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Risk Based Capital in P&C Loss Reserving or Stressing the Triangle

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Risk Based Capital in P&C Loss Reserving or Stressing the Triangle

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Abstract

The traditional chain-ladder method – the most popular run-off techniques in P&C insurance – produces point estimates of ultimate losses without any assumption on the statistic properties of claims. Recently stochastic models for claim payments have been developed which can be considered as the appropriate theoretical framework underlying the chain-ladder algorithm. Applying bootstrap methods to the triangle of past data stochastic chain-ladder models provide the complete predictive distribution of quantities relevant in claims reserving and solvency control, as future payments, ultimate losses and future estimates of reserves. This paper illustrates how risk based capitals for outstanding insurance liabilities can be derived from the tails of the distribution of future reserve assessments generated by the bootstrap procedure; i.e. stressing the triangle.

1. Introduction

In P&C insurance the payment by the insurer of the claim amount due for the occurrence of the insured event is a complex process, starting on the date of occurence of the claim and ending at a later date (the *closure date*), possibly very far in the future. The delay in payment may depend on a number of causes, including administrative and notification delays, investigation, dispute and litigation. Moreover payments (if any) can take place on multiple dates; it can also happen that a claim which is considered as closed at some date must be re-opened at a subsequent date.

In this claims process – which is a (highly) random process – the main concern of the insurer is to evaluate his liabilities generated by claims occurred in the past, in order to determine reserves and prudential margin provisions. Referring to a given line of business (LoB), at the valuation date the *outstanding loss liability* (OLL) by the insurer on the current policy portfolio is a random variable which must be properly modelled based on the LoB's claims experience. The traditional run-off analysis was concerned only with a point estimate of the OLL, providing the *ultimate loss reserve*. This estimate, usually referred to as "best estimate" or "central estimate", was tipically derived by *ad hoc* procedures, without specifying an underlying statistical model; the celebrated *chain-ladder* method is the most popular of these kinds of run-off techniques.

In the past few years more and more attention has been paid on the variability of OLL, both for consistently defining risk adjustments in fair valuation of policy portfolios and in order to determine prudential margin in claims reserving. Examples of an explicit attention paid to the appropriate modelling of OLL uncertainty are the White Paper on Fair Valuation of P&C Insurance Liabilities (CAS, 2000) and the Prudential Standard GPS210 on Liability Valuation for General Insurers (APRA, 2002). In fact the entire probability distribution of OLL must be specified to fulfill the new requirements and an appropriate stochastic model for the claims process is in order. Moreover, determination of capital requirements for one-year exposure, as prescribed in typical definitions of risk based capital, also demands to consider the future assessment of the residual reserve (that is the one-year-later central estimate of OLL) as a random variable itself, since the one-year unexpected loss will be determined both by the claim amount paid during the year and by the year-end level of the reserve.

2. Triangles of paid losses

Let us refer to a specified LoB and to the related portfolio of policies outstanding at the valuation date t. Tipically data on claims experience are organized by classifying payments by period of claim occurence (*accident period*, a.p.), which is the time period in which the event originating the payment took place. Assuming that a history of n periods is relevant for the valuation, we define time 0 as the starting point of the first a.p.. Hence the a.p. i is defined as the time interval:

$$\big[\,(i-1)\,\Delta t,\ i\,\Delta t\,\big)\,, \quad i = 1, \ldots, n\,,$$

where Δt is the length of each period. The current date is $t = n\,\Delta t$.

Since claims originated in each a.p. can be paid both in the period of origin and/or in any of the successive periods, payments in the same a.p. are also classified by their delay, defining the *development period* (d.p.) as the time period in which the payment is effectively made, assuming the a.p. as the first d.p..

We shall denote by:

$$C_{i,j}\,, \quad i = 1, \ldots, n\,, \quad j = 1, \ldots, J\,,$$

the portfolio payments relative to claims originated in the a.p. i, made in the d.p. j. The value J of the maximum delay is highly specific of the LoB; usually $J \geq n$.

For the sake of simplicity, we shall assume here $J = n$; hence the effects of claims originated in the first a.p. are completely observable at time t and no provision is needed on that date for this generation of liabilities. Moreover we shall consider annual periods, assuming $\Delta t = 1$; so we shall refer to *accident years* (a.y.) and *development years* (d.y.).

Since we have an equal number of a.y. and d.y., at time t the data on the paid losses $C_{i,j}$ can be arranged into an $n \times n$ matrix, each row i representing a generation of claims, each column j indicating a delay level. The elements on the diagonal, that is the elements such that $i+j = n+1$, are the payments made in the most recent calendar year $[t-1, t)$. The payments:

$$C_{i,j} \quad \text{such that} \quad i + j \leq n + 1\,,$$

are made before the current date t; they form the "past triangle", which is the set of the observed data. The elements of the "future triangle", that is the paid losses:

$$C_{i,j} \quad \text{such that} \quad i + j > n + 1\,,$$

are random variables at time t. Most of the traditional run-off techniques are methods for "completing the matrix", providing "best estimates" of future payments based on the past claims experience.

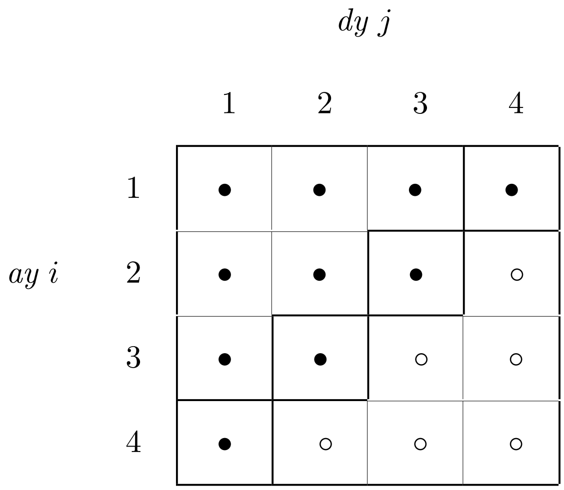

Figure 1. The time structure of the paid losses matrix for $n = 4$. The elements indicated by "•" form the past triangle; the elements indicated by "∘" compose the future triangle; the cells in boxes compose the latest diagonal.

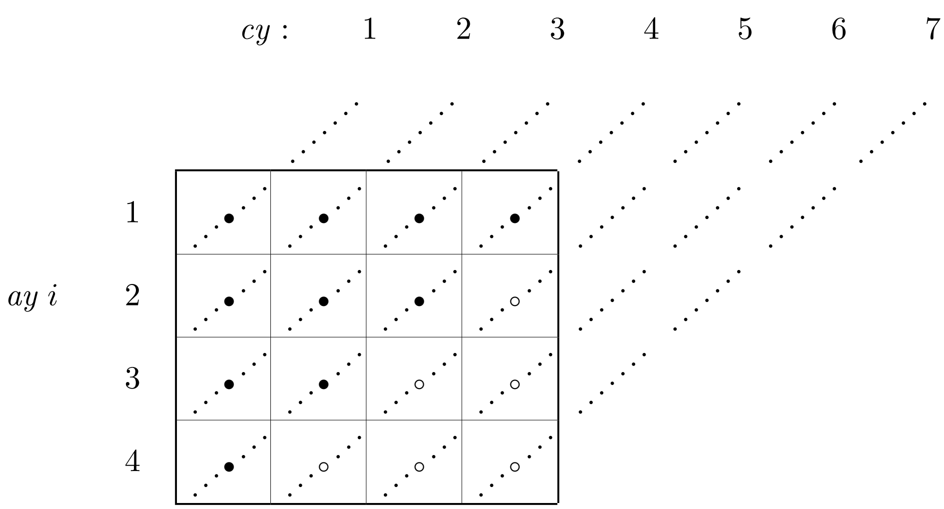

Figure 2. The time measured in calendar years. At time t the c.y. 4 is the year just ended, the c.y. 5 is the first of the future years. The run-off effects extend up to the end of c.y. 7.

Of course, P&C liabilities are essentially real in nature. We shall avoid this problem here, assuming that inflationary effects do not exist, thus interpreting past and future payments as purely nominal amounts. Inflation effects could be modelled by adjusting past data for the historic inflation (so expressing the payments in the current money value) and by incorporating the expected inflation into future payments. This is usually done by assuming a suitable deterministic model for both economic and LoB-specific inflation. It should be stressed, however, that in rescaling for the expected inflation, inflation uncertainty is not taken into account. An appropriate stochastic model should be used for correctly measuring inflation risk (see Moriconi, 1994) for a possible model for discounting under uncertain inflation).

The $C_{i,j}$ are usually referred to as incremental paid losses; the *cumulative* paid losses are defined as:

$$S_{i,j} = \sum_{k=1}^{j} C_{i,k}\,, \qquad i,j = 1,\ldots,n\,.$$

The cumulative paid losses on the diagonal represent the total payments made by the insurer up to the current date. It is useful to introduce a notation denoting, for each a.y., the d.y. most recently observed. For any $k = 1,\ldots,n$, the *diagonal index* defined by:

$$d_k = n - k + 1\,;$$

is the column index of the diagonal element on the row k (or, equivalently, the row index of the diagonal element on the column k). Thus the total payments made up to time t for claims of the a.y. i are given by S_{i,d_i}.

The *ultimate losses* are the cumulative paid losses on the latest d.y.; since we assumed $J = n$, we have:

$$U_i = S_{i,n}\,, \quad i = 1,\ldots,n\,, \qquad U = \sum_{i=1}^{n} U_i\,,$$

where U is the total ultimate loss. The OLLs at time t are the part of the ultimate loss which is not yet paid at time t, i.e.:

$$L_i = U_i - S_{i,d_i}\,, \quad i = 1,\ldots,n\,, \qquad L = \sum_{i=1}^{n} L_i\,.$$

Of course $U_1 = S_{1,n}$ and $L_1 = 0$.

Consistently with the prevailing definition, we shall refer to the expectation of the OLL as the *ultimate loss reserve*, that is:

$$R_i = \mathbf{E}_t[L_i]\,, \quad i = 1,\ldots,n\,, \qquad R = \sum_{i=1}^{n} R_i\,,$$

where $\mathbf{E}_t$ represents the expectation operator conditional on the claims experience up to time t.

