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NONADIABATIC HANNAY'S ANGLE OF SPIN ONE HALF IN GRASSMANNIAN VERSION AND INVARIANT ANGLE COHERENT STATES

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Abstract. We propose to determinate the nonadiabatic Hannay's angle of spin one half in a varying external magnetic field, by using an averaged version of the variational principal. we also show how the evolution and this nonadiabatic Hannay's angle is associated with the evolution of Grassmannian invariant-angle coherent states.

1. Introduction

After the discovery of the adiabatic geometrical phase by Berry [4], there has been a substantial interest in works in this research fields. Indeed Aharonov and Anandan [1] have generalized adiabatic **Berry's phase** to nonadiabatic case in cyclic evolution. Cyclicity means that, after some time, the state returns to itself up to a phase. A way to get this cyclic states is to consider the eigenvectors of a Hermitian periodic invariant [15], which play the same basic role as the Hamiltonian eigenvectors in the adiabatic case. For this reason, invariant theory takes an important place in works on nonadiabatic phases [7, 9, 10, 17].

The classical analogue of Berry's phase is the so-called **Hannay's angle**. Hannay [13] has shown that when the adiabatic excursion takes place on a closed path in the space of parameters, an extra shift analogue to the Berry's phase is realized in the angle variables, which is called adiabatic Hannay's angle. It can be viewed as a semi classical limit of Berry's phase [4]. A geometrical angle can be defined also on a constant-action surface for cyclic evolution [6] in a classical nonadiabatic integrable Hamiltonian system; this angle is the classical counterpart of the geometrical phase [1], so it is called the nonadiabatic Hannay's angle.

2. Grassmannian Version of Spin One Half and Nonadiabatic Hannay's Angle

The first point with which we start is the determination of the nonadiabatic Hannay's angle of a Grassmannian spin $\frac{1}{2}$ in a time-dependent magnetic field by using the conveniently averaged variational principle.
As shown in reference [12], the Hamiltonian of a Grassmannian spin $\frac{1}{2}$ system in time-dependent magnetic field is

$$H = -\frac{\mathrm{i}}{2}\varepsilon_{kml}B(t)\xi_m\xi_l \tag{1}$$

and it involves 3 real Grassmann variables ξ_m which obey to Grassmann algebra rule $\xi_m\xi_l + \xi_l\xi_m = 0$. This leads to Pauli spin in the time-dependent magnetic field

$$\hat{H} = \frac{1}{2}\vec{B}(t)\vec{\sigma} \tag{2}$$

where after the quantization, the anticommuting three-vectors were replaced with the Pauli matrices $\hat{\xi} = \frac{\vec{\sigma}}{\sqrt{2}}$.
The Lagrangian associated to the Hamiltonian (1) is:

$$L = \frac{\mathrm{i}}{2}\xi_k\dot{\xi}_k + \frac{\mathrm{i}}{2}\varepsilon_{kml}B_k(t)\xi_m\xi_l\,. \tag{3}$$

It is easy to show that the system described by the Hamiltonian (1) admits a time dependent invariant

$$I(t) = -\frac{\mathrm{i}}{2}\varepsilon_{kml}R_k(t)\xi_m\xi_l \tag{4}$$

satisfying the relation

$$\frac{\partial I(t)}{\partial t} = -\left\{H(\vec{\xi}), I(\vec{\xi})\right\}_{P.B} = -\mathrm{i}\sum_m H(\vec{\xi})\overleftarrow{\partial}_m\overrightarrow{\partial}_m I(\vec{\xi}) \tag{5}$$

where $\overleftarrow{\partial}_m$ and $\overrightarrow{\partial}_m$ are left and right derivatives with respect to ξ_m, $r(t)$ is the solution of the following auxiliary equation:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\dot{r}}{B_+}\right) + \frac{r}{4}\left[\frac{B_-B_+ + B_3^2}{B_+} - 2\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{B_3}{B_+}\right)\right] - \frac{B_+}{r^3} = 0 \tag{6}$$

where $B_\pm = B_1 \pm \mathrm{i}B_2$, $R_\pm = R_1 \pm \mathrm{i}R_2$, $R_+ = \dfrac{r^2}{2}$, $R_- = \dfrac{r^{*2}}{2}$ and r^* denotes the complex conjugate of r.

