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ENUMERATIVE GEOMETRY FOR COMPLEX GEODESICS ON QUASI-HYPERBOLIC 4-SPACES WITH CUSPS

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Abstract. We introduce orbital functionals $\int \beta$ simultaneously for each commensurability class of orbital surfaces. They are realized on infinitely dimensional *orbital* divisor spaces spanned by (arithmetic-geodesic real 2-dimensional) orbital curves on any orbital surface. We discover infinitely many of them on each commensurability class of orbital Picard surfaces, which are real 4-spaces with cusps and negative constant Kähler–Einstein metric degenerated along an orbital cycle. For a suitable (Heegner) sequence $\int \mathbf{h}_N$, $N \in \mathbb{N}$, of them we investigate the corresponding formal orbital q-series $\sum_{N=0}^{\infty}(\int \mathbf{h}_N)q^N$. We show that after substitution $q = \mathrm{e}^{2\pi \mathrm{i}\tau}$ and application to arithmetic orbital curves $\hat{\mathbf{C}}$ on a fixed Picard surface class, the series $\sum_{N=0}^{\infty}(\int_{\hat{\mathbf{C}}} \mathbf{h}_N)\, \mathrm{e}^{2\pi \mathrm{i}\tau}$ define modular forms of well-determined fixed weight, level and Nebentypus. The proof needs a new orbital understanding of orbital heights introduced in [12] and Mumford–Fulton's rational intersection theory on singular surfaces in Riemann–Roch–Hirzebruch style. It has to be connected with Zeta and Theta functions of hermitian lines, indefinite quaternionic fields and of a matrix algebra along a research marathon over 75 years represented by Cogdell, Kudla, Hirzebruch, Zagier, Shimura, Schoeneberg and Hecke. Our aim is to open a door to an effective enumerative geometry for complex geodesics on orbital varieties with nice metrics.

1. Introduction

In the monograph [12] we defined orbital heights for orbital curves on orbital surfaces. In the most important cases of orbital hyperbolic surfaces (Picard surfaces), which are real 4-dimensional with cusps with negative constant curvature, the orbital curves appear as geodesics of real dimension 2. The orbital heights are rational numbers explicitly defined in algebraic geometric terms. They can also be expressed by zeta function values, see [7], Chapter III, and from the differential geometric viewpoint they are volumes of fundamental domains of discrete subgroups of a unitary group.

In this paper we define groups of orbital divisors and extend the orbital signature heights to functionals $\check{\mathbf{h}}_0$ on the orbital divisor spaces. Additionally, we extend and transfer Mumford–Fulton's rational intersection theory on complex surfaces with (normal) singularities in Riemann–Roch–Hirzebruch style to functionals on the orbital divisor spaces. On orbital Picard surfaces the (arithmetic-geodesic) orbital curves can be normed by positive integers. Using these norms we find all these orbital curves as supports of a well-defined sequence $\mathbf{H}_N$, $N \in \mathbb{N}_+$, of special orbital (Heegner) divisors. They also define orbital functionals $\check{\mathbf{h}}_N$ on orbital divisor spaces, nicely compatibile with finite orbital coverings of orbital surfaces.

Writing $\int\boldsymbol{\beta}$ for an orbital functional $\check{\boldsymbol{\beta}}$ we define formal q-series $\sum_{N=0}^{\infty}(\int \mathbf{h}_N)q^N$. They are applicable to each orbital curve $\hat{\mathbf{C}}$ on any orbital Picard surface defining a formal Taylor series $\Phi_{\hat{\mathbf{C}}}(q)$ in q with rational coefficients $\check{\mathbf{h}}_n(\hat{\mathbf{C}})$, $n = 0, 1, 2, 3, \ldots$.

Substituting $q = \mathrm{e}^{2\pi\mathrm{i}\tau}$ we get convergent series denoted and defined by

$$\begin{aligned} \Phi_{\hat{\mathbf{C}}}(\tau) &= \sum_{N=0}^{\infty}\left(\int\limits_{\hat{\mathbf{C}}} \mathbf{h}_N\right) q^N = \sum_{N=0}^{\infty}\left(\check{\mathbf{h}}_N(\hat{\mathbf{C}})\right) q^N \\ &= \check{\mathbf{h}}_0(\hat{\mathbf{C}}) + \sum_{N=1}^{\infty}(\hat{\mathbf{C}} \cdot \mathbf{H}_N)_{\hat{\mathbf{X}}_\Gamma} q^N \end{aligned} \tag{1.1}$$

on the upper half plane $\mathbb{H}$ with orbital (rational) intersection product $(\cdot)_{\hat{\mathbf{X}}_\Gamma}$ on orbital Picard surfaces $\hat{\mathbf{X}}_\Gamma$ supported by the Baily–Borel compactification $\hat{X}_\Gamma$ of $X_\Gamma := \Gamma\backslash\mathbb{B}$, Γ a Picard modular group acting on the complex two-dimensional unit ball $\mathbb{B}$.

The first sections are dedicated to the construction of orbital functionals and to the proofs of their fundamental properties. At the end of this procedure we can prove that the **orbital Heegner series** $\Phi_{\hat{\mathbf{D}}}(\tau)$ and $\Phi_{\hat{\mathbf{C}}}(\tau)$ are the same up to a degree factor, if $\hat{\mathbf{D}}$ is a finite orbital covering of $\hat{\mathbf{C}}$. The scaling constant term $\check{\mathbf{h}}_0(\hat{\mathbf{C}})$ in (1.1) is in any case the orbital signature height of $\hat{\mathbf{C}}$. If we know one Heegner series $\Phi_{\hat{\mathbf{D}}}(\tau)$ and its properties of the orbital covering class of $\hat{\mathbf{C}}$, then we know (essentially) all. For arithmetic orbital curves $\hat{\mathbf{D}}$ on neat Picard surfaces the modular properties of the Heegner series are known by work of Cogdell. It extends now to the general main result 6.1 of this paper, valid for all arithmetic curves on each Picard surface: The Heegner series are modular forms of explicitly determined weight (three), level and Nebentypus. Level group and Nebentypus depend only on the commensurability class of Picard surfaces.

With help of preparing work by Hecke, Schoeneberg, Kudla and Cogdell we are able to connect our series with congruence Theta functions and — via Mellin transformations — with congruence Zeta functions. This will be summarized in the last sections. More precisely, there are three types of such functions we need. Hecke's congruence Theta and Zeta functions of lattices of hermitian lines sit in cusp lattices of Picard surfaces. One needs Hecke's results of 1926; no later explicit reference seems to be possible. The modular curves on Picard surfaces, characterized by the existence of cusps, are closely connected with Theta and Zeta functions of the matrix algebra $\mathrm{Mat}_2(\mathbb{Q})$ investigated by Cogdell in [4]. Arithmetic-geodesic curves without cusps are Shimura curves. They are connected with congruence Theta and Zeta functions of indefinite quaternion algebras introduced and investigated by Schoeneberg [21] in 1936 following ideas of Hecke. Their application to Picard curves goes essentially back to Kudla's paper [18] transfering ideas and work of Hirzebruch–Zagier [11] from the Hilbert modular to ball cases.

