PREDICTION OF MARKET VOLATILITY WITH A CASCADE MODEL

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- Peter Talkner

TWO PARTS

Part I: A kind of overview

Part II: Financial market modelling inspired by an analogy with turbulence

OVERVIEW

OVERVIEW

- Basic properties of financial markets
- Methodological remarks
- Definitions
- Stylized facts
- The heterogeneous structure of financial markets
- Time series modelling

BASIC FACTS ABOUT THE FX MARKET

- Structure
  - Global market
  - connected by computer network
  - telephone or computer trading
  - size of deals: $\geq 1\,000\,000\,\$$
  - 30 $\sim$ 50 major players
- Traded volume (1998):
  $1.7 \times 10^{12}\ \$$/day
- Prices arrive at intervals of a few seconds
- Data irregularly spaced, with gaps during weekends and nights.

METHODOLOGICAL REMARKS ON THE EMPIRICAL INVESTIGATION OF HIGH FREQUENCY DATA

- Quantity of data: 1 year of high frequency data corresponds to 300 $\sim$ 1000 of daily data!

$\longrightarrow$ New methods for data analysis are needed:
Interpolations, processing, filtering, ...

- Important: Development of statistical methods with as few assumptions as possible concerning the generating process (explorative analysis)
- Demanding task: Modelling daily and weekly seasonalities, scaling laws, correlations, empirical distributions, conditional properties, ...
- Danger to overlook important features in case of (over)hasty modelling

DEFINITIONS

- Middle logarithmic price:

$$x \equiv \frac{1}{2}(\log p_{ask} + \log p_{bid}).$$

  Advantage:

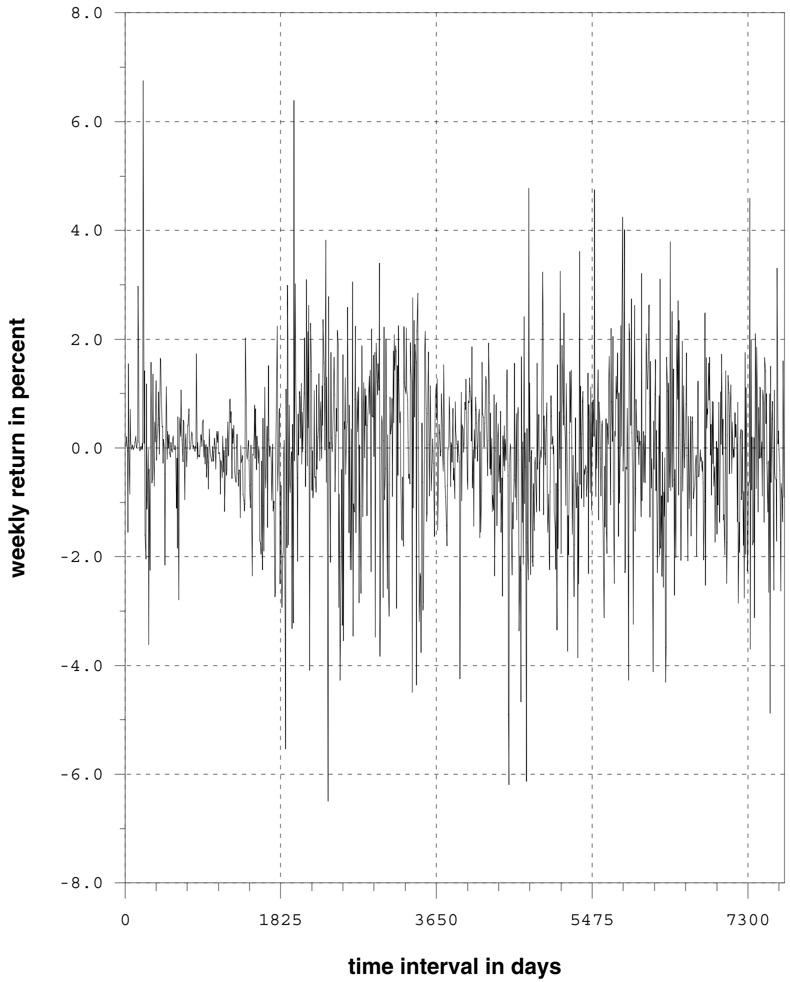
  Logarithmic prices depend only on the *relative* price movement and not on its absolute level.

- The return $r_{\Delta T}(t))$ at time t with respect to a time interval ΔT:

$$r_{\Delta T}(t) \equiv x(t) - x(t - \Delta T).$$

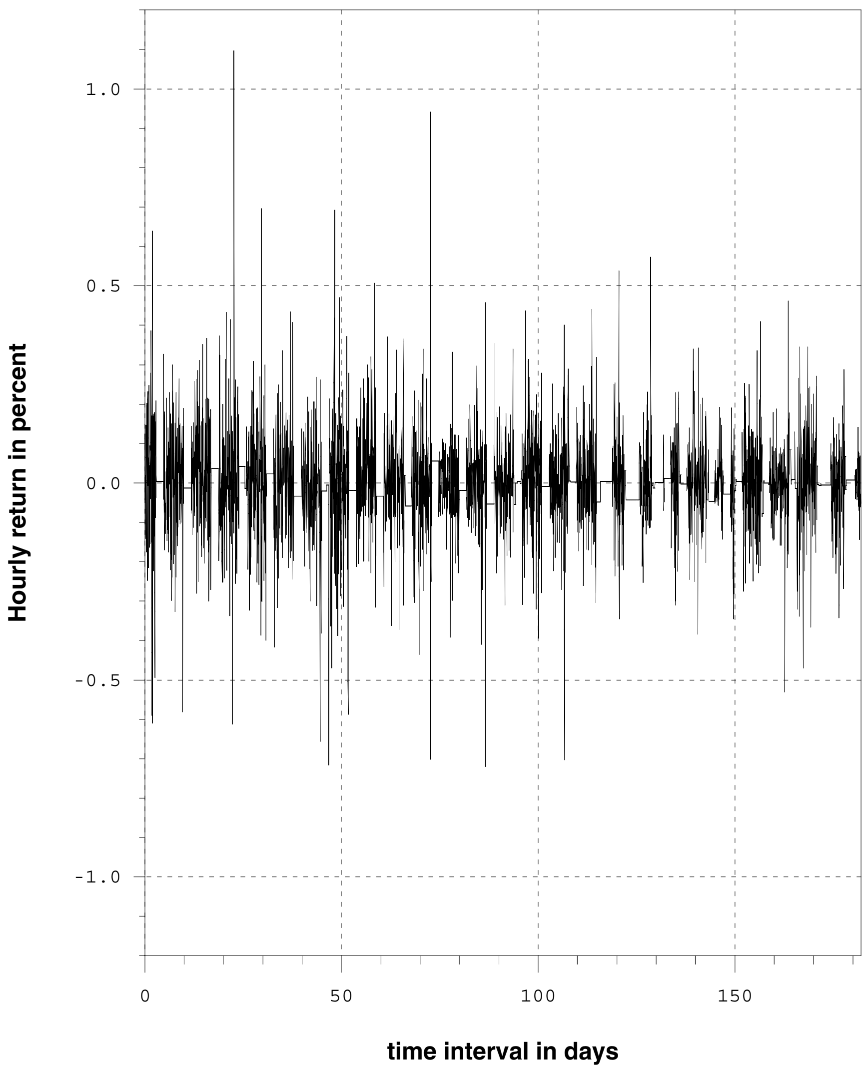
  $r_{\Delta T}(t)$ depends on the time horizon Δt.

CUMULATIVE DISTRIBUTIONS OF USD/JPY RETURNS



USD/JPY Weekly Return over 21 years (Sampling period: 06.06.73 00:00:00 to 01.08.94 00:00:00).

