

**Best estimates, fair values and prudent reserves:
the example of reserving for guaranteed annuity options**

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Methodology and Principles

**Define carefully some insurance liability
(benefits and trial premiums)**

Choose a real world stochastic model

**Choose a suitable investment policy to
invest premiums to meet the liability**

**Use random simulations and calculate
the outcome (of liability and
investments combined)**

There will be a surplus or a deficit

Sort results into order

Choose some security level, e.g. 99%

Choose quantile reserve (QR)

i.e. 1% worst case

or Conditional Tail Expectation (CTE)

i.e. average of 1% worst cases

**This gives us a “prudent” reserve,
similar to Value-at-Risk**

**If a prudent reserve is set up , there will
often be a surplus (99% of cases), and
sometimes a deficit (1% of cases)**

**Surplus/deficit will fall back to
shareholders**

(or perhaps to with-profits policyholders)

**Discount surplus back to time 0 using
some investment policy for surplus**

**Choose two rates of interest, j and k ,
which could both be e.g. 2%**

**Discount surpluses at rate on investments
plus extra of j (giving a lower value)**

**Discount deficits at rate on investments
minus k (giving a higher value)**

**This gives (estimate of) value of
surplus/deficit to shareholders**

**Equivalent to a kinked linear utility
function for shareholders**

**Assume shareholders will pay this amount
to purchase the liability together with the
chosen investment policy**

**Policyholder has to pay the balance to
make up the required prudent reserve**

**So revise trial premium and check that
results still consistent**

Assets that match the liabilities closely will give lower prudent reserves
Therefore usually we get both lower premiums and lower capital requirement (sometimes one reduced, and not the other)
Lowest premiums should get the business, but insurance market is very “inefficient”

Option pricing and hedging:
Assume that hedging is a possible investment policy
Use some option pricing model (with a risk-neutral assumption) to calculate a possibly good hedging strategy
Simulate empirical hedging strategy
Calculate reserves and premiums as before

Option model does not need to be the same as the real-world simulation model
Much better if hedging quantities are calculated by formula, not by simulation
“Deflators” generally not helpful

Real world model should be much more realistic (therefore more complex) than option model,
which is only needed for calculating hedging quantities

Real world model can use annual model (e.g. “Wilkie model”)
with “Brownian bridges” (and “O-U bridges”) as stochastic interpolation methods
Hedging should be at a sensible frequency (e.g. twice a month)
Best to include transaction costs of hedging, but this is complicated

If the real world model is the same as the option pricing model,
and if the parameters are known and fixed
and if there are no transaction costs
and if there are no unhedgeable aspects (such as mortality uncertainty)
and if hedging is very, very frequent
Then empirical hedging outcome is very close to the required liability

“Prudent reserve” is e.g. 99% QR or 99% CTE

Minimum prudent reserve should be defined by the regulator

Management may prefer a stronger level

“Best estimate” is the average value of the outcome (either mean, i.e. 0% CTE, or median, i.e. 50% QR)

“Fair value” is the amount that needs to be paid by the policyholder (allowing for expenses)

“Fair value” is defined (loosely) as the amount paid by one party to another (in an arms-length deal) to transfer liability

Prudent reserve is too big, because then the buyer would expect to get a profit

Best estimate is too low, because then the buyer makes on average zero profit, and he values this as negative (risk-averse utility)

Prudent reserve minus shareholder value gives fair value, for transfer between one insurer and another

Premium is one “fair value”, between policyholder and insurer

Fair value usually different from premium because of commission and expenses

Guaranteed Annuity Options

Paper: “Reserving, pricing and hedging for policies with guaranteed annuity options”

A. D. Wilkie, H. R. Waters & S. Yang (WWY)

Discussed at:

Faculty of Actuaries, January 2003

Institute of Actuaries, October 2003

**British Actuarial Journal, 2003,
Vol 9, pages 263-425**

<http://www.actuaries.org.uk/files/pdf/sessional/sm20031027.pdf>

Guaranteed annuity options (GAOs) have been a problem for some British life offices especially Equitable Life