In Section 7 we present an example on the quasi-hyperbolic Picard plane of a Gauß lattice. The Fourier coefficients of the Heegner series are explicitly described in simple arithmetic terms on two different ways. The N-th coefficient counts our quasi-geodesics of fixed norm N up to intersection multiplicities. For a better understanding of our motivations we recommend the reader to look first to the example in Section 7. It is also closely connected with my lectures [13] delivered at the Varna Conference in 2001.

2. Orbital Divisors on Orbital Surfaces

Let $\hat{X} = X \cup X^\infty$ be a complex compact normal algebraic surface with at most quotient and (ball) cusp singularities. These singularities are precisely defined and classified in my monograph [12]. The cusp singularities (including also some marked smooth points) form a finite set X^∞ of embedded points. Together with an orbital (see below) Weil divisor

$$\hat{B}^1 = v_1\hat{C}_1 + \cdots + v_r\hat{C}_r, \qquad r \geq 0,\ v_i \geq 2$$

$\hat{C}_i$ irreducible, we get an orbital surface $\hat{\mathbf{X}} := (\hat{X}, \hat{B}^1)$. We fix $\hat{B}^1$ and call it the **basic divisor** of $\hat{\mathbf{X}}$. Orbital (in the global sense) means: there is a Galois covering $\hat{p}_G\colon \hat{Y} \to \hat{X} = \hat{Y}/G$ with Galois group $G \subseteq \mathrm{Aut}\,\hat{Y}$ and restriction $p_G\colon Y \to X = X/G$, Y smooth satisfying

Conditions 2.1.

- $\hat{B}^1$ *is the branch divisor of* $\hat{p}_G$
- *the ramification index over* $\hat{C}_i$ *coincides with* v_i, $i = 1, \dots, r$
- *the points of* $Y^\infty := \hat{Y} \setminus Y$ *are (purely) elliptic singularities*
- *the components of the preimage curves* $\hat{p}_G^*(\hat{C}_i)$ *are smooth on* Y
- *their proper transforms on the minimal singularity resolution* Y' *of* $\hat{Y}$ *have only transversal intersections with the exceptional divisor* $T = E(\mu)$ *of* $\mu \colon Y' \to \hat{Y}$, *which is a disjoint sum of elliptic curves on* Y'.

Definition 2.1. *If the above properties are satisfied, we call* $\hat{\mathbf{X}}$ *an* orbital surface *with (defining)* basic orbital divisor $\hat{\mathbf{B}}^1$, *see* (2.6). *The coverings* $\hat{p}_G$ *or the restrictions* p_G *are called* finite uniformization *of* $\hat{\mathbf{X}}$ *or* $\mathbf{X}$, *respectively.*

We will use the notations

$$\hat{\mathbf{p}}_G \colon \hat{Y} \to \hat{\mathbf{X}}, \qquad \mathbf{p}_G \colon Y \to \mathbf{X}$$

with bold letters in order to signalize the orbital structures. Uniformizations are not uniquely determined by the orbital surface $\hat{\mathbf{X}}$. Notice also, that each finite uniformization defines a commutative diagram

$$\begin{array}{ccc} Y' & \xrightarrow{\ \mu\ } & \hat{Y} \\ {\scriptstyle p'_G}\downarrow & & \downarrow{\scriptstyle \hat{p}_G} \\ G\backslash Y' & \xrightarrow[\ \varphi\ ]{} & \hat{X} \end{array} \tag{2.1}$$

with vertical Galois coverings and horizontal birational morphisms.

Restricting $\hat{p} = \hat{p}_G$ to suitable small analytic open neighbourhoods we have around each point $R = \hat{p}(S)$ of $\hat{X}$ a *local finite uniformization* $\hat{U}_S \to \hat{V}_R = G_S \backslash \hat{U}_S$ with branch curve (germes) supported by all components C_i going through R and corresponding ramification indices v_i. Via inductive limit we get a *local orbital morphism* $S \to \mathbf{R}$ from the embedded point $S \in \hat{Y}$ to the embedded *orbital point*

$$\mathbf{R} = \varinjlim (\hat{V}_R, \hat{B}^1 \text{ restricted to } \hat{V}_R)\,. \tag{2.2}$$

An orbital point $\mathbf{R} \in \hat{\mathbf{X}}$ is *trivial*, iff R is a smooth surface point outside of $\hat{B}^1$. In this case we do not distinguish the notations R and $\mathbf{R}$. It is a *basic orbital point* on $\hat{\mathbf{X}}$, iff R is a singular point of $\hat{X}$ or a singular point of $\hat{B}^1$.

The *basic orbital zero cycle* $\hat{\mathbf{B}}^0$ of $\hat{\mathbf{X}}$ is defined as the finite formal sum of all basic orbital points of $\hat{\mathbf{X}}$:

$$\hat{\mathbf{B}}^0 = \sum_{\hat{\mathbf{X}} \ni \mathbf{R} \text{ basic orbital}} \mathbf{R}. \tag{2.3}$$

Now we want to define orbital curves $\hat{\mathbf{C}}$ on $\hat{\mathbf{X}}$ supported by an irreducible curve $\hat{C}$ on $\hat{X}$. For this purpose we look back to an orbital uniformization $\hat{p}_G$ of $\hat{\mathbf{X}}$. The diagram (2.1) restricts to

$$\begin{array}{ccc} D' & \longrightarrow & \hat{D} \\ \Big\downarrow & & \Big\downarrow \\ p'_G(D') =: G\backslash D' & \longrightarrow & \hat{C} \end{array} \tag{2.4}$$

where $\hat{D}$ is an (arbitrary) irreducible component of $p_G^{-1}(\hat{C})$ and D' is the proper transform of $\hat{D}$ on Y'. We assume the following

Conditions 2.2.

- *D' is a smooth curve*
- *D' intersects $T = E(\mu)$ transversally at each common point*
- *$GD' := \bigoplus_{g \in G} gD'$ is a divisor whose support has at most ordinary singularities.*

Uniformizations with these properties are called *$\hat{C}$-uniformization* of $\hat{\mathbf{X}}$. Each singularity S of GD' lying on D' is called a *G-cross point* of D'. If the action of G_S on the set of curve germs of GD' is not transitive, then we call S a *honest* G-cross point of D'. These points are projected along p'_G onto the set of singularities of $G\backslash D'$.

$$\operatorname{Sing} G\backslash D' = p'_G(\{ \text{ honest } G\text{-cross points of } D'\})\,. \tag{2.5}$$

Obviously, the image points of $\operatorname{Sing} G\backslash D'$ on $\hat{X}$ are also curve singularities of $\hat{C}$. These image points are precisely all $\hat{C}$-singularities, which are not resolved by φ. As in (2.2) we define for each singularity R of $\hat{C}$ the *$\hat{C}$-orbital point* $\mathbf{R}$ on $\hat{\mathbf{X}}$ taking into account the branch situation again locally around. The whole set of $\hat{C}$-orbital points on $\hat{\mathbf{X}}$ consists of the $\hat{C}$-orbital points just described and the $\hat{\mathbf{X}}$-orbital points with support on $\hat{C}$.