	1	2	3	4
1	21	17	8	2
2	18	19	6	·
3	27	18	·	·
4	28	·	·	·

$C_{i,j}$

	1	2	3	4
1	21	38	46	48
2	18	37	43	·
3	27	45	·	·
4	28	·	·	·

$S_{i,j}$

Figure 3. Triangles of incremental and cumulative paid losses. The ultimate loss of the first a.y. is $U_1 = 48$.

3. Fair valuation of the liabilities and embedded value

A complete description of the insurer's liabilities at time t is given by the r.v.s $C_{i,j}$ in the future triangle, which in turn are described by their joint probability distribution. We are interested in deriving a valuation of the liabilities which is as much as possible consistent with a mark-to-market valuation – what is usually said a *fair valuation* [(1)]

We can interpret the future incremental paid losses as the payoffs of stochastic zero-coupon bonds having different maturities. Assuming that the payments are made at the end of each d.y., the amount $C_{i,j}$ will fall due after $\tau = j - d_i$ years from the date t. Hence the sums:

$$Y_\tau = \sum_{i=\tau+1}^{n} C_{i,d_i+\tau}\,, \qquad \tau = 1, \ldots, n-1\,, \tag{Yt}$$

will represent the total claims to be paid at time $t + \tau$. Since in our assumptions inflationary effects are ruled out, the stochastic payoffs Y_τ can be interpreted as nominal amounts.

Denoting by $V(t; \mathbf{X})$ the fair value at time t of a stream $\mathbf{X}$ of future cash-flows, we are interested in defining a suitable representation for the fair value:

$$V = V\,(t;\, \mathbf{Y})$$

of the stream $\mathbf{Y}$ of the random amounts Y_τ. We shall assume here the additivity property:

$$V\,(t;\, \mathbf{Y}) = \sum_{\tau=1}^{n-1} V\,(t;\, Y_\tau)\ .$$

Moreover, denoting by:

$$\mathbf{Q}_t[Y_\tau] \tag{Q}$$

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(1) The International Accounting Standards Committee (IASC) in the Insurance Issues Paper (November 1999) defines the fair value as: "*The amount for which an asset could be exchanged, or a liability settled, between knowledgeable, willing parties in an arm's-length transaction.*" A similar definition is adopted by the Financial Accounting Standard Board (FASB) in a Preliminary Views document (December, 1999): "*Fair value is an estimate of the price an entity would have realized if it had sold an asset or paid if it had been relieved of a liability on the reporting date in an arm's-length exchange motivated by normal business considerations. That is, it is an estimate of an exit price determined by market interactions.*" By acknowledging these positions, the CAS Fair Value Task Force (CAS, 2000) defines fair value as: "*the market value, if a sufficiently active market exists, OR an estimated market value, otherwise.*"

the *certainty equivalent* (at time t) of the random amount Y_τ (to be paid at time $t+\tau$), we can adopt the present value representation:

$$V\,(t;\,Y_\tau) = (1 + r_{t,\tau})^{-\tau}\ \mathbf{Q}_t[Y_\tau]\,, \tag{PV1}$$

where $r_{t,\tau}$ is the interest rate prevailing on the market at time t for risk free zero-coupon bonds maturing at time $t+\tau$. The crucial point in this representation is the certainty equivalent definition (Q), which should properly take into account the riskiness of the random payoff Y_τ (2). In particular, risk aversion requires that the *risk premium*, as measured by the difference between the certainty equivalent and the expectation of Y_τ:

$$\mathbf{\Delta}_t[Y_\tau] = \mathbf{Q}_t[Y_\tau] - \mathbf{E}_t[Y_\tau]\,,$$

is non-negative. A fair value representation equivalent to (PV1) is obtained by discounting the expected payoff, that is:

$$V\,(t;\,Y_\tau) = \left(1 + r^*_{t,\tau}\right)^{-\tau}\ \mathbf{E}_t[Y_\tau]\,, \tag{PV2}$$

where $r^*_{t,\tau}$ is now a risk-adjusted interest rate appropriate for the risk of Y_τ. Because of risk aversion, the risk-adjusted rate is lower than the corresponding risk free rate (and possibly negative).

Traditionally the definition of certainty equivalent has been based on the expected utility theory. Most of the traditional premium calculation principles in insurance have been derived within this conceptual framework. In recent years an alternative definition of certainty equivalent has been proposed, based on a distortion of the distribution function of the payoff. Following Wang (1996):

$$\mathbf{Q}_t[Y_\tau] = \mathbf{E}^*_t[Y_\tau]\,, \tag{E$*$}$$

where $\mathbf{E}^*_t$ is the time t expectation under a transformed probability measure. For suitable transformations the definition is consistent with the Yaari's theory of choice under uncertainty (Yaari, 1987). Wang (1996, 2000, 2002) proposed a number of distortion functions providing desirable and reasonable properties for the value functional. In particular, if $F_\tau(y)$ is the cumulative distribution function of Y_τ, the so-called Wang transform is given by:

$$F^*_\tau(y) = \Phi\Big[\Phi^{-1}\big(F_\tau(y)\big) + \pi_\tau\Big]\,,$$

(2) In the Preliminary Views document mentioned above, the FASB states that before discounting the future cash flows are to be adjusted for "*the effects of ... risk, market imperfections, and similar factors if market-based information is available to estimate those adjustments.*"

where $\Phi(\cdot)$ is the standard normal cumulative distribution and π_τ is a "distorsion" parameter determining the risk premium for Y_τ. For negative values of π_τ the Wang transform provide positive values of $\Delta_t[Y_\tau]$. If the random amount Y_τ would be the payoff at time $t+\tau$ of a contract currently traded on a perfect market, then it could be exactly replicated and the arbitrage principle would imply the existence of a unique distorted measure F^*_τ. In this cases F^*_τ is the same as the *risk neutral* probability measure well known in contingent claims pricing, and the risk loading parameter π_τ in the Wang transform can be properly denoted as the *market price of risk*.

When the zero-coupon bond Y_τ cannot be reduced to a traded asset, a market-oriented definition of the certainty equivalent $\mathbf{Q}_t[Y_\tau]$ (or of the risk-adjusted discount rate $r^*_{t,\tau}$) involves a number of difficult issues and a widely adopted standard approach for measuring risk adjustments is currently not available. For example, in the White Paper by the Task Force on Fair Value Liabilities of CAS (CAS, 2000) a number of alternative methods are considered for estimating adjustments for risk in fair valuation of actuarial liabilities. A pragmatic approach to the valuation problem could be to define the fair value as:

$$V(t; Y_\tau) = (1 + r_{t,\tau})^{-\tau}\ \mathbf{E}_t[Y_\tau]\,, \tag{FV2}$$

and then to take into account separately the risk effects by computing the risk loading as the cost of the economic capital required to meet some specified solvency standards. For example, if the required capital is equal to the "unexpected loss liability" $K_\tau(p)$ at a given confidence level p, i.e.:

$$K_\tau(p) = F_\tau^{-1}(p) - \mathbf{E}_t[Y_\tau]\,, \tag{Kt}$$

then a proxy for the risk loading could be chosen as:

$$\Delta_t[Y_\tau] \approx [(1 + k_{t,\tau})^\tau - (1 + r_{t,\tau})^\tau]\ K_\tau(p)\,, \tag{DK}$$

where $k_{t,\tau}$ is the "cost of capital", i.e. the required return from the investment of the capital $K_\tau(p)$ over the orizon $[t, t+\tau]$. Of course it is assumed that $k_{t,\tau} \geq r_{t,\tau}$. A similar interpretation could be proposed with respect to measures of risk based capital defined over a one-year horizon.

Risk measures as defined by (DK) result from a kind of blending of pricing models and capital adequacy models. It is found that similar definitions usually have not all the consistency properties suitable for a risk measure (see Artzner *et al.*, 1999, for a characterization of coherent risk measures).

The specification of an appropriate and coherent risk loading principle is beyond the scope of this exposition; thus we shall assume here the definition (PV1) of the fair value without specifying how the certainty equivalent $\mathbf{Q}_t[Y_\tau]$ is derived. The total value of the otstanding portfolio liabilities can be expressed as the present value:

$$V = \sum_{\tau=1}^{n-1} V(t; Y_\tau) = \sum_{\tau=1}^{n-1} (1 + r_{t,\tau})^{-\tau} \; \mathbf{Q}_t[Y_\tau] \, . \tag{FV3}$$

In particular, if the discounted expectation definition (FV2) is adopted, the only market information needed for the fair valuation of the loss liabilities is the current term structure of interest rates. Following the definitons, the fair value V is the market-determined exit price from the outstanding LoB liabilities. The market price definition has to be interpreted *stricto sensu*: V is the time t price of a default free contract which covers the random liability stream $\mathbf{Y}$ generated by the outstanding policy portfolio. Typically the effective provisions W established by the insurer at time t to meet the outstanding liabilities does not match the fair value V, which is – hopefully – lower than W. The difference:

$$E = W - V \, , \tag{EV}$$

then represents the time t value of the future profits generated by the investment of W and can be qualified as the *Value of Business In Force* (VBIF). This definition is consistent with the definition of VBIF widely used in life insurance; as in life insurance, VBIF represents the liability-side component of the *embedded value* of the insurance business.

4. Reserve process and risk capital

4.1 The year-end insurer's obligations

Since we assumed that all claim payments are made at the end of each year, at the date $t+1$ and immediately before (say at the end of the next year) the insurer will have to meet the following obligations:

a) pay the claim losses:

$$C_{i,d_i+1}\,,\quad i=2,\ldots,n\,,$$

generated by the outstanding portfolio in the calendar year $n+1$;

b) reserve the amounts:

$$R'_i\,,\quad i=2,\ldots,n\,,$$

where $R'_2=0$;

c) pay the claim losses $C_{n+1,1}$ generated by the new business (i.e. in the a.y. $n+1$) and due in the same year;

d) reserve the amount R'_{n+1} for the other claims generated in the a.y. $n+1$.