When expressed in terms of its normal modes, the invariant $I(t)$ takes the form:

$$I = -\frac{1}{2}\psi_1^*\psi_1 + \frac{1}{2}\psi_2^*\psi_2 = -\psi_1^*\psi_1 = \psi_2^*\psi_2 \tag{7}$$

where $\psi_1 = \psi_2^*$ are complex conjugates of each other, while ψ_3 is real. The complex normal coordinates ψ_i are deduced from ξ_i's through the unitary transformation $\psi_m = (S^+)_{ml}\xi_l$ $(m, l = 1, 2, 3)$ which diagonalizes the invariant $I(t)$ and S is the complex unitary matrix given bellow

$$\frac{1}{2}\begin{pmatrix} \frac{1}{\sqrt{2}}\left((1+R_3)\frac{r}{r^*} - (1-R_3)\frac{r^*}{r}\right) & \frac{1}{\sqrt{2}}\left((1+R_3)\frac{r^*}{r} - (1-R_3)\frac{r}{r^*}\right) & \frac{r^2+r^{*2}}{2} \\ \frac{-\mathrm{i}}{\sqrt{2}}\left((1+R_3)\frac{r}{r^*} + (1-R_3)\frac{r^*}{r}\right) & \frac{\mathrm{i}}{\sqrt{2}}\left((1+R_3)\frac{r^*}{r} + (1-R_3)\frac{r}{r^*}\right) & -\mathrm{i}\frac{r^2-r^{*2}}{2} \\ -\frac{rr^*}{\sqrt{2}} & -\frac{rr^*}{\sqrt{2}} & 2R_3 \end{pmatrix}. \tag{8}$$

In normal coordinates the Lagrangian associated to the Hamiltonian (1) takes the form:

$$\begin{aligned} L &= \frac{\mathrm{i}}{2}(\psi_1^*\dot{\psi}_1 + \psi_2^*\dot{\psi}_2 + \psi_3^*\dot{\psi}_3) + \frac{\mathrm{i}}{2}\psi_i^* S_{ki}^+ \dot{S}_{kj}\psi_j + \frac{\mathrm{i}}{2}\varepsilon_{kml}B_k\psi_i^* S_{mi}^+ S_{lj}\psi_j \\ &= \frac{\mathrm{i}}{2}(\psi_1^*\dot{\psi}_1 + \psi_2^*\dot{\psi}_2 + \psi_3^*\dot{\psi}_3) + \frac{1}{2}\left(\frac{B_+}{r^2} + \frac{B_-}{r^{*2}}\right)(\psi_1^*\psi_1 - \psi_2^*\psi_2) \end{aligned} \tag{9}$$

which is independent on θ (the argument of the normal coordinate ψ) but each of the two term $\frac{\mathrm{i}}{2}\psi_i^* S_{ki}^+ \dot{S}_{kj}\psi_j$ and $\frac{\mathrm{i}}{2}\varepsilon_{kml}B_k\psi_i^* S_{mi}^+ S_{lj}\psi_j$ depends on θ, and then we can replace this Lagrangian by its averaged value

$$L = \bar{L} = \frac{1}{2\pi}\int_0^{2\pi} L \,\mathrm{d}\theta \tag{10}$$

Thus can be rewritten also as

$$\begin{aligned} \bar{L} &= \frac{\mathrm{i}}{2}(\psi_1^*\dot{\psi}_1 + \psi_2^*\dot{\psi}_2 + \psi_3^*\dot{\psi}_3) - \frac{\mathrm{i}R_3}{2}\left(\frac{\dot{r}^*}{r^*} + \frac{\dot{r}}{r}\right)(\psi_1^*\psi_1 - \psi_2^*\psi_2) \\ &+ \frac{1}{2}\vec{B}(t)\vec{R}(t)(\psi_1^*\psi_1 - \psi_2^*\psi_2)\,. \end{aligned} \tag{11}$$

The second term $-\frac{\mathrm{i}R_3}{2}(\frac{\dot{r}^*}{r^*} + \frac{\dot{r}}{r})(\psi_1^*\psi_1 - \psi_2^*\psi_2)$ in this equation is a result of averaging $\frac{\mathrm{i}}{2}\psi_i^* S_{ki}^+ \dot{S}_{kj}\psi_j$ over θ and is exactly the term which gives nonadiabatic Hannay's angle (because it becomes null at fixed parameters $(\vec{R} = cte)$).