HOURLY RETURNS



USD/DEM hourly returns from 1/1/1996 to 1/6/1996.
Note the seasonality.

VOLATILITIES

The return can be decomposed into

- the amplitude (*volatility*) and
- its trend (the sign).

We define the *volatility* $v(t)$ as

$$v(t) = v[\Delta T, T](t) \equiv \frac{1}{N} \sum_{k=1}^{N} |r[\Delta T](t - (k-1)\Delta T)|$$

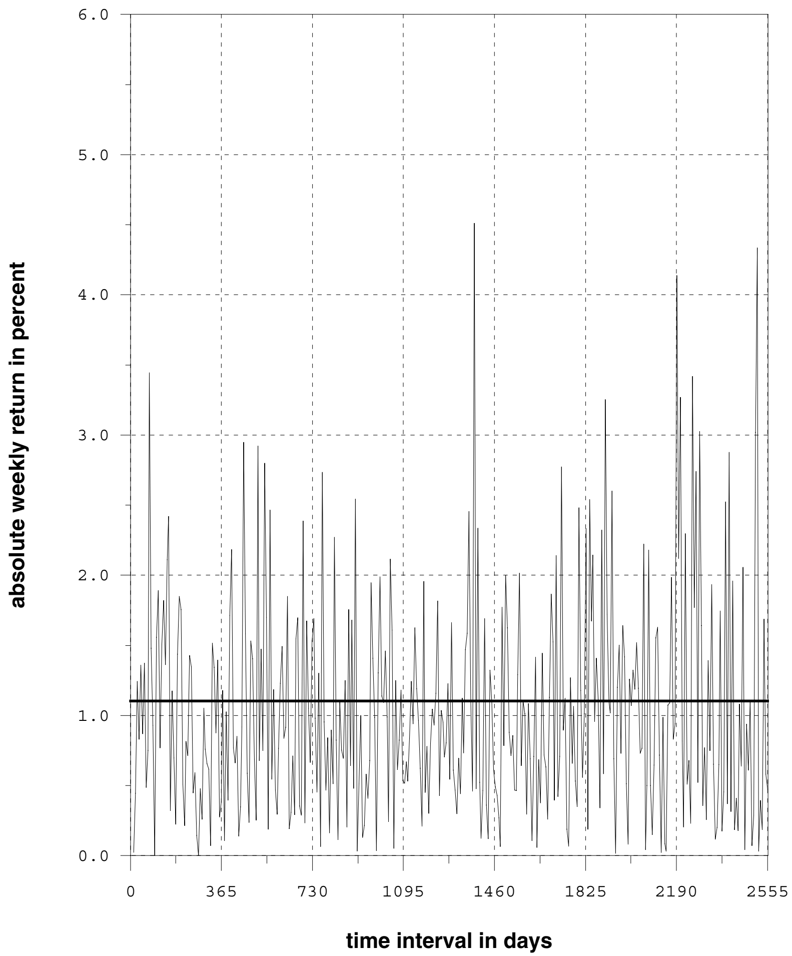
where

T is the sampling period, and

N is a positive integer such that $T = N\Delta T$.

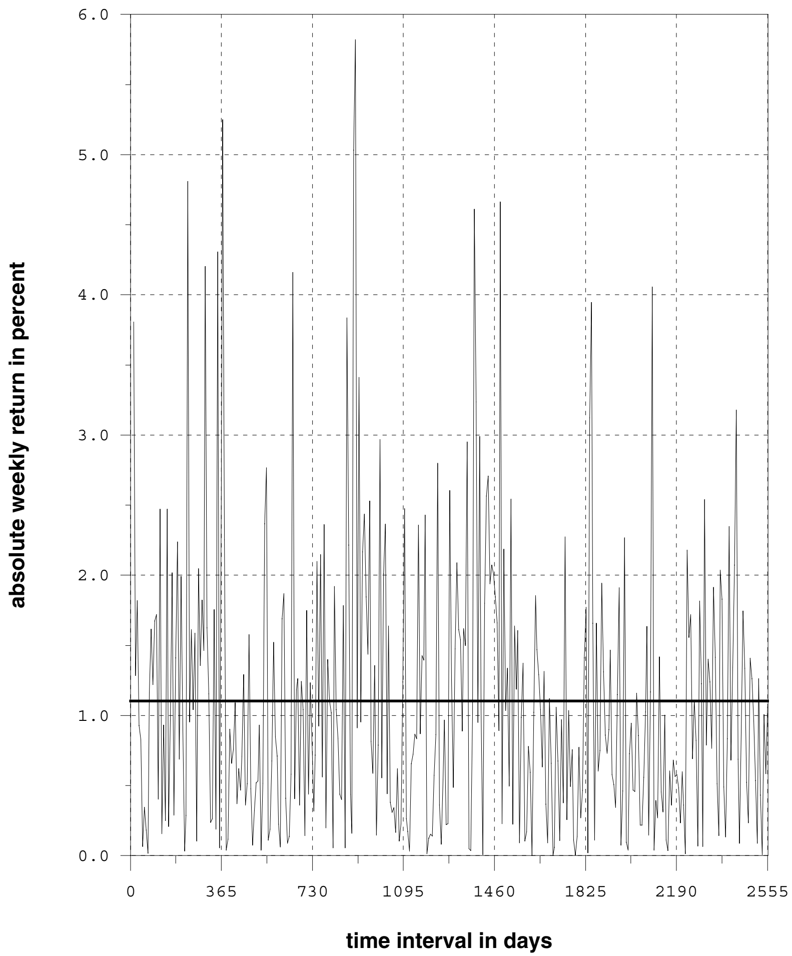
The parameters T, N, and Δt may vary over large ranges.

VOLATILITY OF A GAUSSIAN RANDOM WALK



Weekly Absolute Return (Volatility) as given by a Gaussian random walk simulation over 7 years (Sampling period: 01.01.87 00:00:00 31.12.93 00:00:00). The thick line represents the sample average: 1.033%.

VOLATILITY OF A FINANCIAL TIME SERIES



Weekly Absolute Return (Volatility) USD/JPY over the same sampling period as the previous graph. The thick line represents the sample average: 1.103%. The number of large movements is bigger and a clustering of the movement sizes can be detected.

IMPORTANCE OF VOLATILITY

- Risk management.
- Option pricing.
- LeBaron Effect:
  Market reaction depends on volatility.
  (Blake LeBaron, *Journal of Applied Econometrics*, 1992; *Journal of Business*, 1992.)

STYLIZED FACTS

- Scaling law of absolute returns
- Heavy tails of return distributions
- Seasonalities
- Slowly decaying autocorrelation function of absolute returns

The first two items are related to fractal properties.

SCALING LAW

- Dependence of mean absolute return on the size of the time interval on which the return is observed.