**Interesting in their own right
Several other papers:
Pellser, Boyle & Hardy,
Ballotta & Haberman**

**Various types,
mostly individual pension policies
mostly “with profits”:
guaranteed sum assured
guaranteed added bonus
terminal bonus at end
total converted at better of
market annuity rate and
guaranteed rate**

**WWY assume “unit-linked” policy,
invested in shares, with proceeds
converted to annuity at greater rate.
e.g. entry age x , “retirement” 65,
term $T = 65 - x$,
 $S(0)$ invested in shares
 $S(T)$ at maturity
Guaranteed rate g at age 65,
e.g. £111 annuity per £1,000**

**Value of an annuity at T is $a(T)$
based on age 65
current forecast mortality at T
current interest rates at T
Value of benefit at T is:
 $S(T) \times \text{Max}(1, g \times a(T))$
Value of benefit at 0 is:
 $S(0) \times \text{Max}(1, g \times a(T))$**

**Extra benefit is
 $S(0) \times \text{Max}(1, g \times a(T)) - S(0)$
 $= S(0) \times \text{Max}(0, g \times a(T) - 1)$
Benefit is “maxi option”
equivalent to shares
plus “quanto” (because of $S(T)$)
“call option”
($g \times a(T)$ at exercise price 1)
or “quanto swaption” of cash into
annuity**

**WWY imagine starting in 1985,
before option pricing
methodology well known,
also before quanto options
understood
Use method of British Maturity
Guarantees Working Party
(1980)**

Use a “real world” stochastic model
 (“Wilkie model”, published 1984)

Simulates:

- consumer prices
- share dividends
- share dividend yields
- long-term interest rates “consols”

Wilkie model (simplified)

Consumer prices, $Q(t)$

Inflation, $I(t) = \ln Q(t) - \ln Q(t-1)$

$$I(t) = QMU + QA \times \{I(t-1) - QMU\} + QSD \times QZ(t)$$

$QZ(t)$ distributed normally(0,1)

Dividend, $D(t)$

Smoothed inflation, $DM(t)$:

$$DM(t) = DD \times I(t) + (1-DD) \times DM(t-1)$$

$$K(t) = \ln D(t) - \ln D(t-1) = DW \times DM(t) + (1 - DW) \times I(t) + DMU + DSD \times DZ(t)$$

$DZ(t)$ distributed normally(0,1)

Dividend yield, $Y(t)$

$$\ln Y(t) = \ln YMU + YA \times \{\ln Y(t-1) - \ln YMU\} + YSD \times YZ(t)$$

$YZ(t)$ distributed normally(0,1)

Share price, $P(t)$

$$P(t) = D(t)/Y(t)$$

Total return, $S(t)$

$$S(t) = S(t-1) \times \{P(t) + D(t)\}/P(t-1)$$

Consols yield, $C(t)$

Another smoothed inflation, $CM(t)$:

$$CM(t) = CD \times I(t) + (1-CD) \times CM(t-1)$$

“Real yield”, $CR(t)$

$$\ln CR(t) = \ln CMU + CA \times \{\ln CR(t) - \ln CMU\} + CSD \times CZ(t)$$

$$C(t) = CM(t) + CR(t)$$

$CZ(t)$ distributed normally(0,1)

Parameters of Wilkie model as estimated in 1984

Starting conditions as at end 1984:

Consols yield = 9.90%

**“at the money” annuity rate
about £130 per £1,000**

**Mortality as forecast from 1967-70
experience for each age in each future
year**

**“PMA68Byyyy” for life born in year
yyyy**

Entry in 1985

Allowance for deaths before maturity

£100 single premium

Single entry ages 25 to 55

i.e. terms 40 down to 10

**also Portfolio: 31 policies each for
£100/31, one at each age**

10,000 simulations

Present value of cost as at Jan 1985:

Term	Mean	Q95%	Q97.5%	Q99%	Q99.9%
10	0.26	1.00	3.91	7.84	13.69
20	0.99	7.52	11.03	15.36	26.87
30	1.74	11.37	16.17	21.08	35.39
40	2.30	13.70	18.05	23.69	41.29
Portfolio	1.33	6.72	9.27	12.43	20.95
Sum	1.33	8.74	12.66	17.78	29.84

**Means (A) quite small, but quantiles Q
quite large**

**Portfolio better than sum of individual
terms**

CTEs (not shown) bigger than QRs

Choose desired security level,

say 99% or 99.9%

How much should policyholder pay?