If we are strictly interested in the run-off component of the LoB, only points (a) and (b) are relevant for valuation. So the total amounts needed at time $t+1$ to meet the obligations from the outstanding portfolio are given by the r.v.s:

$$Z_i = C_{i,d_i+1} + R'_i\,,\quad i=2,\ldots,n\,. \tag{Z1}$$

We defined the reserves at time t as:

$$R_i = \mathbf{E}_t\left[\sum_{j=d_i+1}^{n} C_{i,j}\right],\quad i=2,\ldots,n\,.$$

At time $t+1$ (after the claim losses of year $n+1$ have been paid) the reserves are given by:

$$R'_i = \mathbf{E}_{t+1}\left[\sum_{j=d_i+2}^{n} C_{i,j}\right],\quad i=2,\ldots,n\,. \tag{R1R}$$

By the properties of conditional expectations the reserves at time t can also be written as:

$$R_i = \mathbf{E}_t\left[C_{i,d_i+1} + \mathbf{E}_{t+1}\left[\sum_{j=d_i+2}^{n} C_{i,j}\right]\right],\quad i=2,\ldots,n\,,$$

that is:

$$R_i = \mathbf{E}_t\left[C_{i,d_i+1} + R'_i\right] = \mathbf{E}_t[Z_i]\,,\quad i=2,\ldots,n\,. \tag{RZ}$$

Therefore the current reserves are equal to the expectation of the next year obligations generated by the outstanding portfolio.

4.2 The reserve risk capital

The risk based capitals for the reserve process, the "reserve risk capitals", are the amounts K_i^R implicitely defined by:

$$\mathbf{P}\left[Z_i - \overline{Z}_i \leq K_i^R\right] = p\,, \quad i = 2, \ldots, n\,, \tag{KRi}$$

where we denoted by $\overline{Z}_i$ the expectations $\mathbf{E}_t[Z_i]$. The differences $Z_i - \overline{Z}_i = Z_i - R_i$ represent the "unexpected losses" relative to the a.y. i; the probability p (typically very high) is the assumed confidence level.

The total reserve risk capital K^R is defined by referring to the unexpected loss relative to all the a.y.s; that is:

$$\mathbf{P}\left[\sum_{i=2}^{n}(Z_i - \overline{Z}_i) \leq K^R\right] = p\,. \tag{KR}$$

In typical situations (the r.v.s Z_i are weekly correlated or independent), the difference:

$$\Delta K^R = \sum_{i=2}^{n} K_i^R - K^R\,,$$

will be positive, accounting for the diversification effect through a.y.s.

4.3 The premium risk capital

Strictly speaking, only the points (a) and (b) refer to liabilities relevant for the run-off analysis, being determined by decisions taken up to the current date t. However we can assume that also the premium rate π_{n+1} for the policies written in year $n+1$ will be chosen by the insurer at time t. This decision is associated with an additional risk component related to the points (c) and (d), which is usually included in the run-off risk analysis.

The amount needed at time $t+1$ to meet the obligations in (c) and (d) is now:

$$Z_{n+1} = C_{n+1,1} + R'_{n+1}\,, \tag{Z2}$$

where:

$$R'_{n+1} = \mathbf{E}_{t+1}\left[\sum_{j=2}^{n} C_{n+1,j}\right]. \tag{R1P}$$

Since we are referring to a now-starting a.y., the insurer's obligation at time $t+1$ can be written as:

$$Z_{n+1} = \mathbf{E}_{t+1}\left[\sum_{j=1}^{n} C_{n+1,j}\right] = \mathbf{E}_{t+1}\left[U_{n+1}\right]\,, \tag{Zn1}$$

that is it can be expressed as the expectation in $t+1$ of the ultimate loss generated by the new business.

Let P_{n+1} denote the premiums earned in year $n+1$ from the new policies. The "premium risk capital" K^P is defined by:

$$\mathbf{P}\left[Z_{n+1} - P_{n+1} \leq K^P\right] = p\,. \tag{KP}$$

The r.v. $Z_{n+1} - P_{n+1}$ represents the potential loss incurred by the insurer if the ultimate loss estimated at the end of year $n+1$ is greater than the earned premiums. In order that this difference is non-positive one must have:

$$P_{n+1} \geq C_{n+1,1} + R'_{n+1}\,,$$

that is premiums collected must be sufficient to meet both the loss experience in the first d.y. and the expected future liabilities. Thus it is usually claimed that premium risk is both affected by pricing margin risk and by reserve risk for new business.

5. The chain-ladder method

5.1 Projected values

The chain-ladder method is the most popular run-off technique in P&C insurance. It produces estimates of future losses by projecting the data from the past triangle, without any assumption on their statistic properties.

Referring to the triangle of the cumulative paid losses, the *individual development factors* (or *individual link ratios*) λ_j are defined as:

$$\lambda_j = \frac{\sum_{i=1}^{d_j-1} S_{i,j+1}}{\sum_{i=1}^{d_j-1} S_{i,j}}\,, \quad j = 1, \ldots, n-1\,; \qquad \lambda_n = 1\,. \tag{L1}$$

The estimate of the future cumulative paid losses, for $i = 2, \ldots, n$, are given by:

$$\begin{cases} \widehat{S}_{i,d_i+1} &= S_{i,d_i}\,\lambda_{d_i}\,, \\ \widehat{S}_{i,j+1} &= \widehat{S}_{i,j}\,\lambda_j\,, \qquad j = d_i+1, d_i+2, \ldots, n-1\,. \end{cases} \tag{FR}$$

Thus the estimate at time t of future cumulative payments are obtained by a chain rule: λ_j represent the factor linking the estimated payments up to the d.y. $j+1$ with the estimated payments up to the d.y. j.

The *cumulative development factors* Λ_j are the development factors from the d.y. j on, that is:

$$\Lambda_j = \prod_{k=j}^{n} \lambda_k\,, \qquad j = 1, \ldots, n\,.$$

By this definition the cumulative paid losses at the latest d.y. can be expressed starting from the current cumulative payments:

$$\widehat{S}_{i,n} = S_{i,d_i}\,\Lambda_{d_i}\,, \qquad i = 2, \ldots, n\,.$$

Hence the estimated ultimate losses are given by:

$$\widehat{U}_i = \widehat{S}_{i,n}\,, \quad i = 2, \ldots, n\,, \qquad \widehat{U} = S_{1,n} + \sum_{i=2}^{n} \widehat{U}_i\,,$$

and the estimated OLLs are:

$$\widehat{L}_i = \widehat{U}_i - S_{i,d_i}\,, \quad i = 2, \ldots, n\,, \qquad \widehat{L} = \sum_{i=2}^{n} \widehat{L}_i\,.$$

Traditionally, the ultimate loss reserves are chosen to be equal to the estimated OLLs. Obviously this is consistent with our definition of the

21	38		
18	37		
27	45		
28 $\rightarrow$	50.9		

$$\lambda_1 = (38 + 37 + 45)/(21 + 18 + 27)1.818182 \approx 1.82$$

$$\widehat{S}_{4,2} = S_{4,1}\,\lambda_1 = 28 \times 1.82 = 50.9$$

	38	46	
	37	43	
	45 $\rightarrow$	53.4	
	50.9 $\rightarrow$	60.4	

$$\lambda_2 = (46 + 43)/(38 + 37) = 1.1868687 \approx 1.19$$

$$\widehat{S}_{3,3} = S_{3,2}\,\lambda_2 = 45 \times 1.19 = 53.4$$

$$\widehat{S}_{4,3} = S_{4,2}\,\lambda_2 = 50.9 \times 1.19 = 60.4$$

		46	48
		43 $\rightarrow$	44.9
		53.4 $\rightarrow$	55.7
		60.4 $\rightarrow$	63

$$\lambda_3 = 48/46 = 1.0434783 \approx 1.04$$

$$\widehat{S}_{2,4} = S_{2,3}\,\lambda_3 = 43 \times 1.04 = 44.9$$

$$\widehat{S}_{3,4} = S_{3,3}\,\lambda_3 = 53.4 \times 1.04 = 55.7$$

$$\widehat{S}_{4,4} = S_{4,3}\,\lambda_3 = 60.4 \times 1.04 = 63.0$$

$$\widehat{U}_2 = 44.9\,, \quad \widehat{U}_3 = 55.7\,, \quad \widehat{U}_4 = 63\,,$$

$$\widehat{U} = 48 + 44.9 + 55.7 + 63 = 211.6\,.$$

Table 1. The chain-ladder method

21	38	46	48
18	37	43	44.9
27	45	53.4	55.7
28	50.9	60.4	63
j : 1	2	3	4
λ_j : 1.82	1.19	1.04	1

$$\Lambda_1 = \lambda_1\,\lambda_2\,\lambda_3\,\lambda_4 = 1.82 \times 1.19 \times 1.04 \times 1 = 2.25$$

$$\Lambda_2 = \lambda_2\,\lambda_3\,\lambda_4 = 1.19 \times 1.04 \times 1 = 1.24$$

$$\Lambda_3 = \lambda_3\,\lambda_4 = 1.04 \times 1 = 1.04$$

$$\Lambda_4 = \lambda_4 = 1$$

Figure 4. The development factors

reserves only if the estimated payments given by the projection method are equal to their expectations, which makes no sense if a probability distribution for the future payments has not been specified. Nonetheless, it seems that under the chain-ladder method the equalities:

$$\mathbf{E}_t[L_i] = \widehat{L}_i\,, \qquad i = 2, \ldots, n\,,$$

are implicitely assumed.

5.2 Fitted values

Applying the development factors λ with a backward recursion starting from the diagonal elements one can obtain the *fitted* values of the past cumulative paid losses, which, for $i = 1, \ldots, n$, are given by:

$$\begin{cases} \widehat{S}_{i,d_i-1} &= S_{i,d_i} \,/\, \lambda_{d_i-1}\,, \\ \widehat{S}_{i,j-1} &= \widehat{S}_{i,j} \,/\, \lambda_{j-1}\,, \qquad j = d_i - 2, d_i - 3, \ldots, 2\,. \end{cases} \tag{BR}$$

The meaning of the fitted values is apparent if one observes that the link ratio λ_j can be written as the weighted average:

$$\lambda_j = \sum_{i=1}^{d_j-1} \frac{S_{i,j+1}}{S_{i,j}}\, p_{i,j}\,, \qquad j = 1, \ldots, n-1\,.$$

where the weights $p_{i,j}$ are:

$$p_{i,j} = \frac{S_{i,j}}{\sum_{k=1}^{d_j-1} S_{k,j}}\,, \qquad i = 1, \ldots, d_j - 1\,, \quad j = 1, \ldots, n-1\,.$$

Hence, in the past triangle, for any pair of successive columns, j and $j+1$, the ratios between elements on the same row are equal "on the average" to the development factor λ_j (where the average is weighted by the elements on the column j). In the fitted triangle, the data are modified in such a way that the ratio exactly equals λ_j for each pair of the columns j and $j+1$. Obviously, applying the chain-ladder procedure to the fitted data one obtains the same projected values provided by the actual data, by construction.