Definition 2.2. *Let $\hat{C} \subset \hat{X}$ be an irreducible curve allowing a $\hat{C}$-uniformization of $\hat{\mathbf{X}}$. The pair*

$$\hat{\mathbf{C}} = \left(v_{\hat{C}} \hat{C}; \sum_{\hat{C} \ni R \ \hat{C}\text{-orbital}} \mathbf{R} \right)$$

with ***ramification index*** $v_{\hat{C}} \in \mathbb{N}_+$ *of* p_G *at* $\hat{C}$ *is called an* ***orbital curve*** *on* $\hat{\mathbf{X}}$, *and the sum of the second component is the* ***orbital cycle*** *on* $\hat{\mathbf{C}}$.

With the conditions 2.1 it is not difficult to see that each component $\hat{C}_i$ defines an orbital curve $\hat{\mathbf{C}}_i$. These **basic orbital curves** have common $\hat{C}_i$-uniformizations. Namely, each finite $\hat{\mathbf{X}}$- uniformization is a $\hat{C}_i$-uniformization of $\hat{\mathbf{X}}$. We refer to [12] for comparision, where we restricted ourselves essentially to branch curves. The **basic orbital divisor** of $\hat{\mathbf{X}}$ is the formal sum

$$\hat{\mathbf{B}}^1 := \hat{\mathbf{C}}_1 + \cdots + \hat{\mathbf{C}}_r\,, \tag{2.6}$$

and the *basic orbital cycle* of $\hat{\mathbf{X}}$ is defined as formal sum

$$\hat{\mathbf{B}} := \hat{\mathbf{B}}^1 + \hat{\mathbf{B}}^0 = \hat{\mathbf{C}}_1 + \cdots + \hat{\mathbf{C}}_r + \sum_{\hat{\mathbf{X}} \ni \mathbf{R} \text{ basic orbital}} \mathbf{R}\,.$$

Definition 2.3. *The group* $\operatorname{Div}_{\mathbb{Z}} \hat{\mathbf{X}}$ *of* ***orbital divisors*** *on* $\hat{\mathbf{X}}$ *is the free abelian group generated by all orbital curves on* $\hat{\mathbf{X}}$*:*

$$\operatorname{Div}_{\mathbb{Z}} \hat{\mathbf{X}} = \bigoplus_{\hat{\mathbf{X}} \supset \hat{\mathbf{C}} \text{ orbital}} \mathbb{Z} \cdot \hat{\mathbf{C}}$$

Remark 2.1. *As in [12] we can define orbital curves and orbital points on them purely locally on the given orbital surface* $\hat{\mathbf{X}}$ *via local intersections along local finite uniformizations.*

Let $\hat{\mathbf{X}}$ be an orbital surface. If R belongs to X^∞, we call $\mathbf{R}$ an **orbital point at infinity** or **cusp point**. The other orbital points $\mathbf{R} \in \hat{\mathbf{B}}^0$ are called (honest) **finite orbital points** or **quotient points**. Cusp points are supported by cusp singularities, which are locally finite quotients of elliptic points. Quotient points are supported quotient singularities, which are locally finite quotients of smooth surface points. In both cases it may happen that the supporting point is regular.

2.1. Heights of Orbital Curves

Let $\hat{\mathbf{p}} : \hat{Y} \to \hat{\mathbf{X}}$ and $\hat{\mathbf{q}} : \hat{Y} \to \hat{\mathbf{Z}}$ be finite uniformizations with the same covering surface $\hat{Y}$. If the supporting Galois covering $\hat{p}$ factors though $\hat{q}$, then we call the induced orbital morphism $\hat{\mathbf{Z}} \to \hat{\mathbf{X}}$ a *finite orbital surface covering*. Its restriction $\hat{\mathbf{D}} \to \hat{\mathbf{C}}$ to two orbital curves on $\hat{\mathbf{Z}}$ or $\hat{\mathbf{X}}$, respectively, is a *finite orbital curve covering* , by definition. The orbital surfaces together with such finite orbital coverings $\hat{\mathbf{X}} \to \hat{\mathbf{X}}$ as morphisms define the category $\mathbf{OrSf}$ of orbital surfaces. Similarly, we dispose on the category $\mathbf{OrCr}$ of orbital curves with the finite orbital curve coverings as morphisms.

Definition 2.4. *A* height *on* $\mathbf{OrCr}$ *is a non-zero map*

$$h\colon \mathbf{OrCr} \to \mathbb{Q}$$

satisfying

$$h(\hat{\mathbf{D}}) = [\hat{D} : \hat{C}]h(\hat{\mathbf{C}}) \tag{2.7}$$

for all finite orbital curve coverings $\hat{\mathbf{D}} \to \hat{\mathbf{C}}$. *Thereby* $[\hat{D} : \hat{C}]$ *denotes the degree of the underlying curve covering* $\hat{D} \to \hat{C}$.

In the Appendix we prove explicitly that such orbital curve heights exist. Here we need only one type of them, namely the *signature heights* $h_\tau(\hat{\mathbf{C}})$ of orbital curves $\hat{\mathbf{C}}$. The explicit formula looks like

$$h_\tau(\hat{\mathbf{C}}) := \frac{1}{v_{\hat{\mathbf{C}}}}(\tilde{C}^2) + \sum h_\tau(\mathbf{R}) \tag{2.8}$$

where the sum runs through all orbital points $\mathbf{R}$ on $\hat{\mathbf{C}}$, and $\tilde{C}$ is the (smooth) proper transform of $\hat{C} \subset \hat{X}$ on a special well-defined $\hat{\mathbf{C}}$-model of $\hat{X}$, which is smooth along $\tilde{C}$. The contributions $h_\tau(\mathbf{R})$ are rational numbers composed by singularity and weight data of R and the basic orbital curves through R. For branch curves of uniformizations the formulas can be already found in [12]. For the precise contributions in (2.8) we refer to Definition A.3 in the Appendix.

3. Orbital Functionals

Let $\hat{\mathbf{X}}$ be an orbital surface, $\mathrm{Div}_{\mathbb{Z}}\,\hat{\mathbf{X}}$ its orbital divisor group and F a field. We only need the fields $\mathbb{Q}$ and $\mathbb{R}$ of rational and real numbers. The F-vector spaces

$$\mathrm{Div}_F\,\hat{\mathbf{X}} := F \otimes \mathrm{Div}_{\mathbb{Z}}\,\hat{\mathbf{X}}$$

are infinite dimensional in general, at least for our quasihyperbolic cases $\hat{\mathbf{X}} = \hat{\mathbf{X}}_\Gamma$. We call it the F**-divisor space** of $\hat{\mathbf{X}}$.
We associate to each finite orbital covering $\hat{\mathbf{p}}\colon \hat{\mathbf{Y}} \to \hat{\mathbf{X}}$ the F-linear map

$$\hat{p}_\# = \hat{p}_{\#F}\colon \mathrm{Div}_F\,\hat{\mathbf{Y}} \to \mathrm{Div}_F\,\hat{\mathbf{X}}$$