The Euler–Lagrange equation: $\dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial \bar{L}}{\partial \dot{I}_j} - \dfrac{\partial \bar{L}}{\partial I_J} = 0$

$$I_j = \psi_j^*(t)\psi_j(t) \qquad j = 1, 2$$

gives

$$\dot{\theta}_j = \dot{\theta}_j^D + \dot{\theta}_j^H = (-1)^j \left(\frac{B_+}{r^2} + \frac{B_-}{r^{*2}} \right). \tag{12}$$

Finally the geometric part (or nonadiabatic Hannay's angle) is

$$\theta_j^H = (-1)^j (-\mathrm{i}) \int_{(c)} R_3 \left(\frac{\mathrm{d}r^*}{r^*} - \frac{\mathrm{d}r}{r} \right). \tag{13}$$

These above results agree with those obtained by Maamache [18] for the classical bosonic model of spin one half in a varying external magnetic field.

3. Invariant-angle Coherent States and Nonadiabatic Hannay's Angle

The second point which we want to emphasize in this paper is the determination of the nonadiabatic Hannay's angle of a Grassmannian spin $\frac{1}{2}$ in time-dependent magnetic field by using the evolution of the invariant-angle coherent states.

The action-angle coherent states are defined in the classical approximation, that is for small $\hbar$ with respect to the classical action in a way which resemble the definition of the usual (harmonic oscillator) coherent states:

$$|\alpha, \vec{X}(t)\rangle = \mathrm{e}^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} |n, \vec{X}(t)\rangle \tag{14}$$

where $|n, \vec{X}(t)\rangle$ are the eingenstates of the Hamiltonian which depend on parameters $X(t)$ varying slowly in time. We call them "action-angle" coherent states because the complex number α can be related to the classical action-angle variables by $\alpha = \sqrt{\frac{I}{\hbar}}\, \mathrm{e}^{-\mathrm{i}\theta}$. Indeed, when the parameters $\vec{X}$ are fixed, the quantum evolution of $|\alpha, \vec{X}(t)\rangle$ amounts up to a global unessential phase factor, keeps the modulus of α constant and change θ into $\theta + \frac{\partial E_n}{\partial N} t$. This allows the identification of θ with the classical angle variable. Moreover, in the classical limit (when $\hbar$ goes to zero, $|\alpha|$ goes to infinity but the product $\hbar|\alpha|^2$ remaining finite) the sum (14) over n is peaked around $N = |\alpha|^2$ and the relation $I = \hbar|\alpha|^2$ is nothing but the correspondence principle. When the parameters vary slowly with time each eigenfunction $|n, \vec{X}(t)\rangle$ acquires an extra phase $\gamma_N^B(t)$ inducing a change of the coherent state such that the modulus of α remains constant while its argument θ becomes $\theta - \frac{\partial \gamma_N^B}{\partial N}(t)$. Then $\theta_I^H(t) = -\frac{\partial \gamma_N^B}{\partial N}(t)$ defines the corresponding Hannay's angle in classical mechanics. In this way we have exemplified the quantum-classical correspondence at the level of action-angle coherent states. Let us note also that the mean value of the quantum Hamiltonian in these states

$$\langle \alpha, \vec{X}(t)|H(\vec{X}(t))|\alpha, \vec{X}(t)\rangle = H_c(I, \vec{X}(t)) \tag{15}$$

can be identified with the classical Hamiltonian H_c (which is a function of the action only).