(Obviously mean plus some risk loading.)

What is it contract worth to shareholders?

Estimate value to shareholders, say V.

Rest of QR is the premium, $D = QR - V$

Plus expenses, commission, etc

Value to shareholders:
 Discount surpluses at rate on investments plus extra of j (giving a lower value)
 Discount deficits at rate on investments minus k (giving a higher value)
 This gives (estimate of) value of surplus/deficit to shareholders, V
 Equivalent to a kinked linear utility function for shareholders
 Choose " j " and " k ", say 1% or 2% per year

Possible charges % as at Jan 1985:

Term	Mean	D 99% 1%, 1%	D 99% 2%, 2%	D 99.9% 2%, 2 %
10	0.26	0.98	1.63	2.67
20	0.99	3.60	5.73	9.45
30	1.74	6.76	10.48	16.82
40	2.30	9.38	14.13	23.64
Portfolio	1.33	2.39	3.34	4.85

A is "best estimate"

D is "fair value"

$Q_{cc}\%$ is prudent reserve, perhaps including solvency margin

$cc\%$ determined by supervisor, but management may choose a higher value

D is "fair value", i.e. market value of policies. Why?

Business is sold by office X to office Y, along with assets worth, e.g. D
 Y needs to set up prudent reserves, Q so Y requires to provide capital ($Q - D$) which he values at V
 Y requires "rent" for his capital so Y requires to receive $D = Q - V$ which is "best estimate", A plus "rent" for the extra capital

$$Q > D > A$$

WWY paper continues through the years to 2002 (2004 available now)

allowing for:

changes in Wilkie model
(1995 basis instead of 1984)

changes in market interest rates
improvements in mortality rates
(new tables)

Costs and charges as at Jan 2004:

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%, 2%
10	31.15	65.97	79.09	34.46	39.76
20	24.49	67.80	88.73	32.34	45.50
30	22.36	69.95	94.23	34.71	54.56
40	21.76	69.88	97.90	37.65	63.43
Portfolio	24.25	57.27	72.60	27.39	32.94

Option is now “in the money”, and projected mortality has improved

very much higher mean

e.g. term 10: 38.35 cf. 0.26

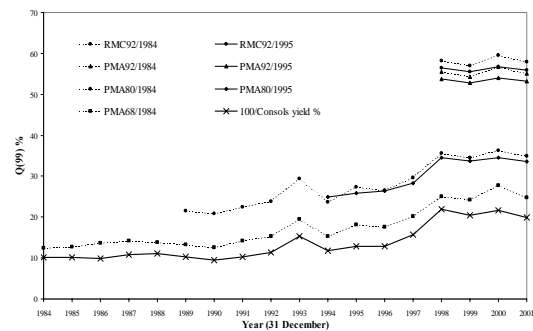
higher quantile reserves

e.g. term 40, Q99.9%: 105.83 cf. 41. 29

very much more expensive

e.g. term 10, D99,2%,2%:
39.76 cf. 2.67

Q99% for the portfolio: various bases



Option pricing:

Quanto “maxi” option

Models for:

shares, $S(t)$

zero-coupon bond yield, term to 65

future life annuity at 65

shares, $S(t)$:

$$dS(t) = \mu_S(t) \cdot S(t) \cdot dt + \sigma_S \cdot S(t) \cdot dW_S$$

zcb yield, $R(t)$

$$dR(t) = \mu_R(t) \cdot dt + \sigma_R \cdot dW_R$$

zcb price, $B(t)$

$$B(t) = \exp(-(T-t) \cdot R(t))$$

future life annuity, $F(t)$

$$dF(t) = \mu_F(t) \cdot F(t) \cdot dt + \sigma_F \cdot F(t) \cdot dW_F,$$

deferred life annuity, $D(t)$

$$D(t) = F(t) \times B(t)$$

W s are Brownian motions

Correlations between W_S, W_R, W_F :

$$\rho_{SR}, \rho_{SF}, \rho_{RF}$$

Value of option, $V(t)$

$$V(t) = S(t) \times \{G \times N(d_1) + N(d_2)\}$$

where

$$d_1 = \log(G)/\Sigma + \Sigma/2$$

$$d_2 = -\log(G)/\Sigma + \Sigma/2$$

$$G = g \times F(t) \times \exp(\text{Cov})$$

$$\text{Cov} = (T-t)^2 \times \rho_{RF} \times \sigma_R \times \sigma_F/2$$

$$+ (T-t) \times \rho_{SF} \times \sigma_S \times \sigma_F$$

$$\text{and } \Sigma = \sigma_F \sqrt{(T-t)}$$

Variables in the option formula:

“Volatilities”, standard deviations:

$$\sigma_S, \sigma_R, \sigma_F$$

Correlations:

$$\rho_{SF}, \rho_{RF} \quad \text{but not } \rho_{SR}$$

Current prices:

$$S(t), F(t) \quad \text{but not } B(t)$$

Not means:

$$\mu_S(t), \mu_R(t), \mu_F(t)$$

Option price is:

$$V(t) = S(t) \times \{G \times N(d_1) + N(d_2)\}$$

Hedging amounts are:

$$\text{in share: } V(t) \quad (\text{i.e. 100\%})$$

$$\text{in dla: } S(t) \times G \times N(d_1)$$

$$\text{in zcb: } -S(t) \times G \times N(d_1)$$

If hedging were perfect, and the model and all the parameters were correctly assessed, proceeds at maturity would exactly match benefit

(assuming mortality correctly forecasted)

But hedging is not perfect and we don't know the parameters

Assume we start with correct amount, $V(0)$, and invest in correct amounts.

Hedge every time step h

Proceeds at h – depend on investment at time 0, and how they have changed over $(0,h)$

Proceeds will (normally) not equal $V(h)$

Reinvest in correct proportions (other strategies possible)

Carry on to time T

Proceeds at T will (normally) not be correct

Examples:

Assume real world model same as option pricing model, with same parameters

No transaction costs

Hedging:

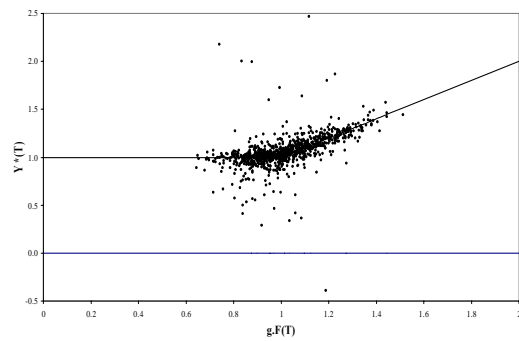
once a year

twice a month

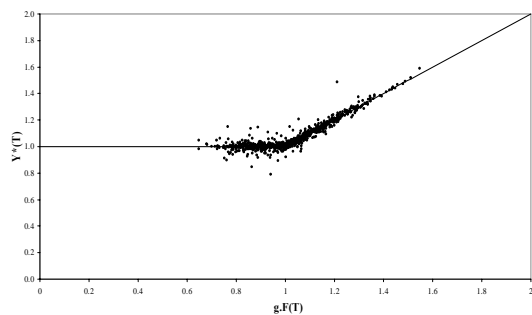
128 times a month

(about every 6 hours)

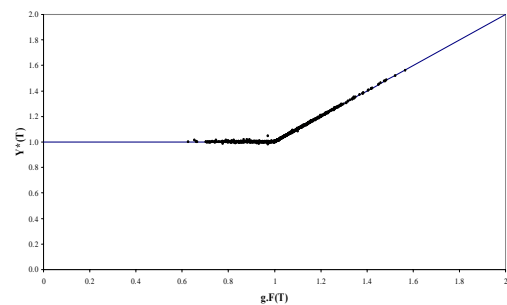
Hedging once a year



Hedging twice a month



Hedging 128 times a month



Use Wilkie model as annual model

Interpolate using Brownian bridges for share price (total return)

Using Ornstein-Uhlenbeck (AR1) bridges for consols yield, and "spread" = $\log(\text{short rate}/\text{long rate})$

Hedge twice per month

Start with basic £100 premium plus theoretical option premium

Hedge in correct *proportions* according to option formula (other choices possible)