extending F-linearly the correspondences $\hat{\mathbf{D}} \mapsto [\hat{D} : \hat{C}]\hat{\mathbf{C}}$, where $\hat{\mathbf{D}}$ is an orbital curve on $\hat{\mathbf{Y}}$ covering the orbital curve $\hat{\mathbf{C}}$ on $\hat{\mathbf{X}}$ supported by $\hat{p}(\hat{D})$. With a little modification we define the *orbital direct image homomorphisms* using the *orbital degree*

$$[\hat{\mathbf{D}} : \hat{\mathbf{C}}] := \frac{v_{\hat{\mathbf{C}}}}{v_{\hat{\mathbf{D}}}}[\hat{D} : \hat{C}]$$

instead of the geometric covering degree $[\hat{D} : \hat{C}]$

$$\hat{\mathbf{p}}_\# \colon \operatorname{Div}_F \hat{\mathbf{Y}} \to \operatorname{Div}_F \hat{\mathbf{X}}, \quad \hat{\mathbf{D}} \mapsto \hat{\mathbf{p}}_\# \hat{\mathbf{D}} := [\hat{\mathbf{D}} : \hat{\mathbf{C}}]\hat{\mathbf{C}}\,.$$

The latter object is called the *orbital direct image* of $\hat{\mathbf{D}}$. After orbitalization of direct images we want to orbitalize also our heights on $\mathbf{OrCr}$ introduced in the last section.

Definition 3.1. *A (rational)* orbital height $\mathbf{h}$ *on* $\mathbf{OrCr}$ *corresponds to each orbital curve* $\hat{\mathbf{C}}$ *and the rational number* $\mathbf{h}(\hat{\mathbf{C}})$ *such that* $\mathbf{h}(\hat{\mathbf{D}}) = [\hat{\mathbf{D}} : \hat{\mathbf{C}}]\mathbf{h}(\hat{\mathbf{C}})$ *for all orbital curve coverings* $\hat{\mathbf{D}} \to \hat{\mathbf{C}}$.

From any *height* h on $\mathbf{OrCr}$ satisfying the degree formula (2.7) it is easy to change to the corresponding *orbital height* $\mathbf{h}$ setting

$$\mathbf{h}(\hat{\mathbf{C}}) := \frac{1}{v_{\hat{\mathbf{C}}}} h(\hat{\mathbf{C}})\,.$$

Namely, from the degree compatibility of h follows immediately

$$\mathbf{h}(\hat{\mathbf{D}}) = \frac{1}{v_{\hat{\mathbf{D}}}} \cdot h(\hat{\mathbf{D}}) = \frac{[\hat{D} : \hat{C}]}{v_{\hat{\mathbf{D}}}} h(\hat{\mathbf{C}}) = \frac{[\hat{D} : \hat{C}] v_{\hat{\mathbf{C}}}}{v_h \hat{\mathbf{D}}} \mathbf{h}(\hat{\mathbf{C}}) = [\hat{\mathbf{D}} : \hat{\mathbf{C}}]\mathbf{h}(\hat{\mathbf{C}})\,.$$

Example 3.1. *The signature height* h_τ *changes in this manner to the* orbital signature height $\mathbf{h}_\tau$.

Remark 3.1. *We will omit the index* $\mathbb{Q}$ *in* $\operatorname{Div}_{\mathbb{Q}} \hat{\mathbf{X}}$ *keeping this base field extension in mind. This will be also done for the field index* $\mathbb{R}$*, if there is no danger of misunderstanding.*

Now we consider the functionals

$$\mathbf{f}_{\hat{\mathbf{X}}} \colon \operatorname{Div} \hat{\mathbf{X}} \to \mathbf{R}$$

which are nothing else but linear maps on the orbital divisor spaces.

Definition 3.2. *A set*

$$\check{\mathbf{f}} = \{\mathbf{f}_{\hat{\mathbf{X}}}\,;\, \hat{\mathbf{X}} \in \mathbf{OrSf}\}$$

is called an orbital functional *on* $\mathbf{OrSf}$ *iff it is compatible with orbital direct images along finite orbital coverings. This means that* $\mathbf{h}_{\hat{\mathbf{X}}} \circ \mathbf{p}_\# = \mathbf{h}_{\hat{\mathbf{Y}}}$ *holds for all finite orbital coverings* $\mathbf{p} \colon \hat{\mathbf{Y}} \to \hat{\mathbf{X}}$.

Example 3.2. *Each orbital height* $\mathbf{h}$ *on* **OrCr** *extends linearly to an orbital functional* $\check{\mathbf{h}}$. *The extension is done objectwise on each orbital divisor space* $\operatorname{Div}\hat{X}$. *The compatibility with orbital finite coverings comes from the defining height property* 3.1 *for the orbital curves generating the orbital divisor spaces. Especially, the orbital signature height extends to the* orbital signature functional.

It is quite natural to give a relative definition on commensurability classes of **OrSf**. Commensurability is the smallest equivalence relation putting an orbital surface and any orbital finite covering of it into the same class. The class of $\hat{\mathbf{X}}$ is denoted by $[\hat{\mathbf{X}}]$. We denote the corresponding full subcategory of **OrSf** by the same symbol. By restriction we define **orbital functionals** on $[\hat{\mathbf{X}}]$ in obvious manner. In the same manner the commensurability class $[\hat{\mathbf{C}}]$ of an orbital curve $\hat{\mathbf{C}}$ on $\hat{\mathbf{X}}$ is well-defined. It consists of orbital curves on objects of $[\hat{\mathbf{X}}]$. It is also considered as a category with finite orbital curve coverings as morphisms.

Now we fix an infinite sequence

$$\check{\mathbf{H}} = (\check{\mathbf{h}}_0, \check{\mathbf{h}}_1, \dots, \check{\mathbf{h}}_N, \dots)$$

of orbital functionals, say with rational values on rational orbital divisors. It defines a very formal series

$$\check{\mathbf{H}}(q) := \sum_{N=0}^{\infty} \check{\mathbf{h}}_N \cdot q^N$$

with a variable q. Applied to orbital curves $\hat{\mathbf{C}}$ we get formal power series

$$\mathbf{H}_{\hat{\mathbf{C}}}(q) := \sum_{N=0}^{\infty} \mathbf{h}_N(\hat{\mathbf{C}}) \cdot q^N \in \mathbb{Q}[[q]] .$$

For orbital finite coverings $\hat{\mathbf{D}} \to \hat{\mathbf{C}}$ we get the relations

$$\mathbf{H}_{\hat{\mathbf{D}}}(q) = [\mathbf{D} : \mathbf{C}] \cdot \mathbf{H}_{\hat{\mathbf{C}}}(q) . \tag{3.1}$$

Now we substitute $q = e^{2\pi i\tau}$, $\tau \in \mathbb{H} := \{z \in \mathbb{C};\, \Im z > 0\}$. For suitable sequences of orbital functionals we expect and will construct convergent series (holomorphic functions)

$$\Phi_{\hat{\mathbf{C}}}(\tau) = \Phi_{\hat{\mathbf{C}}}^{\check{\mathbf{H}}}(\tau) := \sum_{N=0}^{\infty} \mathbf{h}_N(\hat{\mathbf{C}}) \cdot e^{2\pi i N\tau}$$

on the Poincaré upper half plane $\mathbb{H}$. Both, $\mathbf{H}_{\hat{\mathbf{D}}}(q)$ and $\Phi_{\hat{\mathbf{C}}}(\tau)$, are called **orbital series** of the sequence $\check{\mathbf{H}}$ of orbital functionals.

