky asset S as the *default-free market.* We assume that there exists at least a probability Q equivalent to P such that $\widetilde{S}$ is a $\mathcal{F}^S$-martingale, so that the default-free market is arbitrage free.

A default occurs at a random time (i.e. a non-negative random variable) τ. In the defaultable world, the payment of a contingent claim depends whether or not the default has appeared before the maturity. In particular, we shall study a defaultable zero-coupon bond with maturity T (in short DZC) which pays 1 unit at maturity if and only if the default has not appeared before T. More generally, we investigate the case where a promised payoff X is paid at maturity if the defaut has not appeared before maturity and where a compensation is paid at hit (at the default time) if the default occurs before maturity.

2.1 Filtrations and equivalent martingale measures

We assume, as in [11, 14, 17] that the t-time information available by the agents in the default-free market is a σ-algebra $\mathcal{F}_t$. We do not assume that the agent have complete information on the prices: the filtration $\mathbf{F}$ can be smaller or larger than $\mathbf{F}^S$.

We denote by $\mathbf{D}$ the filtration $\mathbf{D} = (\mathcal{D}_t; t \geq 0)$ with $\mathcal{D}_t = \sigma(D_s, s \leq t)$ where D is the default process defined as $D_t = 1\!\!1_{\{\tau \leq t\}}$. At time t, the agent's information on the prices and on default time is $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{D}_t$: at any time the agent knows whether or not the default has appeared. Hence, the default time τ is a $\mathbf{G}$-stopping time where $\mathbf{G} = (\mathcal{G}_t, t \geq 0)$. In fact, $\mathbf{G}$ is the smallest filtration which contains $\mathbf{F}$, satisfying the usual hypotheses, such that τ is a $\mathbf{G}$-stopping time.

We assume that the knowledge of the default time does not induce arbitrages in the market [1], hence there exists a $\mathbf{G}$- equivalent martingale measure, i.e., a probability Q^* on $\mathcal{G}_T$ such that $(\widetilde{S}, t \leq T)$ is a martingale. Let us remark that the restriction of any $\mathbf{G}$ e.m.m. to $\mathcal{F}_T^S$ is a $\mathbf{F}^S$-e.m.m. Indeed, if

$$E_{Q^*}(\widetilde{S}_T|\mathcal{G}_t) = \widetilde{S}_t$$

taking the expectation with respect to $\mathcal{F}_t^S$ of both members, we get

$$E_{Q^*}(\widetilde{S}_T|\mathcal{F}_t^S) = \widetilde{S}_t\,.$$

2.2 Hazard process

In this section, we work under a reference probability P. Latter on, this probability will be either the historical probability, or the risk neutral one.

Let F be the right-continuous version of the submartingale $F_t = P(\tau \leq t|\mathcal{F}_t)$ and G the conditional survival probability $G_t = 1 - F_t$. We assume that $F_t < 1$ a.s. for any t (See Andreasen [1] and Bélanger et al [**?**] for generalization). We introduce the $I\!\!R^+$-valued *hazard process* $\Gamma_t = -\ln(G_t)$. We assume for simplicity that $F_0 = 0$ so that $\Gamma_0 = 0$. For typographical reasons, we shall sometimes use F, G and Γ

[1]This is a restrictive assumption on the time τ. See for example [11] for cases where this assumption is not satisfied

in the same formula. Obviously, the hazard process depends of the reference probability.
We recall a key lemma established in Dellacherie [5, 7] and its corollary.

Lemma 1 *Let $X \in \mathcal{F}_T$ be integrable. Then,*

$$E(X 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} \frac{1}{G_t} E(XG_T|\mathcal{F}_t)\,. \tag{1}$$

Corollary 1 *Let h be a* **F***-predictable bounded process. Then,*
a)

$$E(h_\tau|\mathcal{G}_t) = 1\!\!1_{\{\tau\leq t\}} h_\tau - 1\!\!1_{\{t<\tau\}} \frac{1}{G_t} E\Big(\int_t^\infty h_u dG_u \,|\mathcal{F}_t\Big)\,. \tag{2}$$

b) In particular, if F is increasing and continuous,

$$E(h_\tau|\mathcal{G}_t) = 1\!\!1_{\{\tau\leq t\}} h_\tau + 1\!\!1_{\{t<\tau\}}\, E\Big(\int_t^\infty h_u \exp\big(\Gamma_t - \Gamma_u\big)\, d\Gamma_u \,|\mathcal{F}_t\Big)\,. \tag{3}$$

The proof of this lemma and its corollary is based on the important remark that any $\mathcal{G}_t$-measurable random variable is equal, on the set $\{t < \tau\}$, to an $\mathcal{F}_t$-measurable random variable.
It is also useful to note that for any **G**-predictable process h, there exists an **F**-predictable process h^* such that both processes are equal on the set $\{t \leq \tau\}$, i.e., $h_t 1\!\!1_{\{t\leq\tau\}} = h^*_t 1\!\!1_{\{t\leq\tau\}}$. Moreover, under the hypothesis $\forall t, F_t < 1$, the process $(h^*_t, t \geq 0)$ is unique (See [6], page 186).
In the very particular case when τ is a **F**-stopping time (structural approach), the condition $F_t < 1$ does not holds. However, formulae (1) and (2) are obvious. Indeed, in that case, $F_t = 1\!\!1_{\{\tau\leq t\}}$, the two filtrations **F** and **G** are equal and $\int_0^\infty h_u dF_u = h_\tau$. Hence, setting $e^{-\Gamma_t} = 1 - F_t = 1\!\!1_{\{t<\tau\}}$, the straightforward equalities

$$\begin{aligned} E(X 1\!\!1_{T<\tau}|\mathcal{F}_t) &= 1\!\!1_{\{t<\tau\}} E(X 1\!\!1_{\{T<\tau\}}|\mathcal{F}_t) \\ E(h_\tau|\mathcal{F}_t) &= h_\tau 1\!\!1_{\{\tau\leq t\}} + 1\!\!1_{\{t<\tau\}} E(\int_t^\infty h_u dF_u|\mathcal{F}_t) \end{aligned}$$

are exactly (1) and (2).

