Information Percolation

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1 Introduction

For a setting in which a large number of asymmetrically informed agents are randomly matched into groups over time, exchanging their information with each other when matched, we provide an explicit solution for the dynamics of the cross-sectional distribution of posterior beliefs. We also calculate the rate of convergence of the cross-sectional distribution of beliefs to a common posterior. We show that this convergence rate does not depend on the size of the groups of agents that meet. The convergence rate is merely the mean rate at which an individual agent is matched.

For example, suppose that each agent has λ meetings per year, in expectation. At each meeting, say an auction, $m-1$ other agents are randomly selected to attend. Each agent at the meeting reveals to the others a summary statistic of his or her posterior, such as a bid for an asset, reflecting the agent's originally endowed information and any information learned from prior meetings. Over time, the conditional beliefs held across the population of agents regarding a variable of common concern (such as the payoff of the auctioned asset) converge to a common posterior. We construct an associated mathematical model of information transmission and calculate explicitly the cross-sectional distribution of the posterior beliefs held by the agents at each time. We show that the rate of convergence of these posteriors to a common posterior is λ, regardless of the number m of agents at each meeting.

An important role of markets and organizations, as argued for example by Hayek (1945) and Arrow (1974), is to facilitate the transmission of information that is dispersedly held by its participants. Our results suggest that varying the size of the groups in which individuals exchange information does not facilitate information transmission, at least in terms of the rate of convergence of posteriors. This point is further addressed at the end of the paper.

Previous studies have considered the problem of information percolation in different contexts. For example, Wolinsky (1990) and Blouin and Serrano (2002) study information percolation in decentralized markets, and Banerjee and Fudenberg (2004) study information percolation in a social learning context. In contrast to these and other related studies, we allow for meetings that have more than two agents, and we explicitly characterize the percolation of information and provide rates of convergence of the cross-sectional distribution of beliefs to a common posterior.

Our results extend those of Duffie and Manso (2007), who provided an explicit formula for the Fourier transform of the cross-sectional distribution of posterior beliefs in the same setting, but did not offer an explicit solution for the distribution itself, and did not characterize the rate of convergence of the distribution.

Section 2 provides the model setting. Section 3 provides our results for the traditional search-market setting of bilateral ($m = 2$) contacts. This also serves as an introduction to the results for the case of general m, which are presented in Section 4.

2 The Basic Model

The model of information percolation is that of Duffie and Manso (2007). A probability space $(\Omega, \mathcal{F}, \mathrm{P})$ and a "continuum" (a non-atomic finite measure space $(G, \mathcal{G}, \gamma)$) of agents are fixed. Without loss of generality, the total quantity $\gamma(G)$ of agents is 1. A random variable X of potential concern to all agents has two possible outcomes, H ("high") and L ("low"), with respective probabilities ν and $1 - \nu$.

Each agent is initially endowed with a sequence of signals that may be informative about X. The signals $\{s_1, \ldots, s_n\}$ primitively observed by a particular agent are, conditional on X, independent with outcomes 0 and 1 (Bernoulli trials). The number $n \geq 0$ of signals as well as the probability distributions of the signals may vary across agents. Without loss of generality, we suppose that $\mathrm{P}(s_i = 1 \,|\, H) \geq \mathrm{P}(s_i = 1 \,|\, L)$. A signal i is *informative* if $\mathrm{P}(s_i = 1 \,|\, H) > \mathrm{P}(s_i = 1 \,|\, L)$. For any pair of agents, their sets of originally endowed signals are independent.

By Bayes' rule, conditional on signals $\{s_1, \ldots, s_n\}$, the posterior probability that X has a high outcome is

$$\mathrm{P}(X = H \mid s_1, \ldots, s_n) = \left[1 + \frac{1-\nu}{\nu}\left(\frac{1}{2}\right)^{\theta}\right]^{-1}, \tag{1}$$

where the "type" θ of this set of signals is

$$\theta = \sum_{i=1}^{n}\left(s_i \log_{1/2}\frac{\mathrm{P}(s_i = 1 \mid L)}{\mathrm{P}(s_i = 1 \mid H)} + (1 - s_i)\log_{1/2}\frac{1 - \mathrm{P}(s_i = 1 \mid L)}{1 - \mathrm{P}(s_i = 1 \mid H)}\right). \tag{2}$$

The higher the type θ of the set of signals, the higher is the posterior probability that X is high.