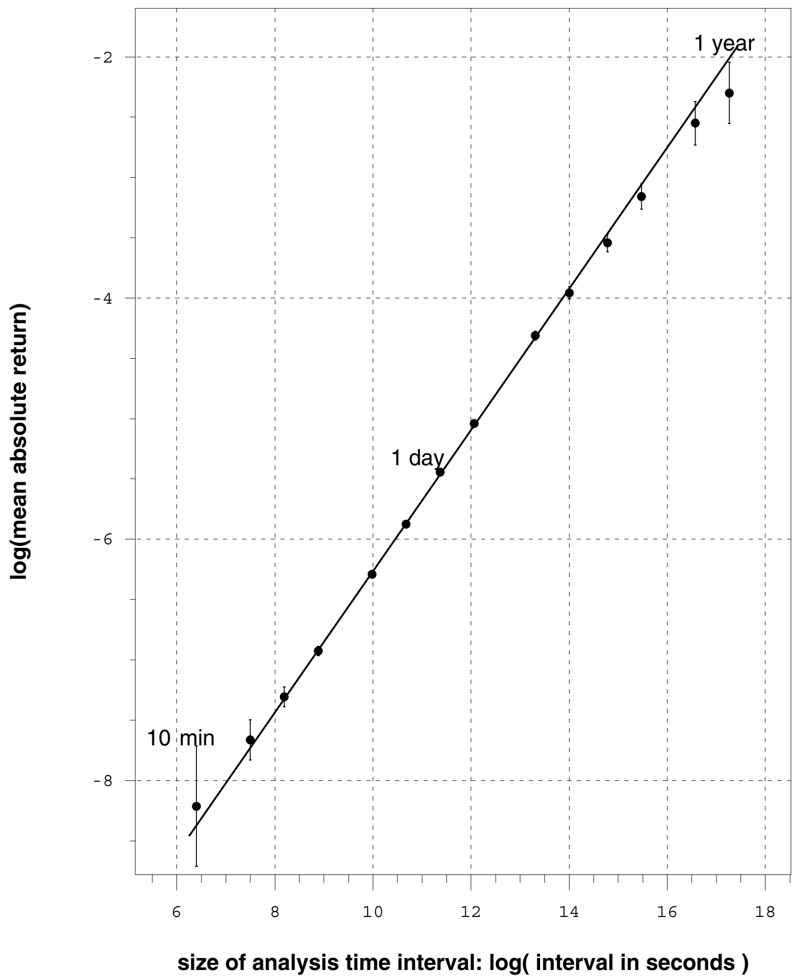
- Empirically, we find a power law for mean absolute returns:

  $$\overline{|r[\Delta t]|} = C \cdot (\Delta t)^D$$

  Δt is the interval size and D the **drift exponent** (a constant).

- The drift exponent is around $D = 0.58$ for freely floating FX rates, with a confidence of typically ± 0.01.
  A Gaussian i. i. d. random walk would have $D = 0.5$.

SCALING LAW GRAPH



Mean absolute change of the logarithmic middle price as a function of the time interval (in seconds). for USD/DEM spot rates (Period: Feb 1986 to Sep 1993, ca. 10 million quotes).

Vertical bars indicate the total error (stochastic and bid-ask uncertainty).

Scaling law:

$\langle r[\Delta T] \rangle = \left(\frac{\Delta T}{11230 \text{ days})} \right)^D$

with drift exponent

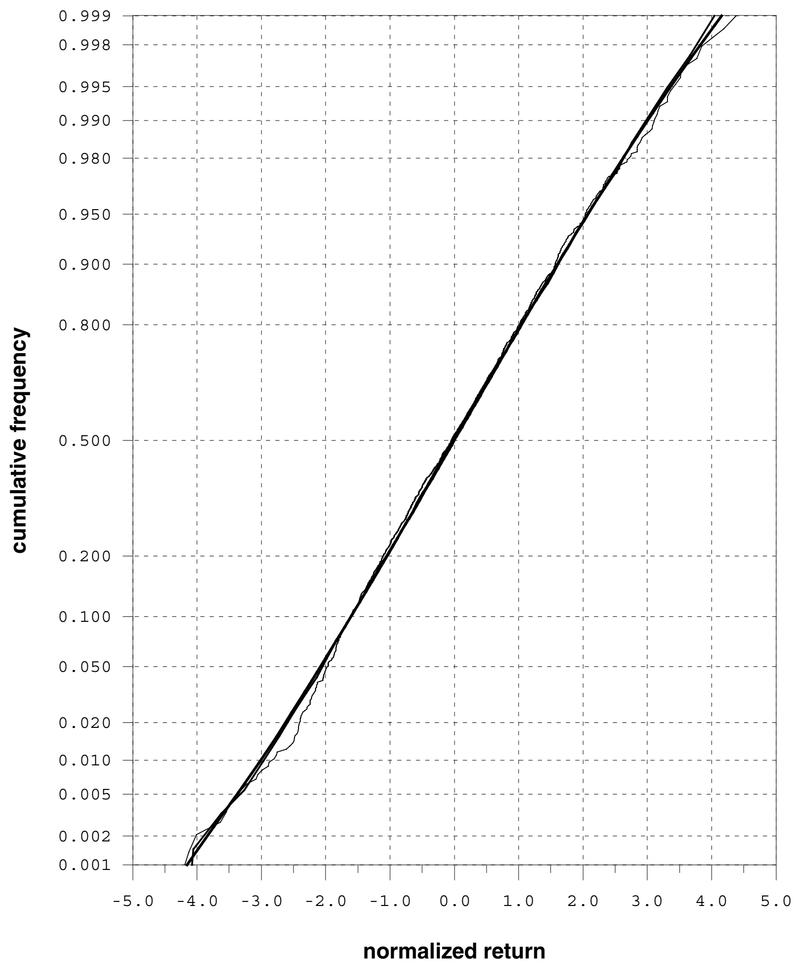
$D = 0.59$.

HEAVY TAILS OF RETURN DISTRIBUTIONS

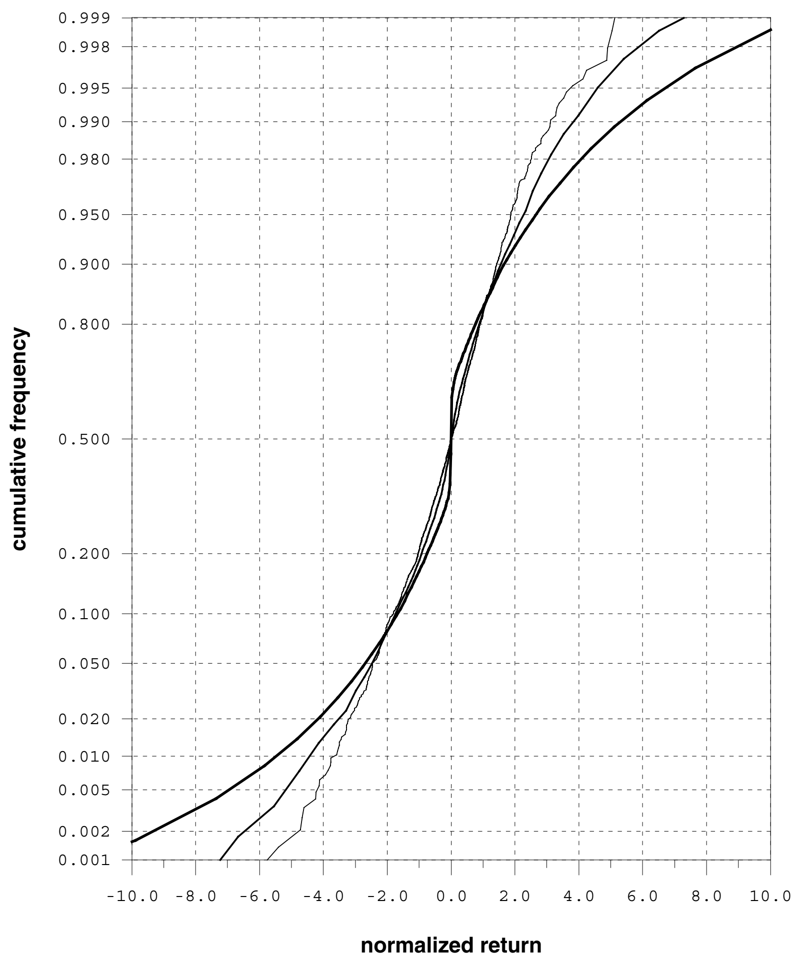
- Return distributions display pronounced heavy tails.
- The tails diminish when the time horizon increases.

$\longrightarrow$ Multifractal properties.