Definition 3.3. *The sequence* $\check{\mathbf{H}}$ *of orbital functionals is called* modular *iff for each orbital curve* $\hat{\mathbf{C}}$ *for which* $\check{\mathbf{H}}$ *is applicable, the attached series* $\Phi_{\hat{\mathbf{C}}}(\tau)$ *is a (holomorphic) modular form.*

This means that there is a congruence subgroup Γ of $\mathbb{S}l_2(\mathbb{Z})$ and a positive integer k (weight) such that

$$\Phi_{\hat{\mathbf{C}}}\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \cdot \Phi_{\hat{\mathbf{C}}}(\tau)$$

for all $\gamma = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$.

Remark 3.2. *If* $\check{\mathbf{H}}$ *is a sequence of orbital functionals on a commensurability class* $[\hat{\mathbf{X}}]$ *and* $\hat{\mathbf{C}}$ *is an orbital curve on* $\hat{\mathbf{X}}$*, then the attached orbital series is uniquely determined up to a rational factor by the orbital series of any representative* $\hat{\mathbf{D}}$ *of* $[\hat{\mathbf{C}}]$*. This follows immediately from the relations (*3.1*). Especially, the proof of modularity can be done by checking this property for only one representative* $\Phi_{\hat{\mathbf{D}}}(\tau)$*.*

Convention 3.1. *At the end of this section we explain the use of integral sign* $\int$ *in orbital series as presented in the abstract and introduction. Classical integrals are understood as functionals. Setting*

$$\int\limits_{\hat{\mathbf{C}}} \mathbf{h} := \mathbf{h}(\hat{\mathbf{C}})$$

is only a converse style of writing (until now), in order to present orbital series in a more familiar manner with a glance to Fourier series.

Notation 3.1. *The orbital signature height will be denoted by* $\check{\mathbf{h}}_0$ *instead of* $\check{\mathbf{h}}_\tau$*. Applied to an orbital curve we identify*

$$\mathbf{h}_0(\hat{\mathbf{C}}) = \mathbf{h}_\tau(\hat{\mathbf{C}}) = \int\limits_{\hat{\mathbf{C}}} \mathbf{h}_0$$

The notation indicates that we constructed the constant term of modular orbital series. The next section prepares the construction of higher terms.

4. Orbital Intersection Products

We will now introduce a bilinear symmetric rational intersection product

$$\operatorname{Div}\hat{\mathbf{X}} \times \operatorname{Div}\hat{\mathbf{X}} \to \mathbb{Q}$$

for orbital divisors on orbital surfaces $\hat{\mathbf{X}}$. Via linear extension it suffices to explain $(\hat{\mathbf{C}}\cdot\hat{\mathbf{D}})$ for each pair $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}$ of orbital curves on $\hat{\mathbf{X}}$. Let $\pi\colon \tilde{X} \to \hat{X}$

be a singularity resolution with exceptional divisor $E = E(\pi) = \sum_{i=1}^{s} E_i$ on $\tilde{X}$. The intersection matrix of the irreducible components is negative definite, see [20]. We say that two $\mathbb{Q}$-divisors

$$A, B \in \operatorname{Div}_{\mathbb{Q}} \tilde{X} := \mathbb{Q} \otimes \operatorname{Div} \tilde{X}$$

are orthogonal, iff its intersection product $(A, B)_{\tilde{X}}$ on $\tilde{X}$ is equal to 0. In this case we write $A \perp B$. For an arbitrary (irreducible) curve $\hat{C}$ on $\hat{X}$ we define

$$\pi^{\#}(\hat{C}) = \tilde{C} + \sum_{i=1}^{s} \lambda_i E_i$$

by the orthogonality conditions

$$\pi^{\#}(\hat{C}) \perp E_1, \ldots, E_s$$

and with proper preimage $\tilde{C}$ of $\hat{C}$ on $\tilde{X}$ and uniquely determined rational coefficients λ_i by these conditions. By $\mathbb{Q}$-linear extension we get a $\mathbb{Q}$-linear map

$$\pi^{\#} \colon \operatorname{Div}_{\mathbb{Q}} \hat{X} := \mathbb{Q} \otimes \operatorname{Div} \hat{X} \to \operatorname{Div}_{\mathbb{Q}} \tilde{X}$$

from Weil to Cartier $\mathbb{Q}$-divisors. The rational intersection product of two curves $\hat{C}$, $\hat{D}$ on $\hat{X}$ is defined as

$$(\hat{C} \cdot \hat{D}) = (\hat{C} \cdot \hat{D})_{\hat{X}} := (\pi^{\#}(\hat{C}) \cdot \pi^{\#}(\hat{D})) .$$

Fulton proved in [8], 8.3.11, that this intersection product does not depend on the choice of the singularity resolution π. Mumford used in [20] the minimal singularity resolution. This works for arbitrary normal compact complex algebraic surfaces $\hat{X}$. By obvious extension we dispose on a $\mathbb{Q}$-bilinear symmetric intersection map

$$(. \cdot .)_{\hat{X}} \colon \operatorname{Div}_{\mathbb{Q}} \hat{X} \times \operatorname{Div}_{\mathbb{Q}} \hat{X} \to \mathbb{Q} .$$

If $\varphi \colon X' \to \hat{X}$ is a birational morphism of normal surfaces with exceptional divisor $E = E(\varphi) = \sum_{i=1}^{s} E_i$ on X' we can now extend the above considerations in order to define

$$\varphi^{\#}(\hat{C}) = C' + \sum_{i=1}^{s} \lambda_i E_i$$

by the orthogonality conditions

$$\varphi^{\#}(\hat{C}) \perp E_1, \ldots, E_s .$$

Thereby C' is the proper transform of the curve $\hat{C}$. Ivinskis proved in [16] that the linear extension to $\varphi^{\#} \colon \operatorname{Div}_{\mathbb{Q}} \hat{X} \to \operatorname{Div}_{\mathbb{Q}} X'$ behaves functorially, which means that

$$(\varphi \circ \psi)^{\#} = \psi^{\#} \circ \varphi^{\#}$$

for any birational morphism ψ from another normal surface onto X'. Applied to singularity resolutions ψ we get the compatibility of $\varphi^{\#}$ with the rational intersection products

$$(\varphi^{\#} D_1 \cdot \varphi^{\#} D_2)_{X'} = (D_1 \cdot D_2)_{\hat{X}}, \qquad D_i \in \mathrm{Div}_{\mathbb{Q}} \hat{X},\ i = 1, 2\,.$$