Using the expression $S_{i,d_i} = \widehat{U}_i/\Lambda_{d_i}$, eq. (FR) and (BR) can be unified in the form:

$$\widehat{S}_{i,j} = \frac{\widehat{U}_i}{\Lambda_j}\,, \qquad i, j = 1, \ldots, n\,,$$

where $\widehat{S}_{i,j}$ is a fitted value or an estimate of a future value depending on whether the cell (i, j) belongs to the past or to the future triangle, respectively. The coefficient $1/\Lambda_j$ is often referred to as the *lag factor*; roughly speaking, the lag factor $1/\Lambda_j$ represents the fraction of the ultimate loss of a given a.y. which has been paid "on the average" at the end of d.y. j.

Of course the estimated incremental payments are obtained by differencing:

$$\begin{cases} \widehat{C}_{i,j} = \widehat{S}_{i,j} - \widehat{S}_{i,j-1}\,, & i = 1, \ldots, n \quad j = 2, \ldots, n\,, \\ \widehat{C}_{i,1} = \widehat{S}_{i,1}\,, & i = 1, \ldots, n\,. \end{cases}$$

The portion of ultimate loss paid on the average in d.y. j for any given a.y. i is immediately obtained by differencing the lag factors:

$$\frac{\widehat{C}_{i,j}}{\widehat{U}_i} = \frac{\widehat{S}_{i,j} - \widehat{S}_{i,j-1}}{\widehat{U}_i} = \frac{1}{\Lambda_j} - \frac{1}{\Lambda_{j-1}}\,, \qquad j = 2, \ldots, n\,,$$

and:

$$\frac{\widehat{C}_{i,1}}{\widehat{U}_i} = \frac{\widehat{S}_{i,1}}{\widehat{U}_i} = \frac{1}{\Lambda_1}\,.$$

	1	2	3	4
1	21.3	38.8	46	48
2	20	36.2	43	·
3	24.8	45	·	·
4	28	·	·	·

Figure 5. The triangle of fitted cumulative paid losses

	1	2	3	2
1	21.3	17.4	7.2	2
2	20	16.3	6.8	1.9
3	24.8	20.3	8.4	2.3
4	28	22.9	9.5	2.6

Figure 6. The triangle of fitted/projected incremental paid losses

6. A stochastic model for the chain-ladder

The chain-ladder method is nothing else that an appealing projection algorithm, based only on arguments of heuristic nature. In order that the projected values $\widehat{C}_{i,j}$ can be consistently interpreted as the expectations of the r.v. $C_{i,j}$ an appropriate stochastic model is needed.

6.1 The over-dispersed Poisson assumption

In 1998 Renshaw and Verral (1998) proposed to model the incremental paid losses using an over-dispersed Poisson (ODP) distribution. They assumed the $C_{i,j}$ as independent r.v.s, with mean and variance given by:

$$\mathbf{E}[C_{i,j}] = x_i\, y_j\,, \qquad \mathbf{V}[C_{i,j}] = \phi\, x_i\, y_j\,, \qquad i,j = 1,\ldots,n\,,$$

with x_i, y_j, $\phi > 0$, and:

$$\sum_{k=1}^{n} y_k = 1\,.$$

For estimation purposes the model can be reparameterised posing:

$$\begin{aligned} &\log(x_i\, y_j) = \eta_{i,j}\,, \\ &\eta_{i,j} = c + \alpha_i + \beta_j\,, \qquad \alpha_1 = \beta_1 = 0\,. \end{aligned}$$

Thus a generalized linear model is assumed where the response is modelled by a logarithmic link function and the variance is proportional to the mean through the scale parameter ϕ.

Since:

$$\mathbf{E}[U_i] = \sum_{j=1}^{n} \mathbf{E}[C_{i,j}] = \sum_{j=1}^{n} x_i\, y_j = x_i\,, \tag{xi}$$

one can see that the "row parameter" x_i represents the expected ultimate loss of the a.y. i. Moreover the "column parameter":

$$y_j \equiv \frac{\mathbf{E}[C_{i,j}]}{\mathbf{E}[U_i]}\,, \tag{yj}$$

represents the proportion of the expected ultimate loss to emerge in the d.y. j for any a.y..

6.2 Connection with the chain-ladder algorithm

In (Verral, 2000) it is shown that, under ODP assumption (and some suitable additional conditions), the maximum likelihood estimators $\widehat{\lambda}_j$ of the individual development factors can be obtained by the maximum likelihood estimates $\widehat{y}_j$ of the column parameters:

$$\widehat{\lambda}_j = \frac{\sum_{k=1}^{j+1} \widehat{y}_k}{\sum_{k=1}^{j} \widehat{y}_k}\,, \qquad j = 1, \ldots, n-1\,.$$

It results that this relations provide the same development factors given by (L1). Thus the chain-ladder technique can be viewed as a method for deriving maximum likelihood estimates which are consistent with the ODP model. So if one considers the chain-ladder as an acceptable projection method, then one can consistently assume the ODP model to describe the stochastic nature of the paid losses. This allows to interpret the chain-ladder projections as expectations and – more interestingly – to obtain informations on the relevant full probability distributions.

It should be pointed out that some controversies have been recently roused discussing which stochastic model exactly underlies the chain-ladder technique. Mack and Venter (2000) argue that the stochastic framework provided by the so-called Distribution-Free Chain-Ladder model (Mack, 1993) is the closest to the classical algorithm. Following Verral and England (2000) we take the less restrictive position of considering the ODP model only as one of the possible stochastic models consistent with the chain-ladder method. Alternative models could be proposed which provide different predictive distributions (see Hess and Schmidt, 2002, for a comparison of various models related to the chain-ladder method).

One can observe that in the matrix:

$$\widehat{C}_{i,j}\,, \qquad i,j = 1, \ldots, n\,,$$

of the incremental payments fitted/projected by the chain-ladder, the ratios:

$$b_{i,j} = \frac{\widehat{C}_{i,j}}{\sum_{k=1}^{n} \widehat{C}_{i,k}}\,, \qquad i,j = 1, \ldots, n\,, \tag{b1}$$

are constant across the a.y.s and equal the estimates of the parameters y_j; that is:

$$b_{i,j} \equiv \widehat{y}_j\,, \qquad j = 1, \ldots, n\,. \tag{b2}$$

The partial sums $\sum_{k=1}^{j} \widehat{y}_k$ provide an estimate of the lag factors $1/\Lambda_j$.

Moreover, the ratios:

$$a_{i,j} = \frac{\widehat{C}_{i,j}}{\sum_{k=1}^{n} \widehat{C}_{k,j}}\,, \qquad i,j = 1, \ldots, n\,,$$

are constant across the d.y.s, i.e.:

$$a_{i,j} \equiv a_{i,1}\,, \qquad i = 1, \ldots, n\,.$$

7. The bootstrap procedure

Since the ODP distribution has not an analytic expression, numerical methods must be used in the model fitting procedure. A suitable method seems to be simulating values of future payments by bootstrapping the observed data.

Essentially, bootstrap methods are based on a resampling-with-replacement procedure applied to the observed data to create a large number of "pseudo-data" , which can be assumed generated by the same underlying distribution. In order to obtain consistent results it is crucial that the data can be considered as observations of independent identically distributed r.v.s. In regression type problems – as the chain-ladder under the ODP assumption is – the data are usually assumed to be independent but are not identically distributed. For this reason bootstrapping is made on the residuals, rather than on the data themselves.

7.1 The sample triangle of residuals

Let us consider the development factors λ_j obtained by applying the chain-ladder algorithm to the past triangle of cumulative paid losses $S_{i,j}$. By the backward recursion given in (BR) one obtaines the past triangle of the fitted cumulative values:

$$\widehat{S}_{i,j}\,, \qquad i=1,\ldots,n,\;\; j=1,\ldots,d_i\,,$$

and the corresponding incremental values:

$$\widehat{C}_{i,j}\,, \qquad i=1,\ldots,n,\;\; j=1,\ldots,d_i\,.$$

Adopting a definition usual in the theory of generalized linear models, the triangle of the unscaled Pearson residuals is given by:

$$r_{i,j}=\frac{C_{i,j}-\widehat{C}_{i,j}}{\sqrt{\widehat{C}_{i,j}}}\,, \qquad i=1,\ldots,n,\;\; j=1,\ldots,d_i\,. \tag{r}$$

As usual the number f of degrees-of-freedom of the estimate is given by the difference:

$$f=m-l\,,$$

where m is the number of observations (the paid losses of the past triangle), given by $m=n\,(n+1)/2$, and l is the number of parameters used in fitting the model; since there are n values for the parameters x_i and n values for the parameters y_j, with the constraint $\sum_{k=1}^{n} y_k=1$, the number of parameters is $l=2n-1$.