CUMULATIVE DISTRIBUTIONS OF GAUSSIANS PROCESSES

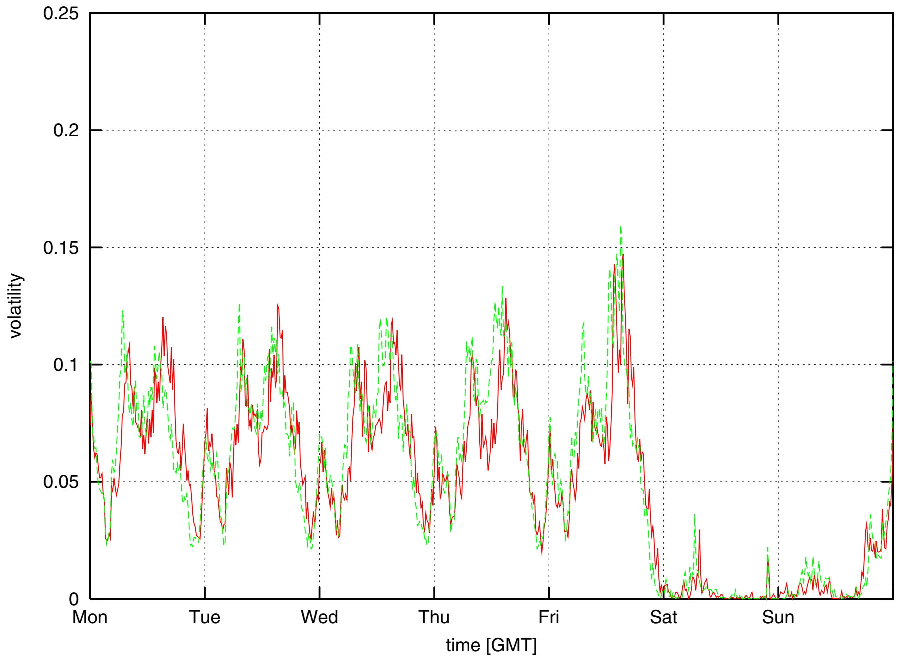


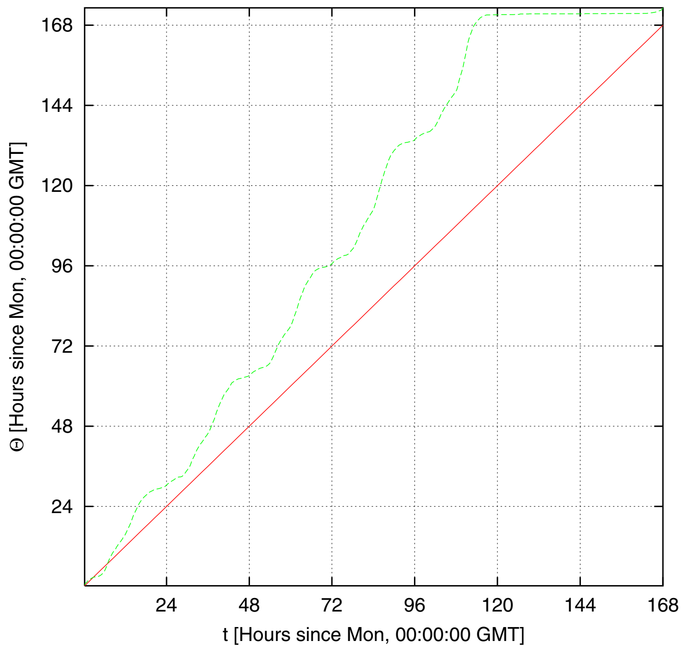
The cumulative distributions for 10 minutes, 1 day and 1 week analysis interval of the **Gaussian simulation** shown against the Gaussian probability on the y-axis. On the x-axis the returns normalized to their mean absolute value are shown.



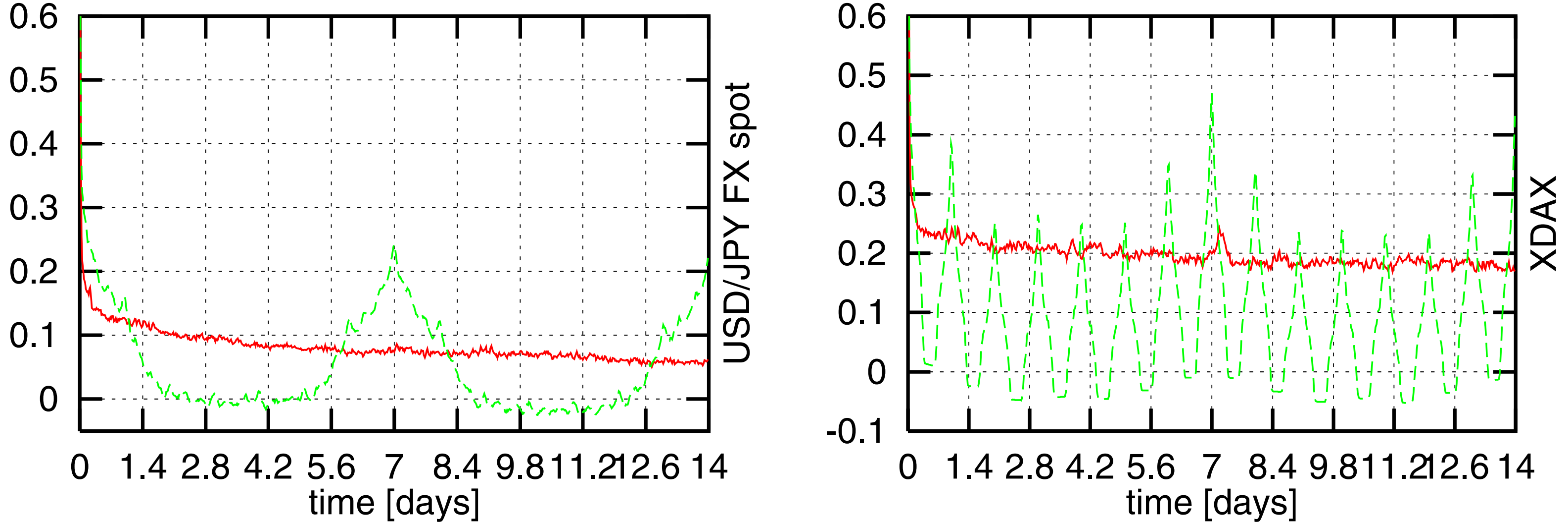
The cumulative distributions for 10 minutes, 1 day and 1 week analysis interval of the **USD/JPY returns** shown against the Gaussian probability on the y-axis. On the x-axis the returns normalized to their mean absolute value are shown.

INTRA-WEEK SEASONALITY AND Θ TIME





AUTOCORRELATION FUNCTIONS OF ABSOLUTE RETURNS

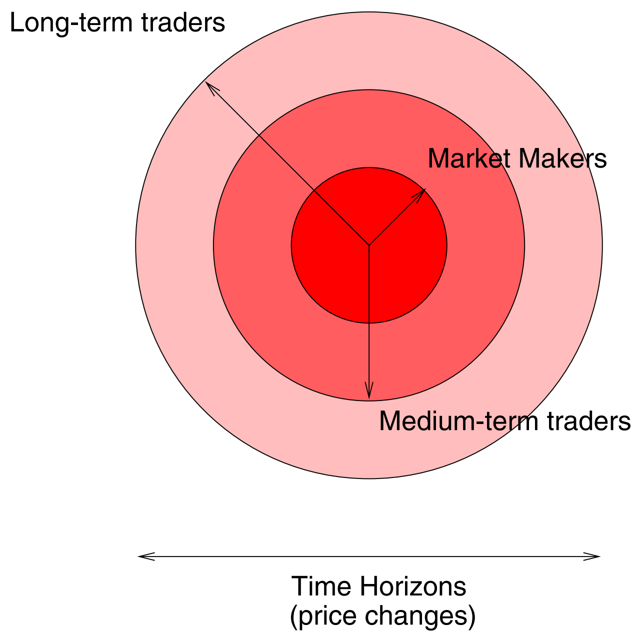


STRUCTURE OF FINANCIAL MARKETS

- Heterogeneous objects.
- Not only the market markers but many participants.
- Different people with different interests, different geographical locations, and different time horizons.
- Time scale (frequency of interventions) is the aspect for which the heterogeneity becomes the most obvious.
- Information flow from long to short time scales.