Let us now present the Grassmanian invariant-angle coherent state approach of this model. We shall find suitable Grassmannian (or fermionic) invariant angle coherent states $|\xi, t\rangle$ which have the following property: every change in the phase of quantum invariant eigenstates $|n,t\rangle \longrightarrow \mathrm{e}^{\mathrm{i}\phi_n}|n,t\rangle$ induces a change $\xi \longrightarrow \xi\,\mathrm{e}^{\mathrm{i}\theta}$ of the arguments of the parameter of Grassmannian invariant-angle coherent states, and the classical fermionic invariants are precisely the expectation value of the corresponding quantum invariants. The difference with the commutative case is that now there is no need of the classical limit $n \longrightarrow \infty$ and $\hbar \longrightarrow 0$. Therefore we can set $\hbar = 1$ in the following considerations.

So let us express the quantum invariant $\hat{I} = \frac{1}{2}\vec{R}(t)\vec{\sigma}$ corresponding to the classical one (4) in terms of fermionic operators $b(t)$ that annihilate the lowest eigenstate $|0,t\rangle$ of I and $b^+(t)$ which brings this state onto the other eigenstate $|1,t\rangle$ as

$$I(t) = b^+(t)b(t) - \frac{1}{2}\,. \tag{16}$$

The time dependent fermionic operators $b(t)$ and $b^+(t)$ are related to the operators $\hat{\xi}_i$ via the time-dependent unitary transformation U as:

$$\begin{pmatrix} b(t) \\ b^+(t) \\ c(t) \end{pmatrix} = U^+(t)\begin{pmatrix} b \\ b^+ \\ c \end{pmatrix} \tag{17}$$

with

$$U = \frac{1}{2}\begin{pmatrix} (1+R_3)\frac{r^*}{r} & -(1-R_3)\frac{r^*}{r} & \frac{{r^*}^2}{\sqrt{2}} \\ -(1-R_3)\frac{r}{r^*} & (1+R_3)\frac{r}{r^*} & \frac{r^2}{\sqrt{2}} \\ -\frac{|r|^2}{\sqrt{2}} & -\frac{|r|^2}{\sqrt{2}} & 2R_3 \end{pmatrix}$$

where the operators $\hat{\xi}_1 = \frac{1}{\sqrt{2}}(b+b^+)$, $\hat{\xi}_2 = \frac{1}{\sqrt{2}}(b-b^+)$, and $\hat{\xi}_3 = c = c^+$ are related to the operators $\hat{\xi}_i$.

The operators $b(t)$ and $b^+(t)$ satisfy the relations

$$\begin{aligned} \{b, b^+\}_+ &= \{c, c^+\}_+ = 1 \\ \{b, b\}_+ &= \{b, c\}_+ = 0\,. \end{aligned} \tag{18}$$

In the matrix notation one has $b^+ = \sigma_+ = \frac{1}{2}(\sigma_1 + \mathrm{i}\sigma_2)$, $b^- = \sigma_- = \frac{1}{2}(\sigma_1 - \mathrm{i}\sigma_2)$ and the Clifford number c is $\frac{1}{\sqrt{2}}\sigma_3$.

Therefore, the time-dependent operators $b(t)$, $b^+(t)$ and $c(t)$ obviously satisfy the algebra isomorphic to that given in (18). The initial Grassmannian invariant-angle coherent states which are taken to be

$$|\xi(0), 0\rangle = \mathrm{e}^{-\frac{1}{2}\xi^*(0)\xi(0)}(|0,0\rangle - \xi(0)|0,0\rangle) \tag{19}$$

are eigenstates of $b(t)$ with eigenvalue $\xi(0)$ and they are created from the ground state $|0,0\rangle$ by the unitary operator $e^{-(\xi(0)b^+(0)+\xi^*(0)b(0))}|0,0\rangle$. According to the Lewis–Riesenfeld theory one can immediately see that the evolution

$$|0,0\rangle \rightarrow \mathrm{e}^{\mathrm{i}\phi_0(t)}|0,t\rangle \quad \text{and} \quad |1,0\rangle \rightarrow \mathrm{e}^{\mathrm{i}\phi_1(t)}|1,t\rangle \tag{20}$$

of the eigenstates of $I(0)$ induces the evolution of Grassmannian invariant-angle coherent states