2.3 A basic martingale

As an application of lemma 1, it is easy to check that the process $(L_t, t \geq 0)$ where

$$L_t = 1\!\!1_{t<\tau} e^{\Gamma_t} = (1 - D_t)\, e^{\Gamma_t} = \frac{1 - D_t}{G_t}$$

is a **G**-martingale. This non-negative martingale is obviously discontinuous.

3 Enlargement of filtration

One major question is to describe the dynamics of the risky asset S in the filtration **G**. As mentioned in Hull and White [12] " When we move from the vulnerable world to a default free world, the stochastic processes followed by the underlying state variables may change." We are studying here the reverse case, i.e. we move from the default-free world to the vulnerable one.

3.1 Decomposition of the F-martingales as G-semi-martingales

We recall some facts on enlargement of filtration. The submartingale F admits a unique Doob-Meyer decomposition as $F_t = Z_t + A_t$, where Z is an **F**-martingale and A an **F**-predictable increasing process. Moreover, the process $M_t = D_t - \Lambda_{t\wedge\tau}$ where $d\Lambda_t = dA_t/G_{t-}$ is a **G**-martingale. (See e.g. [11], [13] and [4] for proofs and comments.)
Moreover, we require that (C) holds, where
(C) One of the following conditions is satisfied

(i) Any **F**-martingale is continuous

(ii) For any **F**-stopping time θ, $P(\tau = \theta) = 0$.

Under this condition, an **F**-martingale has no common jump with the hazard process.

If (C) holds, for any **F**-martingale m, the process

$$\widehat{m}_{t\wedge\tau} = m_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{\Gamma_s} d[m, Z]_s$$

is a stopped **G**-martingale (see Dellacherie et al. [6], page 188 or Yor [19], page 41 for proof and comments on the condition (C)). We shall refer to $\widehat{m}$ as the **G**-martingale part of the **F**-martingale m; in particular $\widehat{Z}$ is defined as

$$\widehat{Z}_{t\wedge\tau} = Z_{t\wedge\tau} + \int_0^{t\wedge\tau} e^{\Gamma_s} d[Z, Z]_s\,.$$

3.2 Representation theorem

Our aim is now to establish a representation theorem for **G**-martingales. Such a representation theorem is difficult to obtain for any **G**-martingale, even when **F** is the natural filtration of a Brownian motion W and τ is an honest time in $\mathcal{F}_\infty$. In that particular case, Azéma et al. [2] have established that any **G**-martingale can be written as a sum of a stochastic integral with respect to the **G**-martingale $\widehat{W}$, a stochastic integral with respect to M and a third martingale $v1\!\!1_{(\tau\le t)}$ where $v \in \mathcal{F}_\tau^+$ such that $E(v|\mathcal{F}_\tau) = 0$, where $\mathcal{F}_\tau^+$ is generated by the random variables h_τ where h is **F**-progressively measurable. For example, if $\tau = \sup\{t \le 1 : W_t = 0\}$, then $v = V\,\mathrm{sgne}W_1$ with $V \in L^2(\mathcal{F}_\tau)$ (See Yor [19], page 74).

In what follows, we restrict our attention to the class of **G** martingales of the form $E(h_\tau|\,\mathcal{G}_t)$. and $E(X1\!\!1_{T<\tau}|\,\mathcal{G}_t)$. As far as we know, the following theorem is new.

Theorem 1 *Suppose that* (C) *holds and let* $F = Z + A$ *be the Doob-Meyer decomposition of* F. *Let* h *be an* **F**-*predictable process, and let* $H_t = E(h_\tau|\,\mathcal{G}_t)$. *Then, the process* H *admits a decomposition in martingales as follows*

$$H_t = m_0^h + \int_0^{t\wedge\tau} e^{\Gamma_{s-}}(d\widehat{m}_s^h - (h_s - J_{s-}^h)d\widehat{Z}_s) + \int_{]0,t\wedge\tau]} e^{\Delta\Gamma_s}(h_s - J_{s-}^h)\,dM_s\,. \tag{4}$$

Here m^h *is the* **F**-*martingale*

$$m_t^h = E\Big(\int_0^\infty h_u dF_u|\,\mathcal{F}_t\Big) = E\Big(\int_0^\infty h_u dA_u|\,\mathcal{F}_t\Big)$$

$\widehat{m}^h$ *and* $\widehat{Z}$ *are the* **G**-*martingale parts of the* **F**-*martingales* m^h *and* Z*,* $J^h_t = e^{\Gamma_t}(m^h_t - \int_0^t h_u dF_u)$ *and* M *is the discontinuous* **G**-*martingale* $M_t = D_t - \Lambda_{t\wedge\tau}$ *where* $d\Lambda_t = \frac{dA_t}{1-F_{t-}}$.
Furthermore,

$$J_t \mathbb{1}_{t<\tau} = H_t \mathbb{1}_{t<\tau}\,.$$

PROOF: The rather technical proof, based on Itô's calculus and property (C) is given in the appendix.