→ Motivates cascade model.

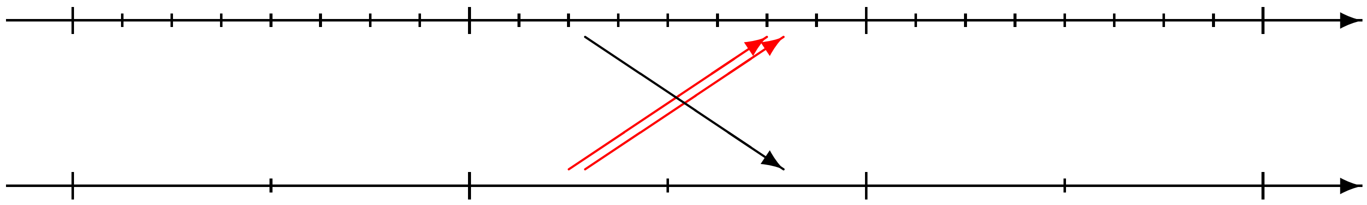
A NON-TRADITIONAL VIEW OF A FINANCIAL MARKET



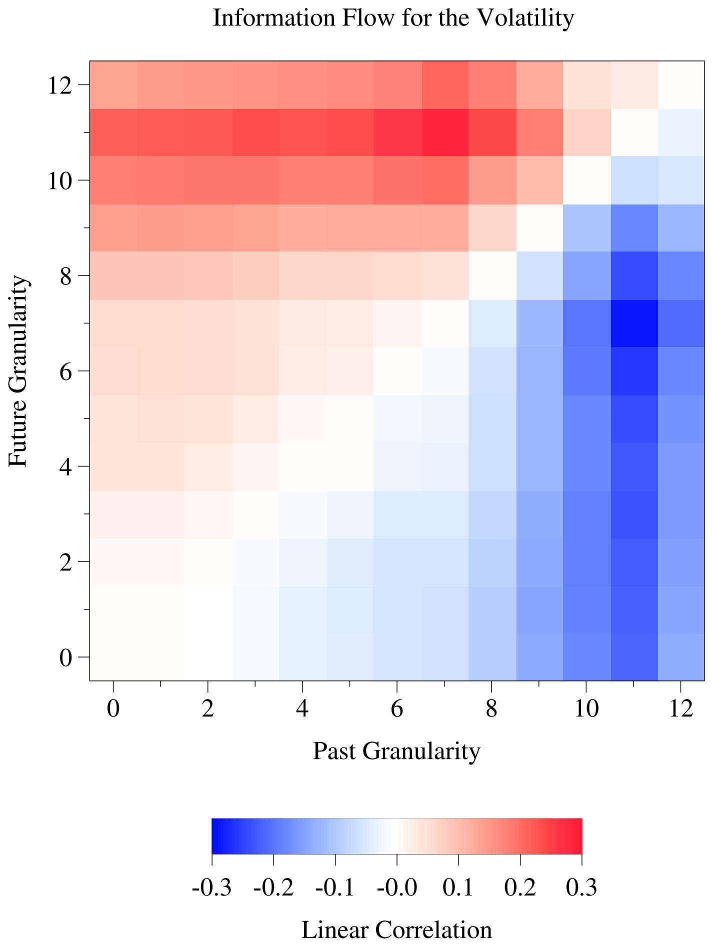
News hit the market at the center. If they are important the price changes will hit different market components and feedback effects will happen in the center. Price changes and time horizons are related through the scaling law $|\Delta x| \propto \Delta t^D$.

We find *asymmetry in the information flow*:

Coarse volatility predicts fine volatility better than the other way around.



INFORMATION FLOW ASYMMETRY, THE HARCH EFFECT



$$I(n, n') = \mathrm{Corr}(\sigma[T, \frac{T}{2^n}](t), \sigma[T, \frac{T}{2^{n'}}](t+T)) - \mathrm{Corr}(\sigma[T, \frac{T}{2^n}](t+T), \sigma[T, \frac{T}{2^{n'}}](t))$$

TIME SERIES MODELLING

- Equation for the returns r_t:

  $$r_t = \sigma_t \xi_t$$

  ξ_t: iid innovations.

- Volatility σ_t usually modeled
  - as deterministic function of past returns (ARCH-type models, used up to now for volatility prediction),
  - or as stochastic process (stochastic volatility models).

- Up to now the volatility could not be observed directly.

  $\longrightarrow$ Stochastic volatility models could not be fitted by maximum likelihood, and not be used for volatility prediction.

FINANCIAL MARKET MODELLING INSPIRED BY ANALOGIES WITH TURBULENCE

- An analogy with hydrodynamic turbulence.
- A stochastic cascade model (SCM) for financial time series.
- Volatility prediction with the SCM.
- Comparison with observations.
- Results and discussion.