	1	2	3	4
1	−0.07	−0.11	0.28	0
2	−0.43	0.67	−0.29	
3	0.45	−0.5		
4	0			

$r_{i,j}$

Figure 6a. The triangle of the Pearson residuals

	1	2	3	4
1	−0.13	−0.19	0.52	0
2	−0.79	1.22	−0.54	
3	0.83	−0.91		
4	0			

$r^*_{i,j}$

Figure 6b. The triangle of the adjusted residuals

An estimator of the scale parameter ϕ is provided by the Pearson scale parameter:

$$\widehat{\phi}_P = \frac{\sum_{i=1}^{n} \sum_{j=1}^{d_i} r_{i,j}^2}{f}\,. \tag{Fip}$$

Moreover the ratio m/f is used for adjusting the residuals, in order to take account of the number of degrees-of-freedom. The adjusted Pearson residuals are then given by:

$$r^*_{i,j} = r_{i,j}\sqrt{m/f}\,, \qquad i = 1,\ldots,n,\ j = 1,\ldots,d_i\,.$$

7.2 The iterative procedure

The iterative simulation procedure is based on the triangle of the adjusted residuals which is used as data sample. Let ν be the number of iterations. In iteration κ (where $\kappa = 1,\ldots,\nu$) a uniform resampling with replacement from the residuals $r^*_{i,j}$ is performed, creating a new past triangle of simulated residuals ${}_\kappa r^*_{i,j}$. Then a triangle of pseudo-incremental data is obtained by eq. (r):

$${}_\kappa C_{i,j} = {}_\kappa r^*_{i,j}\sqrt{\widehat{C}_{i,j}} + \widehat{C}_{i,j}\,, \quad i = 1,\ldots,n,\ j = 1,\ldots,d_i\,.$$

Applying the chain-ladder to this bootstrap data sample one obtaines the estimates of the future incremental payments:

$${}_\kappa\widehat{C}_{i,j}\,, \quad i = 1,\ldots,n,\ j = d_i + 1,\ldots,n\,.$$

The variability of this set of simulated losses is only determined by the variability of the residuals; thus the randomness of the ${}_\kappa\widehat{C}_{i,j}$ values only reflects the uncertainty affecting the estimation procedure. In addition to this *estimate variability* we have only to model the process uncertainty which is determined by the intrinsic variability of data. This *process error* component is added by simulating for each cell (i,j) in the future triangle an ODP incremental payment with mean ${}_\kappa\widehat{C}_{i,j}$; that is a payment value:

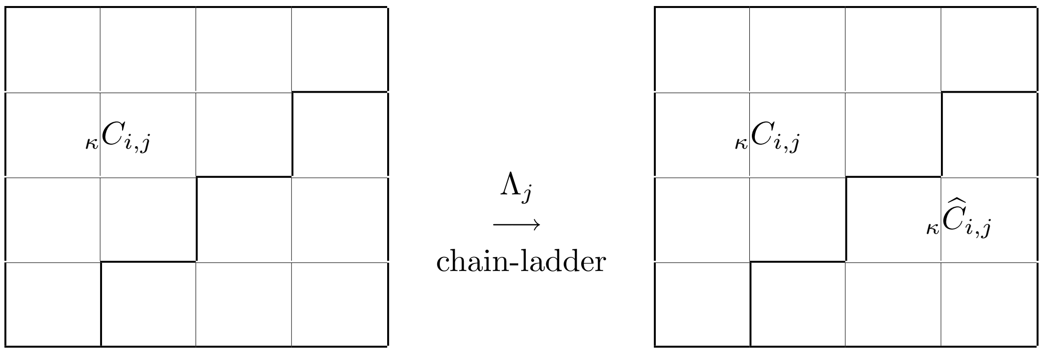
$${}_\kappa C_{i,j}\,, \qquad i = 1,\ldots,n,\ j = d_i + 1,\ldots,n\,, \tag{Ck}$$

is simulated, drawn from an ODP distribution with mean and variance:

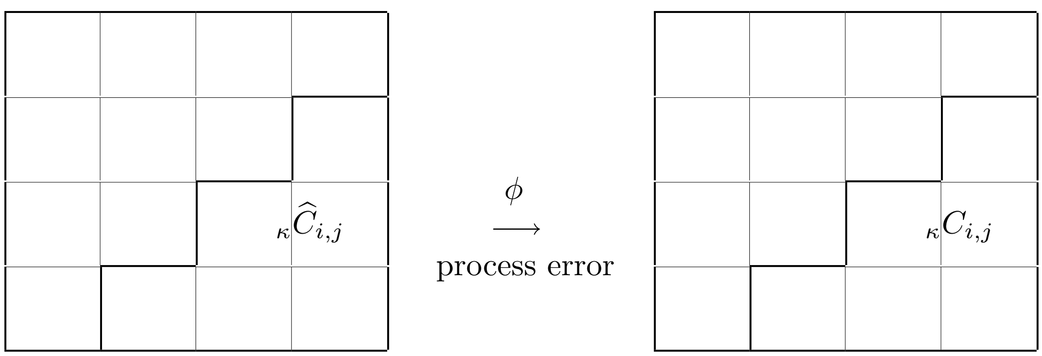
$$\mathbf{E}[C_{i,j}] = {}_\kappa\widehat{C}_{i,j}\,, \qquad \mathbf{V}[C_{i,j}] = \widehat{\phi}_P\ {}_\kappa\widehat{C}_{i,j}\,.$$

By repeating this simulation procedure one obtaines the predictive distribution of the future paid losses $C_{i,j}$ which can be used for deriving summary statistics, as moments and percentiles. For a given past triangle, the adequate number of iterations will depend on the particular application needed and on the desired degree of accuracy in the valuation.

Table 2



From the resampled data to the future values projected by the chain-ladder



The projected data perturbed by the process error

8. Distribution of future payments and loss reserving

Referring to the ODP model, let us consider a bootstrap procedure with ν iterations. The predictive distribution of the future paid losses $C_{i,j}$ is given by the collection of the simulated payments:

$$ {}_{\kappa}C_{i,j}\,, \quad i = 1, \ldots, n,\;\; j = d_i + 1, \ldots, n\,, $$

for $\kappa = 1, \ldots, \nu$. Thus the time t OLLs simulated in the κ–th iteration are:

$$ {}_{\kappa}L_i = \sum_{j=d_i+1}^{n} {}_{\kappa}C_{i,j}\,, \qquad i = 2, \ldots, n\,, $$

and the corresponding ultimate losses can be expressed as:

$$ {}_{\kappa}U_i = S_{i,d_i} + {}_{\kappa}L_i\,, \qquad i = 2, \ldots, n\,. $$

Obviously the simulated total OLL is:

$$ {}_{\kappa}L = \sum_{i=2}^{n} {}_{\kappa}L_i\,, $$

given that the OLL for the first a.y. is zero. The total ultimate loss is:

$$ {}_{\kappa}U = S_{1,n} + \sum_{i=2}^{n} {}_{\kappa}U_i\,. $$

Expectations

The expectations of future values are obtained immediately by computing the sample means on the predictive distribution. For the incremental paid losses one has:

$$ \overline{\overline{C}}_{i,j} = \frac{1}{\nu} \sum_{\kappa=1}^{\nu} {}_{\kappa}C_{i,j}\,, \qquad i = 2, \ldots, n\,, \quad j = d_i + 1, \ldots, n\,. $$

For values of ν sufficiently high one can assume:

$$ \overline{\overline{C}}_{i,j} = \mathbf{E}_t\left[C_{i,j}\right]\,, \qquad i = 2, \ldots, n\,, \quad j = d_i + 1, \ldots, n\,. $$

The same property holds for the sample means expressing the expected ultimate losses:

$$ \overline{\overline{U}}_i = \frac{1}{\nu} \sum_{\kappa=1}^{\nu} {}_{\kappa}U_i\,, \quad i = 2, \ldots, n\,, \qquad \overline{\overline{U}} = S_{1,n} + \sum_{i=2}^{n} \overline{\overline{U}}_i\,, $$

and for the expected OLLs:

$$\overline{\overline{L}}_i = \frac{1}{\nu} \sum_{\kappa=1}^{\nu} {}_{\kappa}L_i\,, \quad i = 2, \ldots, n\,, \qquad \overline{\overline{L}} = \sum_{i=2}^{n} \overline{\overline{L}}_i\,.$$

The expected OLLs give the ultimate loss reserves, by definition:

$$R_i = \overline{\overline{L}}_i\,, \quad i = 2, \ldots, n\,, \qquad R = \overline{\overline{L}}\,.$$

Given the consistency properties of the ODP model, when the number ν of iterations increases all these sample means converge to the corresponding estimates provided by the chain-ladder method.

Prudential reserve margins and fair value of the OLL

Variability measures of claims payments can be immediately derived from the predictive distributions in order to define adequate prudential margins in loss reserving. For example, the standard deviation of the total OLL can be derived as usual by computing the sample standard error:

$$\overline{\sigma}(L) = \sqrt{\frac{\sum_{\kappa=1}^{\nu} ({}_{\kappa}L - \overline{\overline{L}})^2}{\nu - 1}}\,.$$

A risk margin based on the standard deviation principle could then be derived by appropriately choosing a multiple of $\overline{\sigma}(L)$. It should be noted that, given that the paid losses are assumed to be independent, the standard error of the total OLL will result lower than the sum of the standard errors $\overline{\sigma}(L_i)$ on each a.y., the difference:

$$\sum_{i=2}^{n} \overline{\sigma}(L_i) - \overline{\sigma}(L)\,,$$

expressing the diversification effect between a.y.s.

Measures of reserve risk exposure alternative to second order moments can be derived by computing high or extreme percentiles on the OLL distribution. For a given sufficiency level p, the p-th percentile $\overline{L}^{(p)}$ of L can be easily calculated on the predictive distribution. The p% reserve margin is then obtained as $\overline{L}^{(p)} - \overline{\overline{L}}$. Also in this case the diversification effect between a.y.s can be measured by computing separately the risk margins for each generation of claims.

If the time t valuation of the OLLs has to be made on a discounted basis, the simulated future paid losses must be classified by maturity date. Now the predictive distribution of the future cash flow Y_τ due at time $t+\tau$, given by (Yt), is provided by the collection of simulated payments:

$$_\kappa Y_\tau = \sum_{i=\tau+1}^{n} {}_\kappa C_{i,d_i+\tau}\,, \qquad \tau = 1,\dots,n-1\,,$$

for $\kappa = 1,\dots,\nu$. The present value of the OLL can then be derived by computing an "appropriate" summary statistics of the distribution $\{_\kappa Y_\tau\}$, discounting at an "appropriate" interest rate over the horizon τ and then summing over all values of τ (3). If a specified summary statistics $\overline{Q}_\tau$ of $\{_\kappa Y_\tau\}$ is assumed as the certainty equivalent of the r.v. Y_τ, then the fair value of the outstanding insurer's liabilities will be given by:

$$V = \sum_{\tau=1}^{n-1} (1+r_{t,\tau})^{-\tau}\,\overline{Q}_\tau\,,$$

according to (FV3) .

Of course, in all of these computations, for a given predictive distribution the appropriate number ν of iterations depends on the particular summary statistics to be derived and on the degree of accuracy needed. The computation of extreme percentiles could require a very large number of simulations (a useful criterion for determining ν can be found in (Duffie and Pan, 1997)).

(3) For example, following the Prudential Standard GPS210 provided by APRA (APRA, 2000), the appropriate summary statistics should be given by:

$$\overline{Q}_\tau = \max\left\{\overline{Y}_\tau^{(0.75)}\,,\ \left(\overline{\overline{Y}}_\tau + \overline{\sigma}(Y_\tau)/2\right)\right\}\,,$$

where $\overline{Y}_\tau^{(0.75)}$, $\overline{\overline{Y}}_\tau$ and $\overline{\sigma}(Y_\tau)$ is the 75th percentiles, the mean and the standard deviation, respectively, of $\{_\kappa Y_\tau\}$, and the appropriate discount rate is the risk-free interest rate.