Assume extra reserve invested in option proportions (other choices possible)

Calculate present value of deficit (to give QR)

Calculate utility value to shareholders
using QR%, j%, k%

Calculate extra premium to be paid by
policyholder to provide QR%

This also give fair value

**Costs and charges as at Jan 1985
(including option premium)**

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%, 2 %
10	0.42	1.30	1.84	0.50	0.67
20	1.07	2.18	2.60	1.27	1.57
30	1.10	2.56	3.02	1.48	1.96
40	0.53	2.43	3.00	1.16	1.88
Portfolio	0.91	1.60	1.80	1.01	1.14

**Costs and charges as at Jan 1985
(without hedging)**

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%, 2 %
10	0.26	7.84	13.69	0.98	2.67
20	0.99	15.36	26.87	3.60	9.45
30	1.74	21.08	35.39	6.76	16.82
40	2.30	23.69	41.29	9.38	23.64
Portfolio	1.33	12.43	20.95	2.39	4.85

Big differences:

some means higher, others lower

e.g. term 10: 0.42 cf. 0.26

term 40: 0.53 cf. 2.30

very much lower quantile reserves

e.g. term 40, Q99.9%: 3.00 cf. 41.29

very much lower premiums

e.g. term 40, D99,2%,2%:

1.88 cf. 23.64

**Costs and charges as at Jan 2004
(including option premium)**

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%, 2 %
10	43.37	45.57	47.60	43.58	44.13
20	42.64	48.50	51.29	43.70	45.47
30	38.24	45.50	48.03	40.12	42.62
40	30.47	40.48	41.93	33.77	38.17
Portfolio	39.44	44.10	45.34	41.01	42.72

**Costs and charges as at Jan 2004
(without hedging)**

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%, 2 %
10	31.15	65.97	79.09	34.46	39.76
20	24.49	67.80	88.73	32.34	45.50
30	22.36	69.95	94.23	34.71	54.56
40	21.76	69.88	97.90	37.65	63.43
Portfolio	24.25	57.27	72.60	27.39	32.94

Results:

means are higher

e.g. term 10: 43.37 cf. 31.15

quantiles down a lot

e.g. term 40, Q99.9%: 41.93 cf. 97.90

premiums up and down

e.g. D99,2%,2%:

term 10, 44.13 cf. 39.76

term 40, 38.17 cf. 63.43

Complications:

Share prices and interest rate changes are “fat-tailed”

We don’t know the parameters

We don’t know future mortality

Allow for fat-tailed steps in “bridges” by taking difference of two log-normals

$$Z_1 \sim N(\mu_1, \sigma_1^2) \quad Z_2 \sim N(\mu_2, \sigma_2^2)$$

$$Y = \ln(Z_1) - \ln(Z_2)$$

then standardise $X = (Y - E[Y])/SD[Y]$

X is “fat-tailed”

with mean zero, variance 1

chosen skewness (say 0, symmetric)

chosen kurtosis (say 4 to 6)

Costs etc as at Jan 1985 (incl OP)

with specimen fat-tailed bridges

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%, 2 %
10	0.42	1.36	2.08	0.51	0.72
20	1.07	2.37	3.00	1.30	1.70
30	1.09	2.74	3.69	1.52	2.26
40	0.54	2.64	3.34	1.23	2.07
Portfolio	0.91	1.66	1.96	1.02	1.18

Results:

means are almost the same

quantiles up a little

e.g. term 40, Q99.9%: 3.34 cf. 3.00

premiums up a little

e.g. term 10, D99,2%,2%:

0.72 cf. 0.67

Allow for parameter uncertainty in by “hypermodel”

Could be for annual (Wilkie) model parameters

or for bridging parameters

or both

For each simulation, each parameter chosen from a normal distribution

(1 ≥ autoregressive pars ≥ 0)

(variances ≥ 0)

Could be correlated

(multi-variate normal)

but taken as independent

Costs etc as at Jan 1985 (incl OP)
with specimen bridging hypermodel

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%, 2 %
10	0.43	1.75	2.47	0.55	0.79
20	1.08	2.98	3.82	1.42	1.97
30	1.10	3.34	4.35	1.68	2.55
40	0.53	3.06	3.95	1.37	2.40
Portfolio	0.92	2.24	2.87	1.10	1.42