MOTIVATION OF VOLATILITY CASCADE

Heterogeneous market

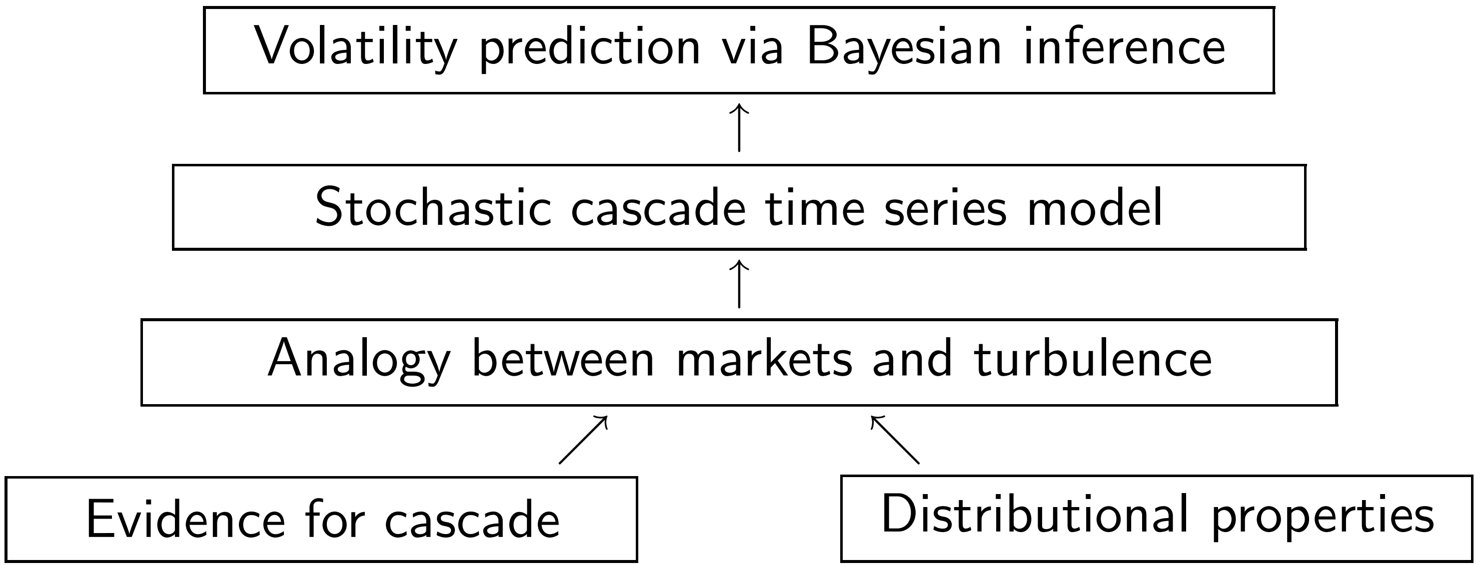
↓

Information flow from long to short time scales

↓

Volatility cascade

EVOLUTION OF THE MODEL

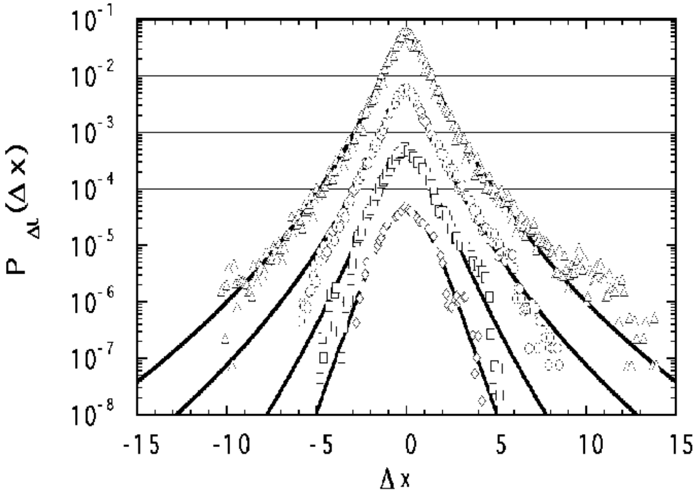


ANALOGY WITH HYDRODYNAMIC TURBULENCE

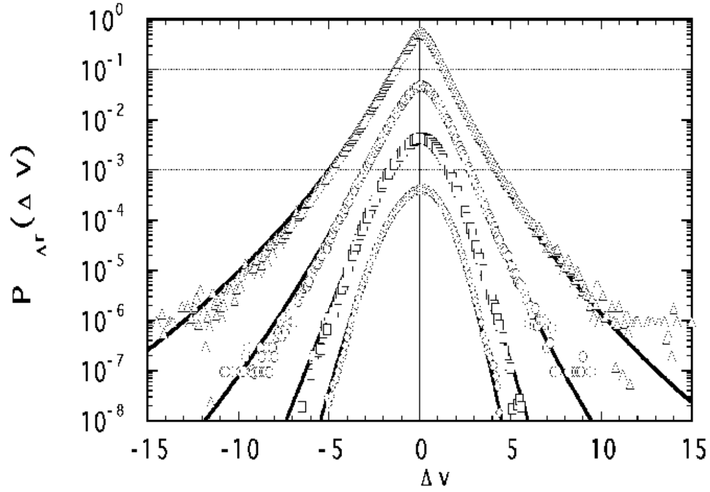
(Ghashghaie et al., Nature **381**, 767–770 (1996))

FX MARKET AND HYDRODYNAMIC TURBULENCE

Pdf of Returns



Pdf of Velocity Differences



FX Market and Turbulence II

hydrodynamic turbulence	FX markets
energy	volatility
spatial distance	time delay
energy cascade in space hierarchy	information cascade in time hierarchy
$\langle(\Delta v)^n\rangle \propto (\Delta r)^{\zeta_n}$	$\langle(\Delta x)^n\rangle \propto (\Delta t)^{\xi_n}$

Correspondence between fully-developed three-dimensional turbulence and FX markets.

THE STOCHASTIC CASCADE MODEL

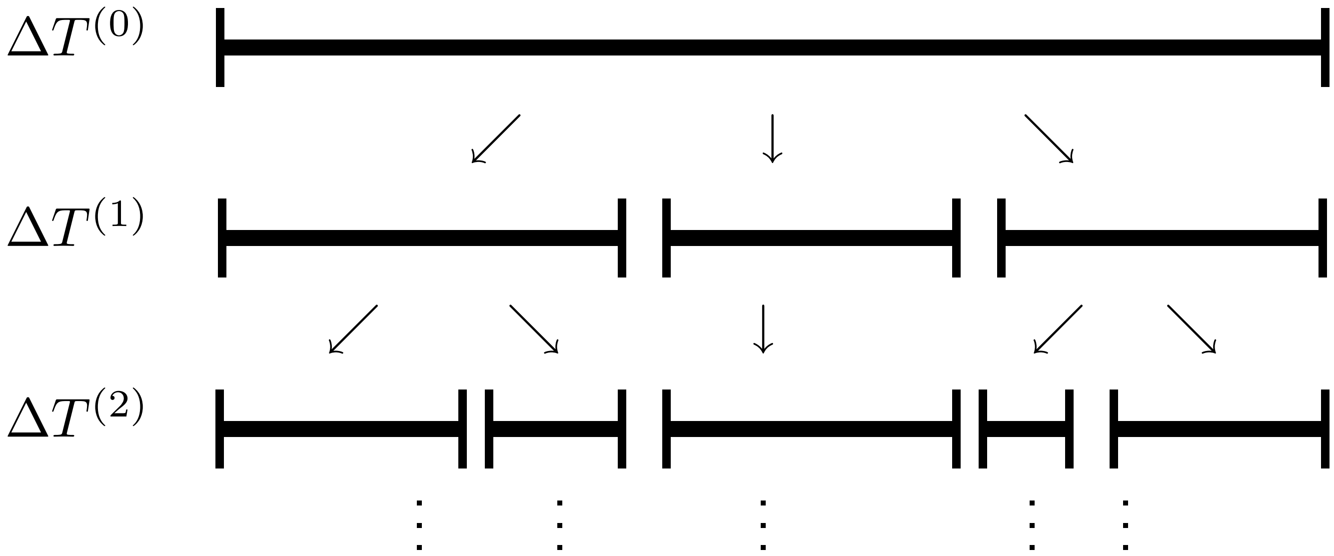
- Equation for the returns r_t:

  $$r_t = \sigma_t \xi_t$$

  ξ_t: iid innovations.

- Volatility σ_t modelled as hierarchical stochastic process with multifractal properties

STOCHASTIC VOLATILITY CASCADE



TIME SERIES MODEL:

W. Breymann, S. Ghashghaie, and P. Talkner (2000)
Int. J. of Applied and Theoret. Finance **3** (in press).

THE VOLATILITY DYNAMICS

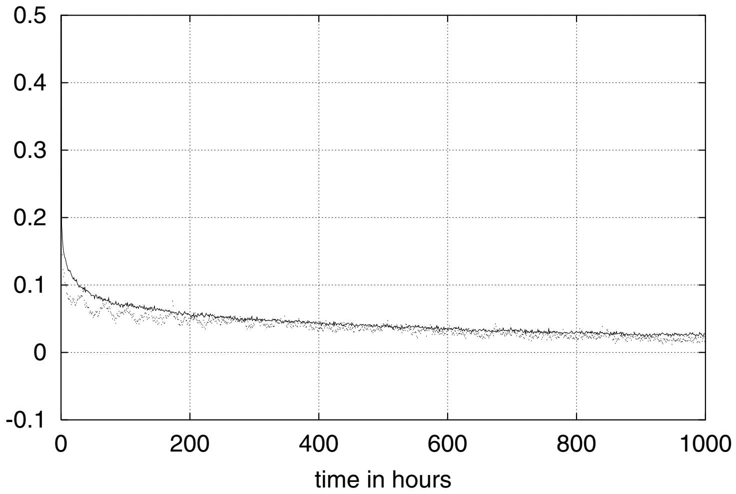
$$\sigma_t \equiv \sigma_t^{(n)} = \sigma_t^{(0)} \cdot \prod_{k=1}^{n} a_t[\Delta T^{(k)}]$$