$\tau =$ 1 2 3

21	17	8	2
18	19	6	3.4
27	18	7.2	2.1
28	25.8	9.3	5.5

$$_\kappa Y_1 = 3.4 + 7.2 + 25.8 = 36.4$$

$$_\kappa Y_2 = 2.1 + 9.3 = 11.4$$

$$_\kappa Y_3 = 5.5$$

Figure 7. Simulated cash-flows with different maturities

9. Distribution of future reserves and risk capital

In order to calculate risk capitals, as defined by (KRi), (KR) and (KP), the probability distribution of the r.v. Z_i given by (Z1) and (Z2) is required; thus, in addition to the predictive distribution of future payments, one also needs the predictive distribution of the future reserves R'_i, that is of the future expectations given by (R1R) and (R1P).

In the ODP model, future expectations can be consistently derived by applying the chain-ladder algorithm to future data, that is to the past triangle of original data updated with the paid losses experienced in the new year. To implement this computation a further step has to be added to the bootstrap procedure. For illustration purposes it is convenient to describe separately reserve risk capital and premium risk capital calculations.

9.1 Reserve risk capital

Referring to the κ-th iteration of the bootstrap procedure, let us consider the simulated paid losses:

$$ {}_{\kappa}C_{i,d_i+1}\,, \quad i = 2, \ldots, n\,, $$

generated in the year $n+1$ by the business in force at time t. Using these sampled payments we can consider at time $t+1$ a new triangle of incremental data:

$$ {}_{\kappa}\Gamma_{i,j}\,, \quad i = 2, \ldots, n+1,\;\; j = 1, \ldots, d_i + 1\,, $$

where:

$$ \begin{cases} {}_{\kappa}\Gamma_{i,j} = C_{i,j}\,, & \text{for} \quad i = 2, \ldots, n,\;\; j = 1, \ldots, d_i\,; \\ {}_{\kappa}\Gamma_{i,d_i+1} = {}_{\kappa}C_{i,d_i+1}\,, & \text{for} \quad i = 2, \ldots, n\,; \\ {}_{\kappa}\Gamma_{n+1,1} = 0\,. \end{cases} \tag{G} $$

Thus the new triangle is obtained by deleting the first row of the original one and by adding below the old diagonal the vector of the next n simulated payments. Since we are only interested at the moment in the reserve risk of the outstanding business, we pose the paid losses of the new business equal to zero.

By the chain-ladder applied to the Γ triangle one obtains the simulated estimates at time $t+1$ of the cumulative development factors, denoted by:

$$ {}_{\kappa}\Lambda'_j\,, \quad j = 1 \ldots, n\,. $$

Applying these development factors to the diagonal of the cumulated payments of the Γ triangle one derives the simulated values of the ultimate loss estimates at time $t+1$:

$$ \begin{cases} {}_{\kappa}\widehat{U}'_2 = {}_{\kappa}S_{2,n} = S_{2,d_2} + {}_{\kappa}C_{2,d_2+1}\,, \\ {}_{\kappa}\widehat{U}'_i = {}_{\kappa}S_{i,d_i+1}\;{}_{\kappa}\Lambda'_{d_i+1} = \left(S_{i,d_i} + {}_{\kappa}C_{i,d_i+1}\right){}_{\kappa}\Lambda'_{d_i+1}\,, \quad i = 3, \ldots, n\,, \\ {}_{\kappa}\widehat{U}'_{n+1} = 0\,, \end{cases} $$

Table 3

	1	2	3	4
1				
2				46.4
3			52.2	
4		53.8		

	1	2	3	4
1	21	38	46	48
2	18	37	43	46.4
3	27	45	52.2	
4	28	53.8		
5	0			

1) The first diagonal of the simulated future cumulative payments and the simulated triangle Γ of the run-off data updated in $t{+}1$

	1	2	3	4
2	18	37	43	46.4
3	27	45	52.2	56.3
4	28	53.8	62.5	67.4
5	0	0	0	0

${}_{\kappa}\widehat{U}_3 = 56.3$

${}_{\kappa}\widehat{U}_4 = 67.4$

2) Simulation of the ultimate losses estimated in $t{+}1$

$$
\begin{aligned}
{}_{\kappa}R_3 &= {}_{\kappa}\widehat{U}_3 - {}_{\kappa}S_{3,3} = 56.3 - 52.2 = 4.1 \\
{}_{\kappa}R_4 &= {}_{\kappa}\widehat{U}_4 - {}_{\kappa}S_{4,2} = 67.4 - 53.8 = 13.6 \\
{}_{\kappa}Z_2 &= {}_{\kappa}C_{2,4} = (46.4 - 43) = 3.4 \\
{}_{\kappa}Z_3 &= {}_{\kappa}C_{3,3} + {}_{\kappa}R_3 = (52.2 - 45) + 4.1 = 11.3 \\
{}_{\kappa}Z_4 &= {}_{\kappa}C_{4,2} + {}_{\kappa}R_4 = (53.8 - 28) + 13.6 = 39.4
\end{aligned}
$$

3) Reserves and estimated overall obligations

and the corresponding estimates of the OLLs:

$$\begin{cases} {}_\kappa\widehat{L}'_2 = 0\,, \\ {}_\kappa\widehat{L}'_i = {}_\kappa\widehat{U}'_i - {}_\kappa S_{i,d_i+1} = \left(S_{i,d_i} + {}_\kappa C_{i,d_i+1}\right)\left({}_\kappa\Lambda'_{d_i+1} - 1\right), \quad i = 3,\ldots,n\,, \\ {}_\kappa\widehat{L}'_{n+1} = 0\,. \end{cases}$$

Under the ODP assumption:

$${}_\kappa\widehat{L}'_i = \mathbf{E}_{t+1}\left[\sum_{j=d_i+2}^{n} {}_\kappa C_{i,j}\right] = {}_\kappa R'_i\,, \qquad i = 3,\ldots,n\,.$$

Then, by (Z1), one obtains the simulated value of the insurer's obligations at time $t{+}1$:

$$\begin{aligned} {}_\kappa Z_i &= {}_\kappa C_{i,d_i+1} + {}_\kappa R'_i \\ &= {}_\kappa C_{i,d_i+1} + {}_\kappa\widehat{L}'_i\,, \qquad i = 3,\ldots,n\,. \end{aligned}$$

The collection of these ${}_\kappa Z_i$ values, for $\kappa = 1,\ldots,\nu$, provides the predictive distribution of the r.v.s Z_i, which can be directly used for computing the reserve risk capitals K_i^R given by (KRi).

9.2 Premium risk capital

The premium risk is determined by the uncertainty of the year-end obligation Z_{n+1} generated by the next year business, as compared with the corresponding earned premiums P_{n+1}. By equation (Zn1), in the ODP model we have:

$$Z_{n+1} = \widehat{U}'_{n+1}\,,$$

where $\widehat{U}'_{n+1}$ is the estimate at time $t+1$ of the ultimate loss projected by the chain-ladder; hence, according to (KP), the premium risk capital K^P is obtained as the p-th percentile of the r.v.:

$$\widehat{U}'_{n+1} - P_{n+1}\,.$$

For the ultimate loss estimates we have:

$$\widehat{U}'_{n+1} = C_{n+1,1}\,\Lambda'_1\,,$$

where $C_{n+1,1}$ is the claim amount from the next year business paid in the first d.y. and Λ'_1 is the first cumulative development factor estimated on the triangle of data observed at the year-end. Therefore, at time t the future insurer's obligation $\widehat{U}'_{n+1}$ is a product of two random variables. The development factors Λ'_j derived at time $t+1$ by the chain-ladder depend only on the paid losses up the a.y. n (that is the elements of the past triangle

Table 4

	1	2	3	4
2	18	37	43	46.4
3	27	45	52.2	56.3
4	28	53.8	62.5	67.4
5	0	0	0	0

$${}_\kappa\widehat{U}_3 = {}_\kappa S_{3,3}\, {}_\kappa\widehat{\Lambda}_3 = 52.2 \times 1.08$$

$${}_\kappa\widehat{U}_4 = {}_\kappa S_{4,2}\, {}_\kappa\widehat{\Lambda}_2 = 53.8 \times 1.25$$

$${}_\kappa\widehat{U}_5 = {}_\kappa S_{5,1}\, {}_\kappa\widehat{\Lambda}_1 = 0 \times 2.33$$

a) The cumulative development factors $\widehat{\Lambda}$ are derived by applying the chain-ladder to the triangle of the simulated run-off data

	1	2	3	4
2	·	·	·	·
3	·	·	·	·
4	·	·	·	·
5	$C_{5,1}$	·	·	$\widehat{U}_5$

$${}_\kappa\widehat{U}_5 = C_{5,1}\, {}_\kappa\widehat{\Lambda}_1 = C_{5,1} \times 2.33$$

b) The estimated ultimate loss $\widehat{U}_5$ is derived by the claim amount paid in $(5, 1)$

without the low-corner element $C_{n+1,1}$); since in the ODP model the paid losses $C_{i,j}$ are assumed to be independent, then $C_{n+1,1}$ and Λ'_1 are also independent. Therefore, in order to model $\widehat{U}'_{n+1}$ we can specify separately the probability distribution of the two factors.

By implementing the bootstrap procedure, the probability distribution of the development factor Λ'_1 is numerically specified by the bootsptrapped projection of past claims experience. As concerning the next year paid losses $C_{n+1,1}$, however, the ODP assumption is not sufficient to completely identify its probability distribution. In the ODP model, the mean and the variance at time t of $C_{n+1,1}$ are given by:

$$\mathbf{E}_t\left[C_{n+1,1}\right] = x_{n+1,1}\, y_1\,, \qquad \mathbf{V}_t\left[C_{n+1,1}\right] = \phi\, x_{n+1,1}\, y_1\,. \tag{EV}$$

By equations (b1), (b2) we have an estimate of the parameter y_1, expressed, for example, by the proportion of fitted incremental payments relative to the first a.y.:

$$\widehat{y}_1 = \frac{\widehat{C}_{1,1}}{\sum_{j=1}^{n} \widehat{C}_{1,j}}\,. \tag{y1}$$

We also have an estimate of the parameter ϕ, given by $\widehat{\phi}_P$ in eq. (Fip). However the parameter x_{n+1} is unspecified and have to be modelled under additional assumptions. In other words, while no prior assumptions on the x_i are made when applying the ODP model to the observed data, we need to assume some prior knowledge about the row parameter x_{n+1} related to the next year.