Corollary 2 *Suppose that* (C) *holds and that* F *is continuous. Let* $Y \in \mathcal{F}_T$ *and*

$$Y_t = E(Y\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{t<\tau}e^{\Gamma_t}E(YG_T|\,\mathcal{F}_t) = \mathbb{1}_{t<\tau}e^{\Gamma_t}m^Y_t\,,$$

where m^Y *is the* **F** *martingale*

$$m^Y_t = E(YG_T|\,\mathcal{F}_t)\,.$$

Then,

$$Y_t = m^Y_0 + \int_0^{t\wedge\tau} e^{\Gamma_{s-}}\left(d\widehat{m}^Y_s + Y_{s-}d\widehat{Z}_s\right) - \int_{]0,t\wedge\tau]} Y_{s-}\,dM_s\,. \tag{5}$$

PROOF: The proof follows from Theorem 1 with $h_t = Y\mathbb{1}_{T<t}$. Nevertheless, a direct proof can be established from the remark that $Y_t = L_t m^Y_t$ and some Itô's calculus which leads in particular to

$$\begin{aligned} dL_t &= -\frac{dD_t}{1-F_t} + (1-D_{t-})\left(\frac{dF_t}{(1-F_t)^2} + \frac{d\langle F\rangle_t}{(1-F_t)^3}\right) \\ &= -\frac{1}{1-F_t}dM_t + \frac{(1-D_{t-})}{(1-F_t)^2}d\widehat{Z}_t\,. \end{aligned}$$

△

4 (H) hypothesis

We introduce an invariance of martingale hypothesis (H), which implies that the dynamics of the asset price is the same in the default free world and in the defaultable world. We discuss the meaning of that hypothesis, its stability under a change of probability measure, its links with arbitrage opportunities in the defaultable world, and we study the hedging of defaultable contingent claims in that setting.
(H) *Any* **F** *square integrable martingale is a* **G** *square integrable martingale.*

4.1 Arbitrage

We discuss now the hypothesis on the modeling of default time that we require in order to avoid arbitrages in the defaultable market. We show that under completion of the default-free market, (H) hypothesis holds under e.m.m. Note that this is not the case in in the Duffie and Lando model [9].

Proposition 1 *Let* S *be a semi-martingale on* $(\Omega, \mathcal{G}, P)$ *and* $(\widetilde{S}_t = S_t R_t, 0 \le t \le T)$ *the discounted price. Assume that there exists a unique probability* Q*, equivalent to* P *on* $\mathcal{F}^S_T$ *such that* $(\widetilde{S}_t, t \le T)$ *is an* $\mathbf{F}^S$-*martingale under the probability* Q*. Assume moreover that there exists a probability* Q^**, equivalent to* P *on* $\mathcal{G}_T$ *such that* $(\widetilde{S}_t, 0 \le t \le T)$ *is a* **G**-*martingale under the probability* Q^**. Then,* (H) *holds under* Q^* *and* Q *and the restriction of any* Q^* *to* **F** *is equal to* Q*.*

PROOF: We give a "financial proof" of this obvious and important resuslt. From the hypothesis, any square integrable random variable $\mathcal{F}^S_T$ measurable (any contingent claim) can be written as a stochastic integral with respect to the discouted price, i.e., there exists x and a square integrable $\mathbf{F}^S$ predictable process θ such that $R_T X = x + \int_0^T \theta_s d\widetilde{S}_s$. The t-time value of the contingent claim is $E_Q(XR_T/R_t|\mathcal{F}^S_t)$. Hence, X is hedgeable by a $\mathcal{G}$-adapted strategy and, from the uniqueness of price for hedgeable claims, for any $\mathbf{G}$-e.m.m. Q^*,

$$E_Q(XR_T|\mathcal{F}^S_t) = E_{Q^*}(XR_T|\mathcal{G}_t)\,.$$

In particular, $E_Q(Z) = E_{Q^*}(Z)$ for any $Z \in \mathcal{F}^S_T$ (take $t = 0$ and $X = ZR_T^{-1}$), hence the restriction of any e.m.m. Q^* to the σ-algebra $\mathcal{F}^S_T$ equals Q. Moreover, since any square integrable $\mathbf{F}^S$-Q-martingale can be written as $E_Q(X|\mathcal{F}^S_t) = E_{Q^*}(X|\mathcal{G}_t)$, we get that any square integrable $\mathbf{F}^S$-Q^*-martingale is a $\mathbf{G}$-Q^*-martingale. △

4.2 Characterization of (H) hypothesis

We assume that (H) holds under the probability P, i.e., any P-square integrable $\mathbf{F}$-martingale is a $\mathbf{G}$-martingale. It is well known [8] that (H) hypothesis holds under P if and only if

$$\forall\, t, \quad P(\tau \le t|\mathcal{F}_\infty) = P(\tau \le t|\mathcal{F}_t). \tag{6}$$

In particular, F and Γ, evaluated under P are increasing processes. This is in particular the case for Cox processes (See e.g. Lando [18]) where τ is defined via a given non-negative $\mathbf{F}$-adapted process γ as

$$\tau = \inf\{t \ge 0\,, \int_0^t \gamma_s ds \ge \Theta\}$$

where Θ is a given random variable, independent of $\mathbf{F}$, generally chosen with an exponential law.

The following interesting lemma, which establishes that working under (H) hypothesis is equivalent to a Cox process modeling is proved in [10].