Any particular agent is matched to other agents at each of a sequence of Poisson arrival times with a mean arrival rate (intensity) λ, which is common across agents. At each meeting time, $m-1$ other agents are randomly selected from the population of agents.[1] The meeting group size m is a parameter of the information model that we shall vary. We assume that, for almost every pair of agents, the matching times and counterparties of one agent are independent of those of the other. We do not show the existence of such a random matching process.[2]

Suppose that whenever agents meet they communicate to each other their posterior probabilities, given all information to the point of that encounter, of the event that X is high. Duffie and Manso (2007) provide an example of a market setting in which this revelation of beliefs occurs through the observation of bids submitted by risk-neutral investors in an auction for a

[1]That is, each of the $m-1$ matched agents is chosen at random from the population, without replacement, with the uniform distribution, which we can take to be the agent-space measure γ. Duffie and Sun (2007) provide a complete construction for independent random matching from a large set (a non-atomic measure space) of agents, for the case $m = 2$.

[2]For the case of groups of size $m = 2$, Duffie and Sun (2007) show existence for the discrete-time analogue of this random matching model. For the case of a finite number of agent types, the associated exact law of large numbers for the cross-sectional distribution of the type processes is provided by Duffie and Sun (2005). Giroux (2005) proves convergence of the naturally associated finite-agent discrete-time model to the analogous continuous-time model matching model assumed by Duffie, Gârleanu, and Pedersen (2005), as the number of agents grows large and the length of a time period shrinks to zero.

forward contract on an asset whose payoff is X. From Proposition 3 in Duffie and Manso (2007), whenever an agent of type θ meets an agent with type ϕ and they communicate to each other their posterior distributions of X, they both attain the posterior type $\theta + \phi$. The same proof implies that whenever m agents of respective types $\phi_1, \ldots, \phi_m$ share their beliefs, they attain the common posterior type $\phi_1 + \cdots + \phi_m$.

We let μ_t denote the cross-sectional distribution of posterior types in the population at time t. That is, for any real interval (a, b), $\mu_t((a, b))$ (also denoted $\mu_t(a, b)$ for simplicity) is the fraction of the population whose type at time t is in (a, b). Because the total quantity $\gamma(G)$ of agents is 1, we can view μ_t as a probability distribution. The initial distribution μ_0 of types is that induced by some particular initial allocation of signals to agents. In the following analysis we assume that there is a positive mass of agents that has at least one informative signal. This implies that the first moment $m_1(\mu_0)$ is strictly positive if $X = H$, and that $m_1(\mu_0) < 0$ if $X = L$. We assume that the initial law μ_0 has a moment generating function, $z \mapsto \int e^{z\theta}\, \mu_0(d\theta)$, that is finite on a neighborhood of $z = 0$.

3 Two-Agent Meetings

We now calculate the explicit belief distribution in the population at any given time, and the rate of convergence of beliefs to a common posterior, in a setting with $m = 2$ agents at each meetings. This is the standard setting for search-based models of labor, money, and asset markets. In this setting, the cross-sectional distribution of types is determined by the evolution equation

$$\mu_t = \mu_0 + \lambda \int_0^t (\mu_s * \mu_s - \mu_s)\, ds, \tag{3}$$

where $*$ is the convolution operator. This is intuitively understood if μ_t has a density $f_t(\,\cdot\,)$, in which case the density $f_t(\theta)$ of agents of type θ is reduced at the rate $\lambda f_t(\theta)$ at which agents of type θ meet other agents and change type, and is increased at the aggregate rate $\lambda \int f_t(\theta - y) f_t(y)\, dy$ at which an agent of some type y meets an agent of type $\theta - y$, and therefore becomes an agent of type θ.

The following result provides an explicit solution for the cross-sectional type distribution, in the form of a Wild summation.[3]

[3]See Wild (1951).

Proposition 1 *The unique solution of (3) is*

$$\mu_t = \sum_{n\geq 1} e^{-\lambda t}(1-e^{-\lambda t})^{n-1}\mu_0^{*n}, \tag{4}$$

where ν^{*n} denotes the n-fold convolution of a measure ν.

Proof As in Duffie and Manso (2007), we write the evolution equation (3) in terms of the Fourier transform $\varphi(\,\cdot\,,t)$ of μ_t, as

$$\frac{\partial\varphi(s,t)}{\partial t} = -\lambda\varphi(s,t) + \lambda\varphi^2(s,t), \tag{5}$$

with solution

$$\varphi(s,t) = \frac{\varphi(s,0)}{e^{\lambda t}(1-\varphi(s,0))+\varphi(s,0)}. \tag{6}$$

This solution can be expanded as

$$\varphi(s,t) = \sum_{n\geq 1} e^{-\lambda t}(1-e^{-\lambda t})^{n-1}\varphi^n(s,t), \tag{7}$$

which is identical to the Fourier transform of the right-hand side of (4). ■

This solution for the cross-sectional distribution of types is converted to an explicit distribution for the cross-sectional distribution π_t of posterior probabilities that $X = H$, using the fact that

$$\pi_t(0,b) = \mu_t\left(-\infty,\, \log_{0.5}\frac{(1-b)\nu}{(1-\nu)b}\right). \tag{8}$$

Like μ_t, the beliefs distribution π_t has an outcome that differs depending on whether $X = H$ or $X = L$.