Results:

means are almost the same

quantiles up a little more

e.g. term 40, Q99.9%: 3.95 cf. 3.00

premiums up a little more

e.g. term 10, D99,2%,2%:

0.79 cf. 0.67

Allow for mortality uncertainty by
stochastic mortality model

Set X(t) as annual random walk

$$X(t) = X(t - 1) + \sigma_X \cdot Z_X(t) - \frac{1}{2}\sigma_X^2$$

$$Y(t) = X(t) + \sigma_Y \cdot Z_Y(t) - \frac{1}{2}\sigma_Y^2$$

$$Z_X(t) \sim N(0,1) \quad Z_Y(t) \sim N(0,1)$$

$$q(x,t) = q_{\text{forecast}}(x,t) \times \exp(Y(t))$$

Same factor for all ages

Costs etc as at Jan 1985 (incl OP)
with specimen stochastic mortality

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%, 2 %
10	0.51	1.80	2.58	0.63	0.88
20	1.27	3.15	3.91	1.61	2.13
30	1.33	3.73	4.65	1.95	2.82
40	0.75	3.53	4.62	1.67	2.87
Portfolio	1.11	2.35	2.86	1.28	1.56

Results:

means are up a bit:

e.g. Portfolio, 1.11 cf. 0.91

quantiles are up a bit:

e.g. term 40, Q99.9%: 4.62 cf. 3.00

premiums are up a bit:

e.g. term 10, D99,2%,2%:

0.88 cf. 0.67

I have assumed that mortality changes continuously, and the current value of the factor is always known:

In practice results are known for a year at a time, and at best, several months in arrears.

Assuming annual changes, but no delay, makes it more expensive.

**Costs etc as at Jan 1985 (incl OP)
with all three complications**

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%,2 %
10	0.51	2.28	3.28	0.68	1.01
20	1.27	4.00	5.63	1.77	2.70
30	1.33	4.37	6.11	2.12	3.47
40	0.76	4.13	5.99	1.87	3.62
Portfolio	1.11	2.94	3.95	1.37	1.84

Results:

means are like stochastic mortality

quantiles are up a lot:

e.g. term 40, Q99.9%: 5.99 cf. 3.00

premiums are up a lot:

e.g. term 10, D99,2%,2%:

1.01 cf. 0.67

roughly additive

In 1984 guaranteed annuity rate was well “out of the money”, so all values are quite low.

By 2002 GAO was well “in the money”

Money value of option is large, hence means are large.

Extra premiums over the mean are also large for longer terms.

**Costs etc as at Jan 2004 (incl OP)
with all three complications**

Term	Mean	Q99%	Q99.9%	D 99% 1%, 1%	D 99.9% 2%,2 %
10	43.35	45.87	52.24	43.60	44.95
20	42.73	59.43	69.49	45.76	51.48
30	38.05	62.22	73.52	44.32	53.94
40	28.15	58.78	74.37	38.26	53.45
Portfolio	39.10	53.10	60.98	41.05	44.72

Results:

means are almost unchanged

quantiles are up a lot:

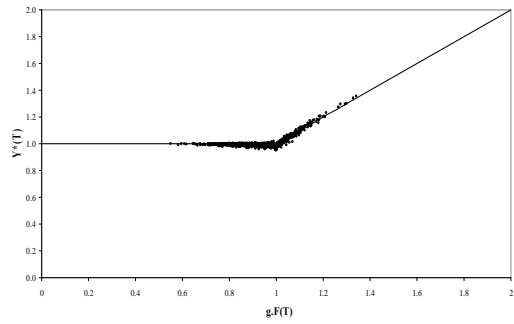
e.g. term 40, Q99.9%: 74.37 cf. 41.93

premiums are up a lot for longer terms:

e.g. term 40, D99,2%,2%:

53.45 cf. 38.17

Hedging Term 20 Jan 1984 with complications



**Model is complicated
needs many parameters**

Many can be estimated from data

**For practical life office use, one needs
to simplify the liability portfolio
perhaps by using “model points”**