Lemma 2 *If* (H) *hypothesis holds and F is continuous, then the random variable Γ_τ is exponentially distributed and independent of $\mathcal{F}_\infty$. Hence,*

$$\tau = \inf\{t\,:\,\Gamma_t \ge \Theta\}$$

where Θ is an exponential random variable, independent of $\mathcal{F}_\infty$.

PROOF: We suppose that (H) holds, which implies that

$$P(\tau \le t|\mathcal{F}_\infty) = e^{-\Gamma_t}\,.$$

Setting $\Theta \stackrel{def}{=} \Gamma_\tau$, leads to

$$\{t < \Theta\} = \{t < \Gamma_\tau\} = \{C_t < \tau\},$$

where C is the right inverse of Γ, so that $\Gamma_{C_t} = t$. Therefore

$$P(\Theta > u|\mathcal{F}_\infty) = e^{-\Gamma_{C_u}} = e^{-u}.$$

We have thus established that Θ is an exponential random variable, independent of the σ-field $\mathcal{F}_\infty$. Furthermore, $\tau = \inf\{t : \Gamma_t > \Gamma_\tau\} = \inf\{t : \Gamma_t > \Theta\}$.

4.3 Brownian filtration, stability of (H) hypothesis

We have seen, that in mainy cases, (H) holds under the e.m.m. However, hypothesis on τ are generally done under the historical probability measure. In general, (H) hypothesis is not stable under a change of probability (See Kusuoka [17] for a counterexample).

In this section, we examine the stability of (H) under a change of probability in a particular case when the filtration $\mathcal{F}$ is a Brownian filtration. We use a characterization of the Radon-Nikodym density, by means of the representation theorem of Kusuoka [17] that we recall now:

Theorem 2 *Under* (H)*, if F is continuous and if* $\mathbf{F}$ *is a Brownian filtration generated by a Brownian motion W any* $\mathbf{G}$*-square integrable martingale admits a representation as a sum of a stochastic integral with respect to the Brownian motion and a stochastic integral with respect to the discontinuous martingale M.*

This theorem admits a straightforward extension to the case where F is discontinuous.

Let Q^* be any probability equivalent to P on the filtration $\mathbf{G}$ and assume that (H) holds under P. Then, using Kusuoka's theorem, the Radon-Nikodym density ζ of Q^* with respect to P can be written as

$$d\zeta_t = \zeta_{t-}(\psi_t dW_t + \phi_t dM_t),\, \zeta_0 = 1\,,$$

or, using Doléans-Dade's exponential

$$\zeta_t = \mathcal{E}(\psi W)_t\, \mathcal{E}(\phi M)_t\,.$$

Let us restrict our attention to the case where ψ is $\mathbf{F}$-adapted. Let R be defined on $\mathcal{G}_t$ by $dR = \mathcal{E}(\phi M)_t dP$. From Girsanov's theorem, the P-$\mathbf{G}$ Brownian motion W is a R-$\mathbf{G}$ Brownian motion therefore a R-$\mathbf{F}$ Brownian motion. Any R-$\mathbf{F}$ martingale can be written as a stochastic integral with respect to W, and (H) holds under R.
Since $\mathcal{E}(\psi W)_t$ is $\mathcal{F}_t$-adapted, for any $t < T$,

$$Q^*(\tau \le t|\mathcal{F}_T) = \frac{E_P(1\!\!1_{\{\tau\le t\}}\, \zeta_T|\mathcal{F}_T)}{E_P(\zeta_T|\mathcal{F}_T)} = \frac{E_P(1\!\!1_{\{\tau\le t\}}\, \mathcal{E}(\phi M)_T|\mathcal{F}_T)}{E_P(\mathcal{E}(\phi M)_T|\mathcal{F}_T)} = E_R(\tau \le t|\mathcal{F}_T)$$

Since (H) holds under R, we obtain

$$Q^*(\tau \le t|\mathcal{F}_T) = E_R(\tau \le t|\mathcal{F}_t) = Q^*(\tau \le t|\mathcal{F}_t)$$

and (H) holds under Q^*.

A particular case is when the underlying asset follows

$$dS_t = S_t(\alpha_t dt + \sigma_t dW_t)$$

where α and σ are $\mathbf{F} = \mathbf{F}^W$-adapted and where the e.m.m. for the filtration $\mathbf{F}$ is unique. The set of equivalent $\mathbf{G}$-martingale measures is characterized by the set of Radon-Nikodym densities ζ^ϕ of the form

$$d\zeta_t^\phi = \zeta_{t-}^\phi(\theta_t dW_t + \phi_t dM_t) \tag{7}$$

where $\theta_t = \dfrac{\alpha_t - r_t}{\sigma_t}$ is $\mathbf{F}$-adapted and ϕ is any $\mathbf{G}$-adapted process, such that $\phi > -1$. In this case, hypothesis (H) holds under any e.m.m.. Note that there exists an infinite number of $\mathbf{G}$ e.m.m.

4.4 Dynamics of defaultable zero-coupon

We assume now that a defaultable zero-coupon bond (DZC in short) is traded on the market. As we shall check, we have to impose some conditions on the risk-neutral dynamics of the defaultable zero-coupon. If the market price of the defaultable zero-coupon is ρ_t, since we have assumed that the market where the asset, a default-free zero-coupon and the DZC are traded, is arbitrage free, there exists at least a **G**-e.m.m. Q^* such that the discounted price of the DZC is a **G**-martingale, i.e., $\rho_t R_t = E_{Q^*}(R_T 1\!\!1_{T<\tau}|\mathcal{G}_t)$. We emphasize that the e.m.m. is chosen by the market which trades the defaultable zero-coupon at the market price ρ_t. We do not assume the uniqueness of e.m.m. Q^*; the DZC being traded, for any e.m.m., the equality

$$\rho_t R_t = E_{Q^*}(R_T 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Gamma_t} m_t \tag{8}$$

where $m_t = E_{Q^*}(R_T G_T|\mathcal{F}_t)$ holds for any e.m.m. Q^*.