We now provide explicit rates of convergence of the cross-sectional distribution of beliefs to a common posterior. In our setting, it turns out that all agents' beliefs converge to that of complete information, in that any agent's posterior probability of the event $\{X = H\}$ converges to 1 on this event and to zero otherwise. In general, we say that π_t converges to a common posterior distribution π_∞ if, almost surely, π_t converges in distribution to π_∞, and we say that the rate of convergence is $\alpha > 0$ if there are constants κ_0 and κ_1 such that, for any b in $(0,1)$,

$$e^{-\alpha t}\kappa_0 \leq |\pi_t(0,b) - \pi_\infty(0,b)| \leq e^{-\alpha t}\kappa_1.$$

Thus, if there is a rate of convergence, it is unique.

Proposition 2 *The cross-sectional distribution of beliefs converges (to that of complete information) at rate λ.*

Proof Because of (8), the rate of convergence of π_t is the same as the rate of convergence to zero or 1, for any a, of $\mu_t(-\infty, a)$. We will provide the rate of convergence to zero of $\mu_t(-\infty, a)$ on the event $X = H$. A like argument gives the same rate of convergence to 1 on the event $X = L$.

From (4),

$$\mu_t(-\infty, a) \geq e^{-\lambda t}\mu_0(-\infty, a). \tag{9}$$

We fix n_0 such that $m_1(\mu_0) > a/n$ for $n > n_0$ and we let $\{Y_n\}_{n\geq 1}$ be independent random variables with distribution μ_0. Then,

$$\begin{aligned}\mu_t(-\infty, a) \quad = \quad & \sum_{n=1}^{n_0} e^{-\lambda t}(1-e^{-\lambda t})^{n-1}\,\mathrm{P}\left[\sum_{i=1}^{n}\left(Y_i - \frac{a}{n}\right) \leq 0\right] \\ & + \sum_{n=n_0+1}^{\infty} e^{-\lambda t}(1-e^{-\lambda t})^{n-1}\,\mathrm{P}\left[\sum_{i=1}^{n}\left(Y_i - \frac{a}{n}\right) \leq 0\right].\end{aligned} \tag{10}$$

It is clear that there exists a constant β such that, for all t, the first term on the right-hand side of equation (10) is less than $\beta e^{-\lambda t}$. Therefore, we only need to worry about the second term on the right-hand side of equation (10). From a standard result in probability theory,[4] if Y is a random variable with a finite strictly positive mean and a moment generating function that is finite on $(-c, 0]$ for some $c > 0$, then $\mathrm{P}(Y \leq 0) \leq \inf_{-c<s<0} E[e^{sY}] < 1$. For $n > n_0$, for some fixed $c > 0$, we then have

$$\mathrm{P}\left[\sum_{i=1}^{n}\left(Y_i - \frac{a}{n}\right) \leq 0\right] \leq \left(\inf_{-c<s<0} E\left[e^{s(Y_1 - a/n)}\right]\right)^n \leq e^{ac}\gamma^n, \tag{11}$$

where $\gamma < 1$. From (11), we conclude that the second term on the right-hand side of equation (10) is bounded by $e^{ac}\frac{\gamma}{1-\gamma}e^{-\lambda t}$. Therefore,

$$\mu_0(-\infty, a)e^{-\lambda t} \leq \mu_t(-\infty, a) \leq \left(\beta + e^{ac}\frac{\gamma}{1-\gamma}\right)e^{-\lambda t}, \tag{12}$$

and the proof is complete. ■

[4] See, for example, Rosenthal (2000), pp. 90-92.

4 Multi-Agent Meetings

For the case of m agents at each meeting, the evolution of the cross-sectional distribution of types is similarly given by:

$$\mu_t = \mu_0 + \lambda \int_0^t (\mu_s^{*m} - \mu_s)\, ds, \tag{13}$$

as explained in Duffie and Manso (2007). A solution for the cross-sectional distribution of beliefs at any time t is given explicitly by (8) and the following extension of the Wild summation formula for the type distribution.

Proposition 3 *The unique solution of (13) is*

$$\mu_t = \sum_{n \geq 1} a_{[(m-1)(n-1)+1]} e^{-\lambda t} (1 - e^{-(m-1)\lambda t})^{n-1} \mu_0^{*[(m-1)(n-1)+1]}, \tag{14}$$

where $a_1 = 1$ and, for $n > 1$,

$$a_{(m-1)(n-1)+1} = \frac{1}{m-1} \left(1 - \sum_{\left\{ \substack{i_1, \ldots, i_{(m-1)} < n \\ \sum i_k = n+m-2} \right\}} \prod_{k=1}^{m-1} a_{[(m-1)(i_k - 1)+1]} \right). \tag{15}$$