Example 4.1. *A* ***neat orbital surface*** *is an orbital surface with only elliptic singularities, and without basic orbital curves. The exceptional locus* $E(\mu)$ *of the minimal resolution* $\mu\colon Y' \to \hat{Y}$ *of singularities is a disjoint sum* $T = T_1 + \cdots + T_h$ *of elliptic curves. For a curve* $\hat{D}$ *with proper transform* D' *on* Y' *one finds*

$$\mu^{\#}(\hat{D}) = D' - \frac{(D' \cdot T_1)}{(T_1^2)} T_1 - \cdots - \frac{(D' \cdot T_h)}{(T_h^2)} T_h =: D' + D^{\infty} \qquad (4.1)$$

because this rational divisor is orthogonal to $T_1, \ldots, T_h$. *Since* $\mu^{\#} D$ *is orthogonal to* $T_1, \ldots, T_h$, *the intersection product with* $\mu^{\#}(\hat{C})$ *for a curve* $\hat{C}$ *on* $\hat{Y}$ *is*

$$\begin{aligned}(\hat{C} \cdot \hat{D}) &= (\mu^{\#}\hat{C} \cdot \mu^{\#}\hat{D}) = ((C' + C^{\infty}) \cdot (D' + D^{\infty})) \\ &= (C' \cdot (D' + D^{\infty})) = (C' \cdot D') + (C' \cdot D^{\infty}) \\ &= (C' \cdot D') - \frac{(C' \cdot T_1)(D' \cdot T_1)}{(T_1^2)} - \cdots - \frac{(C' \cdot T_h)(D' \cdot T_h)}{(T_h^2)}\,.\end{aligned}$$

Especially, we obtain

$$(\hat{C} \cdot \hat{C}) = (C' \cdot C') - k_1^2/s_1 - \cdots - k_h^2/s_h$$

with

$$k_i = k_i(C) := \# C \cap T_i = (C' \cdot T_i), \qquad s_i := (T_i^2)\,.$$

These formulas are valid on the minimal resolution Y' *of singularities of any Picard modular surface* $\hat{Y} = \widehat{\mathbb{B}/\Gamma}$ *with neat* $\mathbb{B}$*-lattice* Γ *and singularity resolving compactification divisor* $T = T_1 + \cdots + T_h$.

Now we come back to the orbital surfaces $\hat{\mathbf{X}}$. Take two orbital curves $\hat{\mathbf{C}}$, $\hat{\mathbf{D}}$ on it and set

$$(\hat{\mathbf{C}} \cdot \hat{\mathbf{D}}) = (\hat{\mathbf{C}} \cdot \hat{\mathbf{D}})_{\hat{\mathbf{X}}} := \frac{1}{v_{\hat{\mathbf{C}}} v_{\hat{\mathbf{D}}}} (\hat{C} \cdot \hat{D})_{\hat{X}}\,.$$

We extend also this **orbital intersection product** to the symmetric bilinear **orbital intersection map**

$$(.\cdot .)_{\hat{\mathbf{X}}}\colon\ \mathrm{Div}\, \hat{\mathbf{X}} \times \mathrm{Div}\, \hat{\mathbf{X}} \to \mathbb{Q}\,.$$

Proof: We need some functorial properties. Let $\hat{p}\colon \hat{Y} \to \hat{X}$ be a finite covering of degree $\deg \hat{p} = [\hat{Y} : \hat{X}]$ of normal surfaces, $\hat{C}_1$, $\hat{C}_2$ two (irreducible) curves on $\hat{X}$ and $\hat{p}^*\hat{C}_1$, $\hat{p}^*\hat{C}_2$ their inverse images on $\hat{Y}$. Then the degree formula

$$(\hat{p}^*\hat{C}_1 \cdot \hat{p}^*\hat{C}_2) = [\hat{Y} : \hat{X}] \cdot (\hat{C}_1 \cdot \hat{C}_2) \tag{4.2}$$

is valid. The proof can be found in [16], p. 38. The formula extends bilinearly to any pair of (Weil) $\mathbb{Q}$-divisors on $\hat{X}$. We remember that

$$\hat{p}^*\hat{C} = \sum_{\hat{D}_i \to \hat{C}} v_i \hat{D}_i = \sum_{i=1}^{h} v_i \hat{D}_i \tag{4.3}$$

where v_i is the ramification index of $\hat{p}$ at the irreducible component $\hat{D}_i$ of $\hat{p}^*\hat{C}$. It holds that

$$[\hat{Y} : \hat{X}] = d_1 v_1 + d_2 v_2 + \cdots + d_h v_h, \qquad d_i := [\hat{D}_i : \hat{C}]. \tag{4.4}$$

As a corollary one gets the *projection formula*

$$(\hat{p}_{\#}\hat{D} \cdot \hat{C}') = (\hat{D} \cdot \hat{p}^*\hat{C}')$$

for curves $\hat{C} = f(\hat{D})$, $\hat{C}' = f(D')$ on $\hat{X}$ and $\hat{D}$, D' on $\hat{Y}$. Thereby the direct image is defined by

$$\hat{p}_{\#}\hat{D} := [\hat{D} : f(\hat{D})] f(\hat{D}) = [\hat{D} : \hat{C}] \cdot \hat{C}\,. \tag{4.5}$$

The projection formula is well-known for divisors on smooth surfaces. For the sake of completeness we prove it for Galois coverings $\hat{p}\colon \hat{Y} \to \hat{X} = \hat{Y}/G$, $G = \mathrm{Gal}(Y/X)$. Then we have $v = v_i$ in (4.3), $d = d_i$ in (4.4), hence $[\hat{Y} : \hat{X}] = vdh$. From the degree formula (4.2) we get

$$[\hat{Y}:\hat{X}](\hat{C}\cdot\hat{C}') = (\hat{p}^*\hat{C}\cdot\hat{p}^*\hat{C}') = v(\hat{D}_1\cdot\hat{p}^*\hat{C}') + \cdots + v(\hat{D}_h\cdot\hat{p}^*\hat{C}') = vh(\hat{D}\cdot\hat{p}^*\hat{C}').$$

We used the G-invariance of $\hat{p}^*C'$, $\{\hat{D}_i;\, i = 1, \ldots, h\}$ and of the intersection product on $\hat{Y}$. On the other hand the Definition (4.5) of direct images yields

$$[\hat{Y} : \hat{X}](\hat{C} \cdot \hat{C}') = \frac{vdh}{d}(\hat{p}_{\#}\hat{D} \cdot \hat{C}')\,.$$

Now the projection formula follows by comparision. □

Now we define for orbital finite surface coverings $\hat{\mathbf{p}}\colon \hat{\mathbf{Y}} \to \hat{\mathbf{X}}$ the **orbital preimage**

$$\hat{\mathbf{p}}^{-1}(\hat{\mathbf{C}}) := \hat{\mathbf{D}}_1 + \cdots + \hat{\mathbf{D}}_h$$

of orbital curves on $\hat{\mathbf{X}}$ with the notations of (4.3).