Proposition 2 *Suppose that* (H) *holds under* Q *and that* F *is continuous. Then*

$$d\rho_t = (L_{t-}/R_t)dm_t + \rho_{t-}(-dM_t + r_t dt)\,.$$

PROOF: Under the hypothesis, $dL_t = e^{\Gamma_t} dM_t$. The result follows from $\rho_t R_t = L_t m_t^R$. △

4.5 Representation theorem

In the case where F is continuous and (H) holds, the representation theorem on the set $\{t < \tau\}$ admits a simple proof that we present now.

Proposition 3 *Suppose that* (H) *holds under* Q *and* F *is continuous. Let* $Y \in \mathcal{F}_T^S$ *be square integrable. Then, the* **G***-martingale* $Y_t = E_Q(Y 1\!\!1_{T<\tau}|\,\mathcal{G}_t)$ *admits, on the set* $\{t < \tau\}$*, a decomposition as follows*

$$Y_t = 1\!\!1_{\{t<\tau\}}\left(m_0^Y + \int_0^t L_{u-} dm_u^Y - \int_{]0,t]} m_u^Y\, e^{\Gamma_u} dM_u\right), \tag{9}$$

where m^Y *is the* **F***-martingale*

$$m_t^Y = E_Q\Big(Y G_T|\mathcal{F}_t\Big)\,,$$

$L_t = e^{\Gamma_t}(1-D_t)$ *and* M *is the discontinuous* **G***-martingale* $M_t = D_t - \Gamma_{t\wedge\tau}$ *where* $\Gamma_t = -\ln Q(\tau > t|\mathcal{F}_t)$.

PROOF: We recall that, if A is a bounded variation process and U a semi-martingale, the integration by parts formula simplifies and can be written as

$$A_t U_t = A_0 U_0 + \int_0^t A_{s-} dU_s + \int_0^t U_s dA_s\,.$$

From lemma 1,

$$E_Q(Y 1\!\!1_{T<\tau}|\,\mathcal{G}_t) = 1\!\!1_{t<\tau} e^{\Gamma_t} E_Q(Y G_T|\,\mathcal{F}_t) = L_t m_t^Y\,.$$

Using $dL_t = -L_t dM_t$, and the fact that L is a process with bounded variation, the integration by part leads to

$$E_Q(Y 1\!\!1_{T<\tau}|\,\mathcal{G}_t) = m_0^Y + \int_0^t L_{u-} dm_u^Y - \int_{]0,t]} m_u^Y e^{\Gamma_u} dM_u\,,$$

on the set $\{t < \tau\}$. △

Remark 1 In the particular case where Γ is deterministic and continuous, we get $m_t^Y = e^{-\Gamma(T)} E_Q(Y|\mathcal{F}_t)$ and

$$E_Q(Y 1\!\!1_{T<\tau}|\mathcal{G}_t) = 1\!\!1_{\{t<\tau\}} e^{-\Gamma(T)} \left(m_0^Y + \int_0^t e^{\Gamma(s)} dY_s - \int_{]0,t]} e^{\Gamma(s)} Y_s dM_s \right),$$

where $Y_t = E_Q(Y|\mathcal{F}_t)$.

4.6 Hedging strategies

We suppose that $\mathbf{F} = \mathbf{F}^S$, and that the default-free market is complete and arbitrage free. We assume that a defaultable zero-coupon is available on the market and H holds under the unique $\mathbf{F}^S$-e.m.m. Q. We also assume that the process F is continuous.
We denote by Q^* a $\mathbf{G}$-e.m.m. We recall that Q^* and Q are equal on the σ-algebra $\mathcal{F}_T$. We now make precise the hedging of a defaultable claim.

4.6.1 Terminal payoff

We study in a first step the case of a terminal payoff of the form $X 1\!\!1_{T<\tau}$. We compute $E_{Q^*}(XR_T 1\!\!1_{T<\tau}|\mathcal{G}_t)$ by mean of our representation theorem, and we give the hedging strategy for $X 1\!\!1_{T<\tau}$ based on savings account, risky asset and defaultable zero-coupon.

We recall that a pair (α, β) of $\mathbf{F}$ adapted processes is an hedging strategy for the contingent claim $Y \in \mathcal{F}_T$ if, denoting by $V_t = \alpha_t S_t^0 + \beta_t S_t$ the value of this strategy, the self-financing relation $dV_t = \alpha_t dS_t^0 + \beta_t dS_t$ holds and $Y = \alpha_T S_T^0 + \beta_T S_T$. A self-financing strategy is characterized by its initial value x and the parameter β via $R_t V_t = x + \int_0^t \beta_s d(RS)_s = E_Q(R_T Y|\mathcal{F}_t)$. The number of shares of riskless asset for this strategy is $\alpha_t = R_t(V_t - \beta_t S_t)$. We shall call "hedging portfolio" the number β of shares of the asset held in the self-financing portfolio.