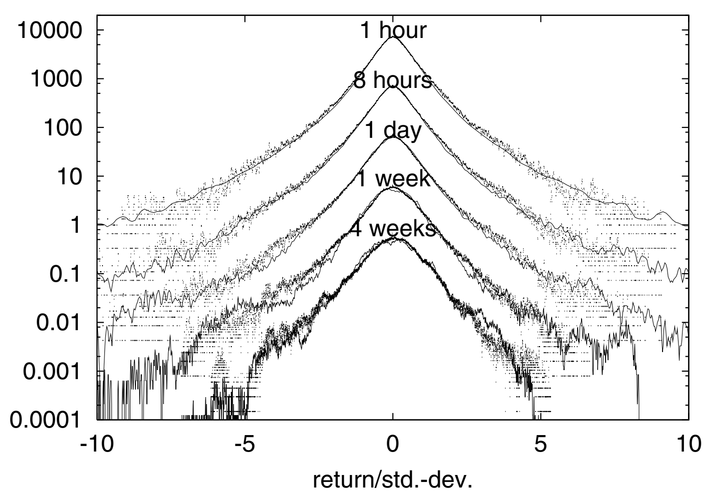
with the following dynamics for the *random volatility factors* $a_t[\Delta T^{(k)}]$:

$$a_t[\Delta T^{(k)}] = \begin{cases} \text{updated} & \text{if } a_t[\Delta T^{(k-1)}] \text{ has been updated} \\ & \text{or with pb. } w[\Delta T^{(k)}] \text{ under the condition} \\ & \text{that } a_t[\Delta T^{(k-1)}] \text{ has not been updated} \end{cases}$$

The $a_t[\Delta T^{(k)}]$ are log-normally distributed.

PROPERTIES OF THE CASCADE MODEL

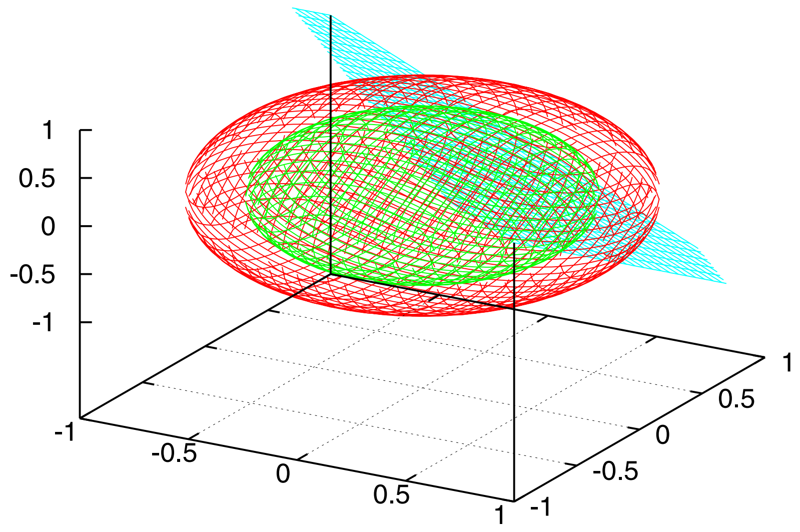


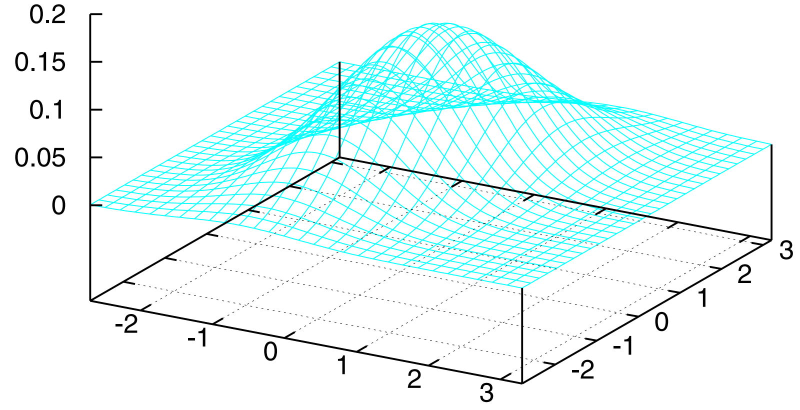


VOLATILITY PREDICTION WITH A CASCADE MODEL: A BAYESIAN APPROACH

- Consider probability density of volatility factors.
- Corresponds to considering phase space densities instead of individual trajectories.
- Volatility is marginal density in appropriate direction of this space.
- In physical terms:
  Dynamics of time series model induces dynamics of the phase space density.
- Log-normal assumption makes model numerically tractable:
  Instead of function space defined on R^N, only $N(N+3)/2$ dimensional vector space to describe the covariance matrix Λ and the mean vector μ of dimension N.

ISO-PROBABILITY SURFACES AND PROJECTED PDF





DYNAMICS OF THE PROBABILITIES

Transition matrices for:

- Inference
- Iteration of the time series process.

THE TRANSITION MATRIX FOR INFERENCE

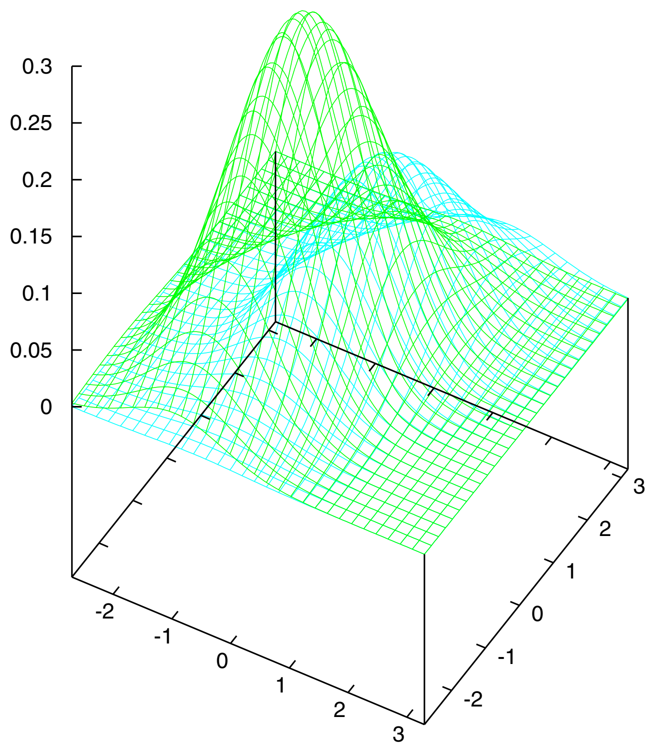
$$w^{(obs)}((\Lambda', \mu'), (\Lambda, \mu)) = w_{\Lambda}^{(obs)}(\Lambda', \Lambda)\, w_{\mu}^{(obs)}(\mu', \mu)$$

with

$$\begin{aligned} w_{\Lambda}^{(obs)}(\Lambda', \Lambda) &= \delta\left(\Lambda' - \Lambda + \frac{\Lambda|\eta\rangle\langle\eta|\Lambda}{\Lambda_{\eta\eta}}\right) \\ w_{\mu}^{(obs)}(\mu', \mu) &= \delta\left(\mu' - \mu + \langle\mu|\eta\rangle\eta\right) \end{aligned}$$

- $\eta = (1, \ldots, 1)/\sqrt{m}$ is the projection of logarithmic volatility on the different logarithmic factors.
- $\langle\mu|\eta\rangle$ and $|\eta\rangle\langle\eta|$ are "bra-ket" notations for scalar product and diadic product used in quantum mechanics.