If $\hat{p}$ is a Galois covering, then $v_i = v(\hat{p}) = v$, $i = 1, \ldots, h$ in (4.3). Uniformizing $\hat{\mathbf{p}}$, which is possible by definition of finite orbital coverings, we see that $v_{\hat{\mathbf{C}}} = v \cdot v_{\hat{\mathbf{D}}}$. Therefore the identity

$$\hat{p}^* \hat{C} = \frac{v_{\hat{\mathbf{C}}}}{v_{\hat{\mathbf{D}}}} \sum_{i=1}^{h} \hat{D}_i = \frac{v_{\hat{\mathbf{C}}}}{v_{\hat{\mathbf{D}}}} \hat{p}^{-1} \hat{C}$$

holds in all Galois cases. We give the following orbital version of the projection formula

Proposition 4.1. *For the finite orbital covering* $\hat{\mathbf{p}}\colon \hat{\mathbf{Y}} \to \hat{\mathbf{X}}$ *supporting the orbital curve covering* $\hat{\mathbf{D}} \to \hat{\mathbf{C}}$ *it holds that*

$$(\hat{\mathbf{D}} \cdot \hat{\mathbf{p}}^{-1} \hat{\mathbf{C}}') = [\hat{\mathbf{D}} : \hat{\mathbf{C}}](\hat{\mathbf{C}} \cdot \hat{\mathbf{C}}') \tag{4.6}$$

for each orbital curve $\hat{\mathbf{C}}'$ *on* $\hat{\mathbf{X}}$.

Proof: If $\hat{\mathbf{p}}$ is supported by a Galois covering $\hat{p}$, then

$$\begin{aligned}
(\hat{\mathbf{D}} \cdot \hat{\mathbf{p}}^{-1} \hat{\mathbf{C}}') &= \frac{1}{v_{\hat{\mathbf{D}}} v_{\hat{\mathbf{D}}'}} (\hat{D} \cdot \hat{p}^{-1} \hat{C}') = \frac{1}{v_{\hat{\mathbf{D}}} v_{\hat{\mathbf{C}}'}} \left(\hat{D} \cdot \frac{v_{\hat{\mathbf{C}}'}}{v_{\hat{\mathbf{D}}'}} \hat{p}^{-1} \hat{C}' \right) \\
&= \frac{1}{v_{\hat{\mathbf{D}}} v_{\hat{\mathbf{C}}'}} (\hat{D} \cdot \hat{p}^* \hat{C}') = \frac{[\hat{D} : \hat{C}]}{v_{\hat{\mathbf{D}}} v_{\hat{\mathbf{C}}'}} (\hat{C} \cdot \hat{C}') \\
&= \frac{[\hat{\mathbf{D}} : \hat{\mathbf{C}}]}{v_{\hat{\mathbf{C}}} v_{\hat{\mathbf{C}}'}} (\hat{C} \cdot \hat{C}') = [\hat{\mathbf{D}} : \hat{\mathbf{C}}](\hat{\mathbf{C}} \cdot \hat{\mathbf{C}}') \,.
\end{aligned}$$

Now let $\hat{\mathbf{p}}$ be an arbitrary finite orbital covering. We take uniformizations

$$\hat{\mathbf{u}} : \hat{\mathbf{Z}} \xrightarrow{\hat{\mathbf{q}}} \hat{\mathbf{Y}} \xrightarrow{\hat{\mathbf{p}}} \hat{\mathbf{X}}, \qquad \hat{\mathbf{E}} \to \hat{\mathbf{D}} \to \hat{\mathbf{C}}$$

For $\hat{\mathbf{u}}$ and $\hat{\mathbf{q}}$ we are in the Galois situation. Therefore

$$\begin{aligned}
(\hat{\mathbf{E}} \cdot \hat{\mathbf{u}}^{-1} \hat{\mathbf{C}}') &= [\hat{\mathbf{E}} : \hat{\mathbf{C}}](\hat{\mathbf{C}} \cdot \hat{\mathbf{C}}') \\
(\hat{\mathbf{E}} \cdot \hat{\mathbf{q}}^{-1} (\hat{\mathbf{p}}^{-1} \hat{\mathbf{C}}')) &= [\hat{\mathbf{E}} : \hat{\mathbf{D}}](\hat{\mathbf{D}} \cdot \hat{\mathbf{p}}^{-1} \hat{\mathbf{C}}') \,.
\end{aligned}$$

The left-hand sides coincide and

$$\frac{[\hat{\mathbf{E}} : \hat{\mathbf{D}}]}{[\hat{\mathbf{E}} : \hat{\mathbf{C}}]} = [\hat{\mathbf{D}} : \hat{\mathbf{C}}] \,.$$

Now (4.6) follows immediately. □

Let us write $\hat{\mathbf{p}}^{\#}$ for the linear extension of $\hat{\mathbf{p}}^{-1}$ from orbital curves to the orbital divisor groups. We dispose on linear homomorphisms

$$\hat{\mathbf{p}}^{\#}\colon \operatorname{Div}\hat{\mathbf{X}} \to \operatorname{Div}\hat{\mathbf{Y}}, \qquad \hat{\mathbf{p}}_{\#}\colon \operatorname{Div}\hat{\mathbf{Y}} \to \operatorname{Div}\hat{\mathbf{X}}$$

with nice functorial behaviour. Namely, for $\mathbf{A} \in \operatorname{Div}\hat{\mathbf{X}}$ and $\hat{\mathbf{D}} \in \operatorname{Div}\hat{\mathbf{Y}}$ the relations (4.6) extend bilinearly to the
Orbital Projection Formula:

$$(\hat{\mathbf{p}}_{\#}\hat{\mathbf{D}} \cdot \mathbf{A})_{\hat{\mathbf{X}}} = (\hat{\mathbf{D}} \cdot \hat{\mathbf{p}}^{\#}\mathbf{A})_{\hat{\mathbf{Y}}}\,. \tag{4.7}$$

Theorem 4.1. *Let $\hat{\mathbf{X}}$ be an orbital surface. Each orbital divisor $\mathbf{A}$ on it defines an orbital functional $\check{\mathbf{h}}_{\mathbf{A}}$ on the relative category $\mathbf{OrSf}_{\hat{\mathbf{X}}}$ of all orbital surfaces $\hat{\mathbf{Y}}$ covering $\hat{\mathbf{X}}$. It extends linearly the basic correspondence*

$$\hat{\mathbf{D}} \mapsto \mathbf{h}_{\mathbf{A}}(\hat{\mathbf{D}}) := (\hat{\mathbf{D}} \cdot \hat{\mathbf{p}}^{\#}\mathbf{A})_{\hat{\mathbf{Y}}}$$

for orbital curves $\hat{\mathbf{D}} \subset \hat{\mathbf{Y}}$ along orbital finite surface coverings $\hat{\mathbf{p}}\colon \hat{\mathbf{Y}} \to \hat{\mathbf{X}}$.