$$|\xi(0), 0\rangle \rightarrow \mathrm{e}^{\mathrm{i}\phi_0(t)}|\xi(t), t\rangle \tag{21}$$

with $\xi(t) = \xi(0)\,\mathrm{e}^{\mathrm{i}(\phi_0(t)-\phi_1(t))}$. The argument of the parameter ξ changes during the evolution. As is well known, the global phases $\phi_n(t)$ $(n = 0,1)$ contain a dynamical part $\phi_n^D = -\int_0^t \mathrm{d}t'\langle n,t'|\hat{H}(t)|n,t'\rangle$ and a geometrical one $\phi_n^G = \mathrm{i}\int_0^t \mathrm{d}t'\langle n,t'|\frac{\partial}{\partial t'}|n,t'\rangle$. The main point of this elementary result is that the argument of the parameter $\xi(t)$ contains a dynamical part $\phi_1^D(t) - \phi_0^D(t)$ and a geometrical part $\phi_1^G(t) - \phi_0^G(t)$. This geometrical part is nothing but (minus) Hannay's angle [13] in a cyclic evolution. The second key property $I_c = \langle\xi(t),t|I(t)|\xi(t),t\rangle + \frac{1}{2} = \xi^*(t)\xi(t)$ is an immediate consequence of (16) and (21). It allows the identification of the ξ's entering into the definition of $|\xi,t\rangle$ with the classical normal modes and justifies the Grassmannian invariant-angle coherent states denomination of $|\xi,t\rangle$: ξ^* is the classical invariant variable.

Let us embark on the calculation of these angles. From equations (2) and (17), we have

$$\begin{aligned}\theta^D = \int_0^t \mathrm{d}t' \Bigg(&\langle 1,t'|(B_+(t'), B_-(t'), \sqrt{2}B_3(t'))U(t') \begin{pmatrix} b(t') \\ b^+(t') \\ c(t') \end{pmatrix} |1,t'\rangle \\ &- \langle 0,t'|(B_+(t'), B_-(t'), \sqrt{2}B_3(t'))U(t') \begin{pmatrix} b(t') \\ b^+(t') \\ c(t') \end{pmatrix} |0,t'\rangle \Bigg).\end{aligned} \tag{22}$$

We see that only the term proportional to $c(t)$ contributes to the dynamical angle and yields the correction

$$\theta^D = \int_0^t \vec{B}(t')\vec{R}(t')\,\mathrm{d}t' \tag{23}$$

to the geometrical angle. Using equation (17), $\frac{\partial b^+(t)}{\partial t}$ can be expressed as

$$\frac{\partial b^+(t)}{\partial t} = R_3\left(\frac{\dot{r}^*}{r^*} - \frac{\dot{r}}{r}\right) b^+(t) + \frac{1}{2\sqrt{2}}\left(r\dot{r}^* - r^*\dot{r} - \frac{4\dot{R}_3}{|r|^2}\right) c(t) \tag{24}$$

so that

$$\begin{aligned} \theta^G &= \mathrm{i}\int_0^t \mathrm{d}t' \left(\langle 0,t'|\frac{\partial}{\partial t'}|0,t'\rangle - \langle 1,t'|\frac{\partial}{\partial t'}|1,t'\rangle\right) \\ &= -\mathrm{i}\int_0^t \mathrm{d}t'\langle 1,t'|\frac{\partial b^+(t')}{\partial t'}|0,t'\rangle = -\mathrm{i}\int_0^t \mathrm{d}t' R_3\left(\frac{\dot{r}^*}{r^*} - \frac{\dot{r}}{r}\right) \end{aligned} \tag{25}$$

for a cyclic evolution of duration T the nonadiabatic Hannay's angle is

$$\theta^G = -\mathrm{i}\int_{(c)} R_3\left(\frac{\mathrm{d}r^*}{r^*} - \frac{\mathrm{d}r}{r}\right). \tag{26}$$

We note here that r must return to its original value, and indeed there do exist such solutions to equation (6).

4. Conclusion

These above results agree with those obtained by using the classical Grassmannian approach and by Maamache [18] for the classical bosonic model of spin one half.

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