Since any contingent claim in $\mathcal{F}_T$ is hedgeable in the default-free market, for any square integrable random variable $X \in \mathcal{F}_T$, there exists a predictable process $(\mu_t^X, t \geq 0)$ (the hedging portfolio) and a constant x (the price) such that

$$XG_T R_T = x + \int_0^T \mu_s^X d\widetilde{S}_s .$$

Hence, the t-time price m_t^X of XG_T is given via its discounted price $\widetilde{m}_t^X = R_t m_t^X$ by

$$\widetilde{m}_t^X = E_Q(XG_T R_T|\mathcal{F}_t) = m_0^X + \int_0^t \mu_s^X d\widetilde{S}_s , \tag{10}$$

where Q is the $\mathbf{F}$-e.m.m.
The strategy $(R_t(m_t^X - \mu_t^X S_t), \mu_t^X)$ is now a self-financing strategy hedging the contingent claim XG_T.

In the same way, a triple (a_t, b_t, c_t) of $\mathbf{G}$-adapted processes is an heging strategy for a $\mathcal{G}_T$-measurable contingent claim Y if the process $V_t = a_t S_t^0 + b_t S_t + c_t \rho_t$ satisfies the self-financing relation $dV_t = a_t dS_t^0 + a_t dS_t + c_t d\rho_t$ and $Y = a_T S_T^0 + b_T S_T + c_t \rho_T$.

Theorem 3 *We assume that* (H) *holds under Q and that $F_t = Q(\tau \leq t|\mathcal{F}_t^S)$ is continuous. We denote by Q^* any* $\mathbf{G}$*-e.m.m. Let X_t^d be the value of the defaultable contingent claim $X 1\!\!1_{T<\tau}$, i.e.*

$$X_t^d R_t = E_{Q^*}(XR_T 1\!\!1_{T<\tau}|\mathcal{G}_t)$$

The self financing hedging strategy (a_t, b_t, c_t) *for the defaultable contingent claim* $X\mathbb{1}_{T<\tau}$*, based on the riskless bond, the asset and the defaultable zero-coupon satisfies*

$$a_t S_t^0 + b_t S_t + c_t \rho(t,T) = X_t^d$$

and consists of

(i) $c_t = \dfrac{X_t^d}{\rho(t,T)}$,

(ii) a position on the savings account and the risky asset such that

$$a_t S_t^0 + b_t S_t = 0$$

More precisely, the self financing hedging strategy is made of

(i) a long position of $\dfrac{m_t^X}{m_t}$ *defaultable zero-coupon,*

(ii) a number of asset'shares equal to $e^{\Gamma_t}(\mu_t^X - \dfrac{m_t^X}{m_t}\mu_t)$

(iii) an amount in the riskless bond equal to

$$-e^{\Gamma_t}(\mu_t^X - \frac{m_t^X}{m_t}\mu_t)S_t \,,$$

where the different values (m_t, μ_t) *and* (m_t^X, μ_t^X) *are defined in (8) and (10).*

Proof: ¿From the representation theorem applied to

$$\widetilde{X}_t^d = X_t^d R_t = E_{Q^*}(XR_T\mathbb{1}_{T<\tau}|\mathcal{G}_t) = (1-D_t)e^{\Gamma_t}\widetilde{m}_t^X = L_t\widetilde{m}_t^X \,,$$

we obtain

$$\widetilde{X}_T^d = m_0^X + \int_0^{T\wedge\tau} e^{\Gamma_u} d\widetilde{m}_u^X - \int_{]0,T\wedge\tau]} \widetilde{m}_u^X e^{\Gamma_u} dM_u \,.$$

Let us denote as in (8) $\widetilde{m}_t = E_Q(R_T G_T|\mathcal{F}_t) = \widetilde{m}_t^1$ the discounted price of G_T and $\mu_t = \mu_t^1$ its hedging portfolio. Then, using that $d\widetilde{\rho}_t - L_{t-}d\widetilde{m}_t = -L_t\widetilde{m}_t dM_t$,

$$\begin{aligned}
\widetilde{X}_T^d &= m_0^X + \int_0^{T\wedge\tau} e^{\Gamma_s}(\mu_s^X - \frac{\widetilde{m}_s^X}{\widetilde{m}_s}\mu_s)d\widetilde{S}_s + \int_{]0,T\wedge\tau]} \frac{\widetilde{m}_u^X}{\widetilde{m}_u} d\widetilde{\rho}_u \\
&= m_0^X + \int_0^{T\wedge\tau} e^{\Gamma_s}(\mu_s^X - \frac{m_s^X}{m_s}\mu_s)d\widetilde{S}_s + \int_{]0,T\wedge\tau]} \frac{m_u^X}{m_u} d\widetilde{\rho}_u \,. \qquad (11)
\end{aligned}$$

The equality (11) characterizes the hedging portfolio in terms of risky investments. It remains to note that $\rho_t = L_t m_t$ and $X_t^d = L_t m_t^X$. ¿From the construction, this strategy is self-financing. △

Let us give a direct proof in the simple case $r = 0$. The price of the defaultable claim $X\mathbb{1}_{T<\tau}$ is X_t^d defined by

$$\begin{aligned}
X_t^d &= E_{Q^*}(X\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \mathbb{1}_{t<\tau}e^{\Gamma_t}E_Q(Xe^{-\Gamma_T}|\mathcal{F}_t) \\
&= \mathbb{1}_{t<\tau}e^{\Gamma_t}m_t^X = L_t m_t^X
\end{aligned}$$

and the price of the DZC satisfies

$$\rho(t,T) = \mathbb{1}_{t<\tau}e^{\Gamma_t}E_Q(e^{-\Gamma_T}|\mathcal{F}_t) = \mathbb{1}_{t<\tau}e^{\Gamma_t}m_t$$

Hence

$$X_t^d = E_{Q^*}(X\mathbb{1}_{T<\tau}|\mathcal{G}_t) = \frac{m_t^X}{m_t}\rho(t,T)$$

We now check that there exists a triple (a,b,c) determining a self financing portfolio such that $a_t S_t^0 + b_t S_t = 0$. The self financing condition reads

$$b_t S_t \sigma d\widetilde{W}_t + c_t(L_{t-} dm_t + m_t dL_t) = dX_t^d = L_t dm_t^X + m_t^X dL_t$$

¿From the choice of c, this equality reduces to

$$b_t S_t \sigma d\widetilde{W}_t + c_t L_{t-} dm_t = L dm_t^X$$

Hence the form of b given in the theorem.