Since $x_{n+1} = \mathbf{E}[U_{n+1}]$ by eq. (xi), an assessment of the parameter x_{n+1} could be given by an estimate $\widehat{x}_{n+1}$ of the ultimate loss for the new year business made by the insurer at time t. So, assuming the value of the parameters y_1 and ϕ as given and equal to their estimated values $\widehat{y}_1$ and $\widehat{\phi}_P$, the estimates of the mean and the variance of the ODP r.v. $C_{n+1,1}$ are obtained as $\widehat{C}_{n+1,1} = \widehat{x}_{n+1}\,\widehat{y}_1$ and $\widehat{\phi}_P\,\widehat{C}_{n+1,1}$, respectively. The process error is then added by simulating ${}_\kappa C_{n+1,1}$ in each iteration of the bootstrap procedure as a r.v. drawn by the corresponding ODP distribution.

In principle, if we had to model the variability of $\widehat{U}'_{n+1}$ we should also model the variability of $\widehat{C}_{n+1,1}$, in order to include in the simulation the prior's uncertainty concerning x_{n+1}. However we are interested here on the difference $\widehat{U}'_{n+1} - P_{n+1}$ and we can avoid to add prior's uncertainty to the premium risk if we make the natural assumption that also P_{n+1} is determined by x_{n+1}. Let N_{n+1} be the number of units of risk written in the year $n+1$. We assumed that the premium rate π_{n+1}, that is the premium required by the insurer for bearing a unit of risk in the a.y. $n+1$, is fixed at the beginning of the year. Therefore the earned premiums are given by:

$$P_{n+1} = N_{n+1}\,\pi_{n+1}\,,$$

and the uncertainty on P_{n+1} is only given by the uncertainty on N_{n+1}. One can assume:

$$\pi_{n+1} = \frac{\mathbf{E}_t[U_{n+1}]}{\mathbf{E}_t[N_{n+1}]}\,(1+\ell)\,,$$

where ℓ is a risk loading coefficient fixed at time t; hence the earned premiums can be expressed as:

$$P_{n+1} = N_{n+1} \frac{\mathbf{E}_t[U_{n+1}]}{\mathbf{E}_t[N_{n+1}]} (1+\ell) .$$

If a time t estimate $\widehat{N}_{n+1}$ of the number of risks written is available, P_{n+1} can be specified as:

$$P_{n+1} = N_{n+1} \frac{\widehat{x}_{n+1}}{\widehat{N}_{n+1}} (1+\ell) ,$$

and can be easily modelled by assuming for the r.v. N_{n+1} a suitable distribution, based on past underwriting experience [4].

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[4] A simple model can be obtained by assuming that the r.v. $Q_{n+1} = N_{n+1}/\widehat{N}_{n+1}$ representing the prediction error factor on N_{n+1} is lognormal. The parameters of this distribution can be easily estimated if a reliable time series of error factors $Q_i = N_i/\widehat{N}_i$ is available for $i \leq n$.

10. Example

For illustration we use the data from Taylor and Ashe (1983), also considered by several Authors in subsequent papers. The data are arranged into a triangle with $n = 10$, which is shown in incremental form in Table 1A (tables are reported in the Appendix). The cumulative paid losses are reported in Table 2A. In Table 3A the individual and cumulative development factors obtained by the chain-ladder algorithm are given. The completed matrix of the cumulative payments is shown in Table 4A; the matrix of fitted/projected incremental payments is given in Table 5A.

In Table 6A the triangle of unscaled Pearson residuals $r_{i,j}$ is shown; as required by the "corner constraints" used in the reparameterisation of the ODP model, the residuals at either end of the latest diagonal are zero. Since we have $m = 55$ observations and $l = 19$ parameters, there are $f = 36$ degrees-of-freedom. The estimate of the scale parameter ϕ is $\widehat{\phi}_P = 52601.362$. The triangle of the adjusted residuals $r^*_{i,j}$ which is used as the data sample for the bootstrap procedure is given in Table 7A.

In the simulation procedure the realization from an ODP distribution with mean $\mu_{i,j} = {}_\kappa \widehat{C}_{i,j}$ and variance $\widehat{\phi}_P \, \mu_{i,j}$ required to generate the process error is obtained by sampling from a Poisson distribution with mean $\mu_{i,j}/\widehat{\phi}_P$ and then multiplying the result by $\widehat{\phi}_P$ [(5)]. Since the resampling-with-replacement can occasionally give negative values of ${}_\kappa\widehat{C}_{i,j}$, the effective mean of the ODP distribution is defined as $\mu_{i,j} = |{}_\kappa\widehat{C}_{i,j}|$ and the simulated ODP variable is then adjusted by subtracting the correction term $2 \max\{-{}_\kappa\widehat{C}_{i,j}, 0\}$.

In order to obtain reliable results in the estimation of extreme percentiles, a high number of simulations is required. Following considerations on risk capital computations (see below) we assumed $\nu = 110,000$. The results of the simulation for the OLLs are shown in Table 8A, where the sample mean, the standard deviation and the 75^{th} percentile of the predictive distributions for the L_i and for the overall loss L are reported. The OLLs provided by the deterministic chain-ladder and the differences from the sample means are also given in the table. An histogram of the distribution of L is shown in Figure 1A (together with a smoothed density line).

Summing the simulated incremental paid losses by calendar year, instead of by a.y., one obtains the predictive distribution of Y_τ. The sample mean, the standard deviation and the 75^{th} percentile of these distributions are reported in Table 9A.

The risk capitals are computed assuming a confidence level $p = 0.9993$. From historical analyses on corporate default rates provided by rating agencies, it results that a tipical figure for the one-year default probability of an

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(5) An alternative method based on sampling from a *Gamma* distribution with mean $\mu_{i,j}$ and variance $\widehat{\phi}_P \, \mu_{i,j}$ provides similar results.

"A" rated company is $0.0007 = 1 - p$. Thus the risk capital at the 99.93^{th} percentile can be interpreted as the adequate cushion for the insurer to maintain a credit rating "A". The number $\nu = 110,000$ of simulations has been chosen in connection with this value of p. It can be shown that, assuming that the pseudo-random numbers generator is unbiased, with 110,000 iterations there is a probability lower than 1% that a "true" 99.91^{th} percentile is overestimated as (not greater than) the 99.93^{th} percentile [(6)].

The results of the reserve risk capital computation are shown in Table 10A. As usual, the sample mean, the standard deviation and the 99.93^{th} percentile of the predictive distribution are reported for the obligations Z_i of each a.y. and for the overall obligations Z. The estimated risk capitals are given by the difference between the percentile and the mean and are also expressed as a percent of the ultimate loss reserves. Since the sum over i of the risk capital K_i^R is equal to 19,395,128 (the 103.8% of the total reserve), the difference $8,562,666 = 19,395,128 - 10,832,463$ expresses the diversification benefit across a.y.s (the diversified risk capital is the 55.85% of the undiversified one).

The premium risk capital K^P is derived assuming for the row parameter estimate $\widehat{x}_{n+1}$ the same value of the chain-ladder estimate $\widehat{U}_n$ of the ultimate loss for the last a.y. and choosing a risk loading $\ell = 10\%$. For semplicity, we assumed $N_{n+1} = \widehat{N}_{n+1}$, considering the uncertainty on N_{n+1} a second order effect. Under these assumptions the earned premiums are $P_{n+1} = 5,466,807$ and the premium risk capital, given by the 99.93^{th} percentile of the predictive distribution of $\widehat{U}'_{n+1} - P_{n+1}$, is $K^P = 7,517,060$, which corresponds to the 137.5% of the earned premiums.

(6) Since 99.91% is the typical one-year survival probability of a firm rated "A$^-$", one can say that with 110,000 simulations there is a probability less than 1% to attribute a solvency rating of "A" (corresponding to a survival probability of 99.93%), or lower, where the firm should be correctly rated as "A$^-$".

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Appendix

Table 1A - Incremental paid losses

a.y.	d.y. 1	d.y. 2	d.y. 3	d.y. 4	d.y. 5	d.y. 6	d.y. 7	d.y. 8	d.y. 9	d.y. 10
1	357848	766940	610542	482940	527326	574398	146342	139950	227229	67948
2	352118	884021	933894	1183289	445745	320996	527804	266172	425046	.
3	290507	1001799	926219	1016654	750816	146923	495992	280405	.	.
4	310608	1108250	776189	1562400	272482	352053	206286	.	.	.
5	443160	693190	991983	769488	504851	470639	.	.	.	.
6	396132	937085	847498	805037	705960	.	.	.	.	.
7	440832	847631	1131398	1063269	.	.	.	.	.	.
8	359480	1061648	1443370	.	.	.	.	.	.	.
9	376686	986608	.	.	.	.	.	.	.	.
10	344014	.	.	.	.	.	.	.	.	.

Table 2A - Cumulative paid losses

a.y.	d.y. 1	d.y. 2	d.y. 3	d.y. 4	d.y. 5	d.y. 6	d.y. 7	d.y. 8	d.y. 9	d.y. 10
1	357848	1124788	1735330	2218270	2745596	3319994	3466336	3606286	3833515	3901463
2	352118	1236139	2170033	3353322	3799067	4120063	4647867	4914039	5339085	.
3	290507	1292306	2218525	3235179	3985995	4132918	4628910	4909315	.	.
4	310608	1418858	2195047	3757447	4029929	4381982	4588268	.	.	.
5	443160	1136350	2128333	2897821	3402672	3873311	.	.	.	.
6	396132	1333217	2180715	2985752	3691712	.	.	.	.	.
7	440832	1288463	2419861	3483130	.	.	.	.	.	.
8	359480	1421128	2864498	.	.	.	.	.	.	.
9	376686	1363294	.	.	.	.	.	.	.	.
10	344014	.	.	.	.	.	.	.	.	.