Proof From (13), the Fourier transform of μ_t satisfies

$$\frac{\partial \varphi(s,t)}{\partial t} = -\lambda \varphi(s,t) + \lambda \varphi^m(s,t), \tag{16}$$

whose solution satisfies

$$\varphi(s,t)^{m-1} = \frac{\varphi(s,0)}{e^{(m-1)\lambda t}(1 - \varphi^{m-1}(s,t)) + \varphi^{m-1}(s,t)}. \tag{17}$$

Following steps analogous to those of Proposition 1,

$$\mu_t^{*(m-1)} = \sum_{n \geq 1} e^{-(m-1)\lambda t} (1 - e^{-(m-1)\lambda t})^{n-1} \mu_0^{*(m-1)n}. \tag{18}$$

Let ν_t denote the right-hand side of (14). By recursively calculating the convolution,

$$\begin{aligned}
\nu_t^{*(m-1)} &= \left(\sum_{n\geq 1} a_{[(m-1)(n-1)+1]} e^{-\lambda t}(1-e^{-(m-1)\lambda t})^{n-1}\mu_0^{*[(m-1)(n-1)+1]}\right)^{*(m-1)} \\
&= \sum_{n\geq 1} \beta_n e^{-(m-1)\lambda t}(1-e^{-(m-1)\lambda t})^{n-1}\mu_0^{*(m-1)n} \qquad (19)\\
&= \sum_{n\geq 1} e^{-(m-1)\lambda t}(1-e^{-(m-1)\lambda t})^{n-1}\mu_0^{*(m-1)n}, \qquad (20)
\end{aligned}$$

where

$$\beta_n = \sum_{\left\{\substack{i_1,\ldots,i_{(m-1)} \\ \sum i_k = n+m-2}\right\}} \prod_{k=1}^{m-1} a_{[(m-1)(i_k-1)+1]},$$

and where the last equality follows from the definition of $a_{[(m-1)(n-1)+1]}$ for $n \geq 1$. Thus, $\nu_t^{*(m-1)} = \mu_t^{*(m-1)}$, and it remains to show that the distribution μ_t is uniquely characterized by its convolution of order $m-1$. This follows[5] from the fact that μ_0, and therefore μ_t^{*k} for any t and k, have a moment generating function in a neighborhood of zero and a non-zero first moment on the event $\{X = H\}$. ■

Proposition 4 *For any meeting group size m, the cross-sectional distribution of beliefs converges (to that of complete information) at rate λ.*

Proof Again, it is enough to derive the rate of convergence of $\mu_t(-\infty, a)$ to zero on the event $\{X = H\}$. From (14),

$$\mu_t(-\infty, a) \geq e^{-\lambda t}\mu_0(-\infty, a). \qquad (21)$$

[5] Because, on $\{X = H\}$, the derivative of the moment generating function of μ_0 at zero is the first moment of μ_0, which is positive, the moment generating function is strictly less than 1 in an interval $(-\epsilon, 0]$, for a sufficiently small $\epsilon > 0$. This implies that there is an analogous explicit solution for the moment generating function of μ_t^{*n}, for any n and t, on a small negative interval. The $(m-1)$-st root of the moment generating function of $\mu_t^{*(m-1)}$, on such an interval, uniquely determines the associated measure μ_t. For additional details, see Billingsley (1986), p. 408.

Now, from (18) and our analysis in Section 3, we know that for some constant $\kappa > 0$,

$$\mu^{*(m-1)}(-\infty, (m-1)a) \leq \kappa e^{-(m-1)\lambda t}. \tag{22}$$

From the fact that

$$(\mu(-\infty, a))^{m-1} \leq \mu^{*(m-1)}(-\infty, (m-1)a), \tag{23}$$

we conclude that

$$\mu(-\infty, a) \leq \kappa^{\frac{1}{m-1}} e^{-\lambda t}. \tag{24}$$

From (21) and (24), it follows that the rate of convergence of $\mu_t(-\infty, a)$ to zero is λ, completing the proof. ■

Because the expected rate at which a particular individual enters meetings is λ per year, independence and a formal application of the law of large numbers implies that the total quantity of m-agent meetings per year is λ/m, almost surely. So the total annual attendance at meetings is almost surely λ per year, invariant[6] to m. Our results show that total attendance at meetings is what matters for information convergence rates.

We have not shown that our invariance result extends from the case of a constant group size to a model in which the group size varies at random from meeting to meeting, say with the same mean group size across meetings.

[6]This is not about large numbers, or uncertainty. For example, suppose each member of a group $\{A, B, C, D\}$ of 4 agents holds one meeting with a different member of the group. For example, A meets with B, and C meets with D. Then there is a total of two meetings, and each individual attends one meeting. If the 4 agents meet all together, once, we have the same total attendance, and the same rate at which each individual attends a meeting.

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