It is difficult to make explicit the hedging, since it requires the hedging of claims of the form $\exp(\int_0^T \gamma_s ds)$ where γ is a random process. However, in the very particular case where γ and r are deterministic we get

$$X_T^d = h + \int_0^{T\wedge\tau} e^{\Gamma_s} \mu_s^X d\widetilde{S}_s + \int_{]0,T\wedge\tau]} R_T E_Q(X|\mathcal{F}_s) d\widetilde{\rho}_s \,.$$

4.6.2 Rebate part

We compute the quantity $E_{Q^*}(h_\tau 1\!\!1_{\tau\le T} R_\tau|\mathcal{G}_t)$, which corresponds to the price of the rebate, when the compensation is payed at hit.

Proposition 4 *Let C_t^h be the value of the rebate, i.e.*

$$C_t^h R_t = h_\tau R_\tau 1\!\!1_{\tau\le t} + 1\!\!1_{t<\tau} e^{\Gamma_t} E_Q(\int_t^T R_u h_u dF_u|\mathcal{F}_t) = ,$$

The hedging strategy before default time of the rebate part, paid at hit, consists of

(i) $c_t = \frac{1}{\rho_t}(C_t^h - h_t)$ defaultable zero-coupon

(ii) a position on savings account and risky asset such that

$$a_t S_t^0 + b_t S_t = h_t$$

and the strategy (a_t, b_t, c_t) is self financing.

More precisely the hedging strategy of the rebate part is made of (before the default time)

(i) $\frac{1}{m_t L_t}(C_t^h - h_t)$ defaultable zero-coupon

(ii) $e^{\Gamma_t}[\mu_t^h - \frac{1}{m_t}\mu_t C_t^h] + \frac{1}{m_t}\mu_t h_t$ shares of the asset

(iii) a cash amount of $S_t[\frac{1}{m_t}\mu_t h_t - e^{\Gamma_t}[\mu_t^h + \frac{1}{m_t}\mu_t C_t^h]] + h_t$,

where $C_t^h = (R_t)^{-1} E_Q(\int_t^T h_u R_u f_u du|\mathcal{F}_t)$.

Proof: We denote by C_t^h the price of the rebate

$$\widetilde{C}_t^h = E_{Q^*}(h_\tau 1\!\!1_{\tau\le T} R_\tau|\mathcal{G}_t) = h_\tau R_\tau 1\!\!1_{\tau\le t} + 1\!\!1_{t<\tau} e^{\Gamma_t} E_Q(\int_t^T R_u h_u dF_u|\mathcal{F}_t)\,,$$

and by μ^h the hedging strategy for the discounted claim $\int_0^T R_u h_u dF_u$ in the default-free world, i.e.,

$$m_0^h + \int_0^t \mu_s^h d\widetilde{S}_s = E(\int_0^T R_u h_u dF_u|\mathcal{F}_t)\,.$$

The representation theorem states that

$$E_{Q^*}(h_\tau R_\tau \mathbb{1}_{\tau\le T}|\mathcal{G}_t) = C_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u}\mu_u^h d\widetilde{S}_u + \int_{[0,t\wedge\tau[}(h_u R_u - J_{u-})dM_u\,.$$

Now, on the set $t<\tau$,

$$J_t = E_{Q^*}(h_\tau R_\tau \mathbb{1}_{\tau\le T}|\mathcal{G}_t) = e^{\Gamma_t}E_Q(\int_t^T h_u R_u dF_u|\mathcal{F}_t) = \widetilde{C}_t^h\,.$$

Hence, introducing $\widetilde{m}_t$, the discounted price of G_T

$$E_{Q^*}\left(h_\tau R_\tau \mathbb{1}_{\tau\le T}|\mathcal{G}_t\right) = C_0^h + \int_0^{t\wedge\tau} e^{\Gamma_u}\mu_u^h d\widetilde{S}_u - \int_{[0,t\wedge\tau[}(h_u R_u - \widetilde{C}_u^h)\frac{1}{L_u\widetilde{m}_u}[d\widetilde{\rho}_u - L_u d\widetilde{m}_u]$$

which leads to

$$E_{Q^*}\left(h_\tau R_\tau \mathbb{1}_{\tau\le T}|\mathcal{G}_t\right) \quad = \quad m_0^h + \int_0^{t\wedge\tau}\left[e^{\Gamma_u}\left(\mu_u^h - C_u^h\frac{\mu_u^h}{m_u}\right) + \frac{\mu_u h_u}{m_u}\right]d\widetilde{S}_u - \int_{[0,t\wedge\tau[}(h_u - C_u^h)\frac{1}{L_u m_u}d\widetilde{\rho}_u$$

$\triangle$

Corollary 3 *Under* (H)*, if F is continuous, the defaultable-market is complete as soon as a defaultable zero-coupon is traded.*