Table 3A - Chain-ladder development factors

d.y.	individual factors	cumulative factors
1	3.49061	14.4466
2	1.74733	4.1387
3	1.45741	2.3686
4	1.17385	1.6252
5	1.10382	1.3845
6	1.08627	1.2543
7	1.05387	1.1547
8	1.07656	1.0956
9	1.01772	1.0177
10	1.00000	1.0000

Table 4A - Past and projected cumulative paid losses

a.y.	d.y. 1	d.y. 2	d.y. 3	d.y. 4	d.y. 5	d.y. 6	d.y. 7	d.y. 8	d.y. 9	d.y. 10
1	357848	1124788.00	1735330.00	2218270.00	2745596.00	3319994.00	3466336.00	3606286.00	3833515.00	3901463.00
2	352118	1236139.00	2170033.00	3353322.00	3799067.00	4120063.00	4647867.00	4914039.00	5339085.00	5433718.81
3	290507	1292306.00	2218525.00	3235179.00	3985995.00	4132918.00	4628910.00	4909315.00	5285148.49	5378826.29
4	310608	1418858.00	2195047.00	3757447.00	4029929.00	4381982.00	4588268.00	4835457.98	5205637.33	5297905.82
5	443160	1136350.00	2128333.00	2897821.00	3402672.00	3873311.00	4207459.08	4434133.22	4773589.08	4858199.64
6	396132	1333217.00	2180715.00	2985752.00	3691712.00	4074998.58	4426546.12	4665023.44	5022155.14	5111171.46
7	440832	1288463.00	2419861.00	3483130.00	4088678.10	4513179.11	4902528.20	5166648.75	5562182.46	5660770.62
8	359480	1421128.00	2864498.00	4174756.15	4900544.65	5409336.50	5875996.53	6192562.05	6666634.74	6784799.01
9	376686	1363294.00	2382128.11	3471744.08	4075312.72	4498426.08	4886502.44	5149759.61	5544000.38	5642266.26
10	344014	1200817.52	2098227.65	3057983.91	3589619.64	3962306.63	4304132.31	4536014.66	4883270.07	4969824.69

Table 5A - Fitted/projected incremental paid losses

a.y.	d.y. 1	d.y. 2	d.y. 3	d.y. 4	d.y. 5	d.y. 6	d.y. 7	d.y. 8	d.y. 9	d.y. 10
1	270061.42	672616.73	704494.15	753437.75	417350.16	292570.58	268343.51	182034.68	272606.02	67948.00
2	376125.01	936779.40	981176.32	1049341.97	581259.75	407474.40	373732.42	253526.75	379668.98	94633.81
3	372325.32	927315.87	971264.28	1038741.31	575387.75	403358.01	369956.90	250965.57	375833.49	93677.80
4	366723.96	913365.09	956652.33	1023114.21	566731.47	397289.78	364391.17	247189.98	370179.35	92268.49
5	336287.25	837559.23	877253.79	938199.60	519694.89	364316.23	334148.08	226674.15	339455.86	84610.55
6	353798.10	881171.87	922933.37	987052.68	546755.98	383286.58	351547.54	238477.32	357131.70	89016.32
7	391841.66	975923.40	1022175.47	1093189.47	605548.10	424501.00	389349.09	264120.55	395533.72	98588.16
8	469647.52	1169707.19	1225143.29	1310258.15	725788.49	508791.86	466660.02	316565.53	474072.69	118164.27
9	390560.78	972733.22	1018834.11	1089615.97	603568.64	423113.36	388076.36	263257.17	394240.77	98265.88
10	344014.00	856803.52	897410.13	959756.26	531635.73	372686.99	341825.67	231882.35	347255.41	86554.62

Table 6A - Pearson residuals

a.y.	d.y. 1	d.y. 2	d.y. 3	d.y. 4	d.y. 5	d.y. 6	d.y. 7	d.y. 8	d.y. 9	d.y. 10
1	168.926	115.010	-111.936	-311.631	170.234	521.036	-235.516	-98.6386	-86.9097	0
2	-39.145	-54.510	-47.734	130.760	-177.747	-135.474	252.024	25.1140	73.6433	.
3	-134.088	77.347	-45.707	-21.672	231.270	-403.768	207.213	58.7655	.	.
4	-92.665	203.918	-184.507	533.159	-390.865	-71.769	-261.916	.	.	.
5	184.294	-157.749	122.493	-174.180	-20.591	176.152	.	.	.	.
6	71.172	59.564	-78.522	-183.206	215.307	.	.	.	.	.
7	78.263	-129.865	108.031	-28.617	.	.	.	.	.	.
8	-160.756	-99.913	197.158	.	.	.	.	.	.	.
9	-22.201	14.068	.	.	.	.	.	.	.	.
10	0.000	.	.	.	.	.	.	.	.	.

Table 7A - Adjusted Pearson residuals

a.y.	d.y. 1	d.y. 2	d.y. 3	d.y. 4	d.y. 5	d.y. 6	d.y. 7	d.y. 8	d.y. 9	d.y. 10
1	208.798	142.156	-138.356	-385.186	210.415	644.018	-291.105	-121.921	-107.423	0
2	-48.384	-67.376	-59.000	161.624	-219.701	-167.451	311.510	31.042	91.026	.
3	-165.737	95.604	-56.495	-26.787	285.857	-499.071	256.122	72.636	.	.
4	-114.537	252.050	-228.056	659.002	-483.122	-88.709	-323.737	.	.	.
5	227.794	-194.983	151.405	-215.292	-25.451	217.729	.	.	.	.
6	87.971	73.623	-97.055	-226.448	266.126	.	.	.	.	.
7	96.735	-160.518	133.530	-35.371	.	.	.	.	.	.
8	-198.700	-123.496	243.694	.	.	.	.	.	.	.
9	-27.442	17.388	.	.	.	.	.	.	.	.
10	0.000	.	.	.	.	.	.	.	.	.

Table 8A - Summary statistics of simulated OLLs

a.y.	sample mean (a)	standard deviation	coefficient of variation (%)	75-th percentile	chain-ladder estimate (b)	difference (a-b)	% difference (a-b %)
2	95,799	113,751	118.739	157,804	94,634	1,165	1.23158
3	473,390	220,270	46.530	611,806	469,511	3,878	0.82607
4	715,282	263,929	36.899	894,223	709,638	5,644	0.79530
5	993,036	307,840	31.000	1,171,173	984,889	8,148	0.82727
6	1,427,756	379,230	26.561	1,683,244	1,419,459	8,297	0.58450
7	2,189,934	500,080	22.835	2,524,865	2,177,641	12,293	0.56451
8	3,941,573	796,550	20.209	4,442,865	3,920,301	21,272	0.54261
9	4,315,458	1,057,527	24.506	4,944,528	4,278,972	36,486	0.85269
10	4,721,919	2,037,379	43.147	5,891,352	4,625,811	96,108	2.07766
overall	18,874,147	3,014,992	15.974	20,724,936	18,680,856	193,292	1.03471

Figure 1A – Predictive aggregate distribution of total losses

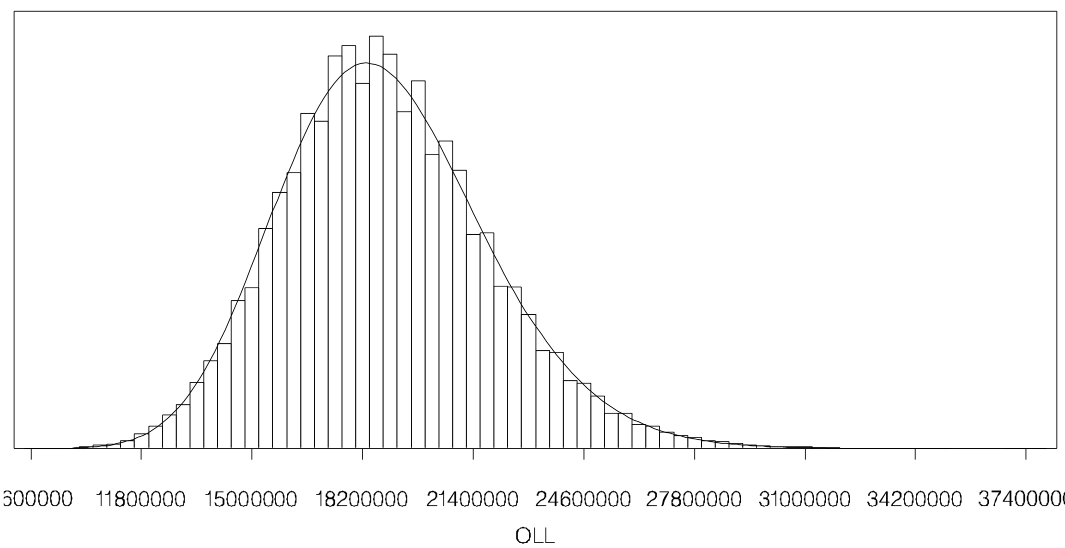

Table 9A - Summary statistics of future cash flows

maturity year	sample mean	standard deviation	75-th percentile
1	5,263,759	758,857	5,733,548
2	4,216,754	723,831	4,681,521
3	3,165,165	656,144	3,576,893
4	2,151,276	489,175	2,472,264
5	1,581,550	412,779	1,841,048
6	1,192,529	372,754	1,420,237
7	757,224	303,205	946,825
8	456,101	261,966	595,133
9	89,790	118,664	157,804

Tabl 10A - Summary statistics of next-year reserve obligations and Reserve Risk Capitals

a.y.	sample mean	standard deviation	99.93-th percentile	chain-ladder estimate (a)	Reserve Risk Capital (b)	% Reserve Risk Capital (b/a %)
2	95,799	113,751	683,818	94,634	588,018	621.362
3	473,753	219,475	1,406,745	469,511	932,992	198.715
4	738,242	209,325	1,588,045	709,638	849,803	119.752
5	1,027,302	229,412	1,963,330	984,889	936,028	95.039
6	1,504,246	261,467	2,530,014	1,419,459	1,025,769	72.265
7	2,193,826	344,128	3,530,334	2,177,641	1,336,508	61.374
8	3,903,243	603,494	6,225,292	3,920,301	2,322,049	59.231
9	4,349,468	810,060	7,597,180	4,278,972	3,247,711	75.899
10	4,805,473	1,820,252	12,961,723	4,625,811	8,156,250	176.320
overall	19,091,352	2,680,710	29,923,815	18,680,856	10,832,463	57.987