EFFECT OF A NEW OBSERVATION ON THE VOLATILITY FACTOR PDF



A new volatility observation displaces the maximum of the volatility factor density and makes it narrower.

TRANSITION MATRIX FOR ITERATION

$$w^{(it)}((\Lambda,\mu)^2,(\Lambda,\mu)^1)) = \sum_{k=1}^{N} \pi^{(k)} w_{\mu,k}^{(it)}(\mu^2,\mu^1) \cdot w_{\Lambda,k}^{(it)}(\Lambda^2,\Lambda^1)$$

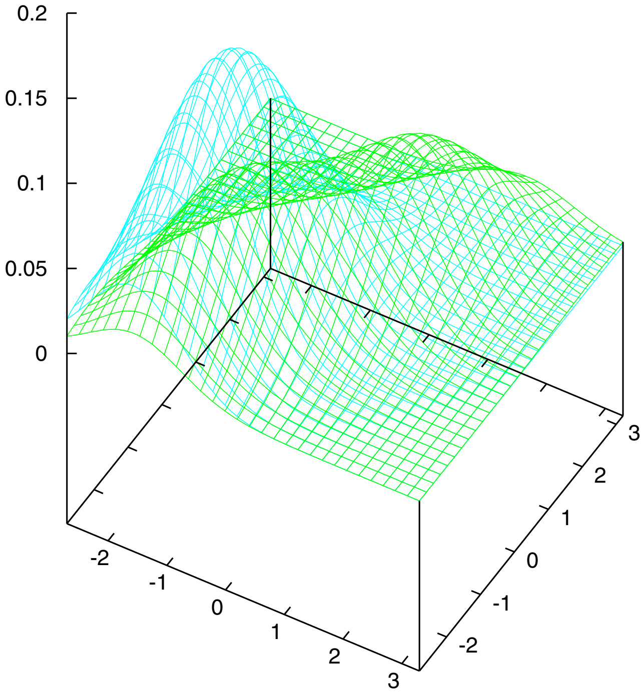
with

$$w_{\mu,k}^{(it)}(\mu^2,\mu^1) = \left\{ \prod_{i=1}^{k-1} \delta(\mu_i^2 - \mu_i^1) \prod_{i=k}^{N} \delta(\mu_i^2 - \mu_i^{(0)}) \right\}$$

and

$$w_{\Lambda,k}^{(it)}(\Lambda^2,\Lambda^1)) = \left\{ \prod_{i=1}^{k-1} \left(\prod_{j=1}^{k-1} \delta(\Lambda_{ij}^2 - \Lambda_{ij}^1) \right) \left(\prod_{j=k}^{N} \delta(\Lambda_{ij}^2 - \Lambda_{ii}^{(0)}\delta_{ij}) \right) \right\}$$
$$\times \left\{ \prod_{i=k}^{N} \left(\prod_{j=1}^{k-1} \delta(\Lambda_{ij}^2 - \Lambda_{ii}^{(0)}\delta_{ij}) \right) \left(\prod_{j=k}^{N} \delta(\Lambda_{ij}^2 - \Lambda_{ii}^{(0)}\delta_{ij}) \right) \right\}$$

EFFECT OF THE TIME SERIES DYNAMICS ON THE VOLATILITYy FACTOR PDF

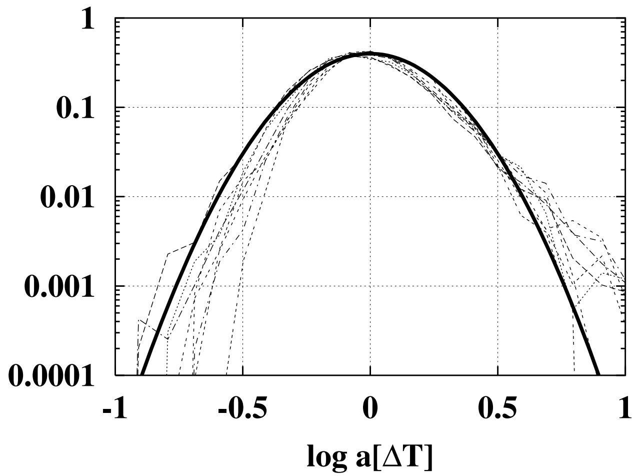


The time series dynamics causes relaxation of part of the volatility factor density.

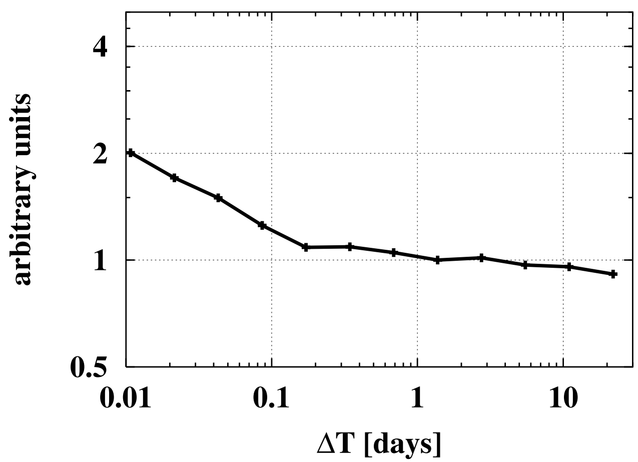
LOGARITHMIC VOLATILITY FACTORS

$$\log a_t[\Delta T] = \log \frac{\sigma_t[\Delta T]}{\sigma_t[0.5\Delta T]}$$

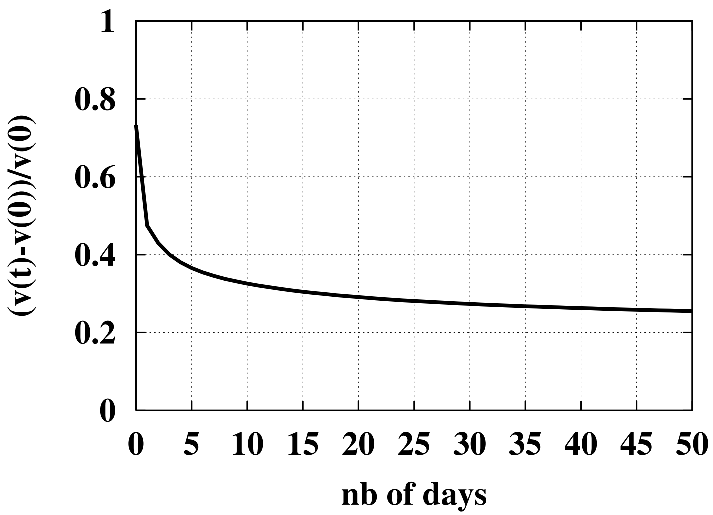
PROBABILITY DENSITIES



SCALING OF VARIANCE



IMPULS RESPONSE



MODEL TESTED WITH:

- Input data:
  Daily realized volatilites computed on the basis of high-frequency returns.
- 1 day prediction.

NUMERICAL RESULTS (preliminary)

Prediction error (RMSE):

$$\sqrt{\frac{1}{N}\sum_{t=1}^{N}\left(\sigma_t^{(pred)} - \sigma_t^{(obs)}\right)^2}$$

Model	RMSE (annualized)
previous day volatility	0.037
RiskMetrics	0.035
SCM	0.033

DISCUSSION

- The SCM model relates to the internal structure of the market.
- The prediction algorithm is based on phase-space densities instead of individual trajectories (Bayesian approach).
- The parameters are adjustment directly to observed quantities.
- Prediction is better than with standard methods.
- The model can be extended to work directly with high-frequency data.