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Appendix

Proof of theorem

As usual, G^c is the martingale continuous part of the semi-martingale G. We recall that $d[G]_t = d[G^c]_t + (\Delta G_t)^2$. Then, Itô's formula leads to

$$\begin{aligned} d(e^{\Gamma_t}) = d(G_t^{-1}) &= -\frac{1}{(G_{t-})^2} dG_t + \frac{1}{(G_{t-})^3} d[G^c]_t + \left(e^{\Gamma_t} - e^{\Gamma_{t-}} + \frac{1}{(G_{t-})^2} \Delta G_t \right) \\ &= \frac{1}{(G_{t-})^2} \left(-dG_t + \frac{1}{G_{t-}} d[G^c]_t \right) + \frac{1}{G_t\,(G_{t-})^2} (\Delta G_t)^2 \\ &= \frac{1}{(G_{t-})^2} \left(-dG_t + \frac{1}{G_{t-}} d[G^c]_t + \frac{1}{G_t} (\Delta G_t)^2 \right) \end{aligned} \tag{12}$$

The quadratic variation of the processes

$$Y_t = m_t^h - \int_0^t h_u dF_u = m_t^h + \int_0^t h_u dG_u$$

and e^{Γ_t} is

$$\begin{aligned}
d[e^{\Gamma}, Y]_t &= d[e^{\Gamma}, m^h]_t + h_t d[e^{\Gamma}, G]_t \\
&= \frac{1}{(G_{t-})^2}\left[-d[G, m^h]_t + \frac{1}{G_t}(\Delta G_t)^2 \Delta m_t^h - h_t d[G]_t + \frac{h_t}{G_t}(\Delta G_t)^3\right] \\
&= \frac{1}{(G_{t-})^2}\left[-d[G, m^h]_t + \frac{1}{G_t}(\Delta G_t)^2 \Delta m_t^h - h_t d[G^c]_t - h_t(\Delta G_t)^2 + \frac{h_t}{G_t}(\Delta G_t)^3\right]
\end{aligned}$$

¿From integration by part formula, the dynamics of $J_t = Y_t e^{\Gamma_t}$ are

$$\begin{aligned}
dJ_t &= e^{\Gamma_{t-}} dY_t + Y_{t-} de^{\Gamma_t} + d[e^{\Gamma}, Y]_t \\
&= -e^{\Gamma_{t-}}(J_{t-} - h_t) dG_t + \frac{1}{(G_{t-})^2}(J_{t-} - h_t) d[G^c]_t + \frac{1}{G_t G_{t-}}(J_{t-} - h_t)(\Delta G_t)^2 \\
&\quad + e^{\Gamma_{t-}}\left(dm_t^h + \frac{1}{G_t G_{t-}}(\Delta G_t)^2 \Delta m_t^h\right) - \frac{1}{(G_{t-})^2} d[G, m^h]_t \\
&= e^{\Gamma_{t-}}(J_{t-} - h_t)\left(-dG_t + e^{\Gamma_{t-}} d[G^c]_t + \frac{1}{G_t}(\Delta G_t)^2\right) \\
&\quad + e^{\Gamma_{t-}}\left(dm_t^h + \frac{1}{G_t G_{t-}}(\Delta G_t)^2 \Delta m_t^h - \frac{1}{G_{t-}} d[G, m^h]_t\right)
\end{aligned}$$

The decomposition of F in the filtration $\mathbf{G}$ is

$$F_{t\wedge\tau} = Z_{t\wedge\tau} + A_{t\wedge\tau} = \widehat{Z}_{t\wedge\tau} - \int_0^{t\wedge\tau} \frac{1}{G_s} d[Z]_s + A_{t\wedge\tau}$$

Then, on the set $\{\tau > t\}$

$$dJ_t^h = e^{\Gamma_{t-}}\left[(J_{t-}^h - h_t) d\widehat{Z}_t + d\widehat{m}_t^h + (J_{t-}^h - h_t) dC_t + dK_t\right]$$

where

$$\begin{aligned}
dC_t &= \frac{1}{G_t}(\Delta G_t)^2 + e^{\Gamma_{t-}} d[G^c] + dA_t - \frac{1}{G_t} d[Z]_t \\
dK_t &= \frac{1}{G_t G_{t-}}(\Delta G_t)^2 \Delta m_t^h - \frac{1}{G_{t-}} d[G, m^h]_t + \frac{1}{G_t} d[G, m^h]_t
\end{aligned}$$

If G has no jump at time t, $dK_t = 0$, if G has a jump

$$dK_t = -(\Delta G_t)(\Delta m_t^h)\frac{1}{G_t G_{t-}}\left(-\Delta G_t + G_t - G_{t-}\right) = 0\,.$$

The first term is equal to

$$\begin{aligned}
dC_t &= \frac{1}{G_t}(\Delta G_t)^2 + e^{\Gamma_t} d[G^c]_t + dA_t - \frac{1}{G_t} d[Z]_t \\
&= \frac{1}{G_t} d[G]_t + dA_t - \frac{1}{G_t} d[Z]_t \\
&= \frac{1}{G_t} d[A]_t + dA_t = \frac{1}{G_t}(\Delta A_t)^2 + dA_t \\
&= e^{\Delta\Gamma_t} dA_t
\end{aligned}$$

where we have used that, in the case where $\mathbf{F}$ martingales are continuous $\Delta A = \Delta F$ and that A is continuous in the case $P(\tau = \theta) = 0$ (See Jeulin [15] page 65). It remains to compensate the jump at time τ in order to obtain the result. △