5. Arithmetic Orbital Divisors

Let K be a fixed imaginary quadratic number field with ring of integers $\mathfrak{O} = \mathfrak{O}_K$. A *Picard lattice* (over $\mathfrak{O}$) is a hermitian $\mathfrak{O}$-lattice Λ with a hermitian form $\langle .\,, .\rangle\colon \Lambda \times \Lambda \to \mathfrak{O}$ of signature (2, 1). A *Picard modular group* corresponding to Λ is a subgroup Γ of $\operatorname{Aut} V$ commensurable with the automorphism group $\Gamma_1 := \Gamma_1(\Lambda)$ of Λ, called the *full Picard modular group* of Λ. The lattice defines the the hermitian K- or $\mathbb{C}$-vector spaces $V := \mathbb{Q} \otimes \Lambda$ or $V_{\mathbb{R}} := \mathbb{R} \otimes V$, respectively, isomorphic to K^3 respectively $\mathbb{C}^3$ (forgetting the hermitian structure). Through the paper we will use the notations

$$\begin{aligned}
\Lambda^- &:= \{\lambda \in \Lambda; \langle\lambda,\lambda\rangle < 0\}, \quad \Lambda^+ := \{\lambda \in \Lambda; \langle\lambda,\lambda\rangle > 0\}\\
\Lambda^0 &:= \{\lambda \in \Lambda; \langle\lambda,\lambda\rangle = 0\}\\
V^- &:= \{v \in V; \langle v,v\rangle < 0\}, \quad V^+ := \{v \in V; \langle v,v\rangle > 0\}\\
V^0 &:= \{v \in V; \langle v,v\rangle = 0\}\\
V_{\mathbb{R}}^- &:= \{v \in V_{\mathbb{R}}; \langle v,v\rangle < 0\}, \quad V_{\mathbb{R}}^+ := \{v \in V_{\mathbb{R}}; \langle v,v\rangle > 0\}\\
V_{\mathbb{R}}^0 &:= \{v \in V_{\mathbb{R}}; \langle v,v\rangle = 0\}\,.
\end{aligned}$$

The corresponding elements are called *negative*, *positive*, or *isotropic*, respectively. Projectivising we get embeddings

$$\mathbb{B} := \mathbb{P}V_{\mathbb{R}}^- \subset \mathbb{P}V_{\mathbb{R}} = \mathbb{P}^2, \qquad \partial\mathbb{B} = \mathbb{P}V_{\mathbb{R}}^0 \subset \mathbb{P}^2.$$

The elements of

$$\partial_K\mathbb{B} := \mathbb{P}^2(K) \cap \partial\mathbb{B}$$

are called *K-rational boundary points* of $\mathbb{B}$. $\mathbb{B}$ is isomorphic to the standard complex unit ball $\{(z_1, z_2) \in \mathbb{C}^2;\ |z_1|^2 + |z_2|^2 < 1\}$. The elements of the Picard modular group $\Gamma \subset \mathbb{U}(V_\mathbb{R}) \cong \mathbb{U}((2,1),\mathbb{C})$ act on $\mathbb{B}$ by fractional linear transformations via embedding

$$\mathbb{P}\Gamma \subset \mathbb{PG}l(V_\mathbb{R}) \cong \mathbb{PG}l_3(\mathbb{C})\,.$$

This action is properly discontineous. The Picard modular groups Γ are arithmethmetic ball lattices. The quotient surface $X_\Gamma := \Gamma\backslash\mathbb{B}$ and its Baily–Borel compactification $\hat{X}_\Gamma$ is called *Picard modular surfaces* of Γ. The compactification locus consists of finitely many normal points coming from rational boundary points, precisely

$$\hat{X}_\Gamma = X_\Gamma \sqcup \Gamma\backslash\partial_K\mathbb{B}\,.$$

Endowed with the compactified branch divisor of the infinite locally finite covering $p_\Gamma\colon \mathbb{B} \to X_\Gamma$ we get the *orbital Picard surfaces* $\mathbf{X}_\Gamma$ and $\hat{\mathbf{X}}_\Gamma$. Each sublattice Γ' of Γ induces an orbital finite covering $\hat{\mathbf{X}}_{\Gamma'} \to \hat{\mathbf{X}}_\Gamma$. If Γ' is a neat sublattice of Γ, then we write $\hat{X}_{\Gamma'}$ instead of $\hat{\mathbf{X}}_{\Gamma'}$ because $p_{\Gamma'}$ is a universal covering, which has no ramification. If, moreover, Γ' is a normal subgroup of Γ, then $\hat{X}_{\Gamma'} \to \hat{\mathbf{X}}_\Gamma$ is a finite uniformization with Galois group $G = \Gamma/\Gamma'$.

Let L be a line in $\mathbb{P}^2$ defined over K (K-line). A **K-disc** $\mathbb{D}$ on $\mathbb{B}$ is a non-void intersection of $\mathbb{B}$ with a K-line. The group $N_\Gamma(\mathbb{D})$ of all elements of Γ acting on $\mathbb{D}$ is an arithmetic $\mathbb{D}$-lattice. Conversely, each linear subdisc $\mathbb{D}$ of $\mathbb{B}$, for which $N_\Gamma(\mathbb{D})$ is a $\mathbb{D}$-lattice, must be a K-disc. An **arithmetic curve** on $\hat{X}_\Gamma$ is the closure $\widehat{\Gamma\backslash\mathbb{D}}$ of a quotient curve $\Gamma\backslash\mathbb{D} \subset X_\Gamma$, $\mathbb{D}$ a K-disc. The corresponding **orbital arithmetic curve** is denoted by $\widehat{\mathbf{\Gamma}\backslash\mathbb{D}}$. The notations are justified by the following

Proposition 5.1. *Each arithmetic curve $\hat{C} = \widehat{\Gamma\backslash\mathbb{D}}$ on $\hat{X}_\Gamma$ has a $\widehat{\Gamma\backslash\mathbb{D}}$-uniformization realized by a surface $\hat{X}_{\Gamma'}$ with a suitable neat normal sublattice Γ' of Γ. Therefore arithmetic curves are orbital in the global sense.*

Proof: This has been proved in [12], Prop. 4.4.12 Namely, we constructed there $\mathbb{D}$-neat ball lattices Γ' by means of principal congruence subgroups. The curve $\hat{D} = \widehat{\Gamma'\backslash\mathbb{D}}$ satisfies the conditions 2.2 by definitions. □

The ball $\mathbb{B}$ has a hermitian metric with negative constant holomorphic sectional curvature (hyperbolic, Bergman metric). For the explicit construction we refer to [1]. The above subdiscs are geodesics. These structures go down to X_Γ and $\Gamma\backslash\mathbb{D}$, if Γ is neat and the curve smooth. In general we have to move some curves and points (branch locus, degeneration locus) from X_Γ to preserve this nice metric together with the geodesic property of the embedded quotient curve.

We say that X_Γ has a quasi-hyperbolic structure and $\Gamma\backslash\mathbb{D}$ is quasi-geodesic, in general. There exists a finite covering with complete hyperbolic structure and complete geodesic covering of $\Gamma\backslash\mathbb{D}$.
The orbital curves have moduli but the arithmetic curves are rigid by the arithmetic nature of definition: you cannot move K-discs on $\mathbb{B}$ without leaving this set.

Definition 5.1. *The group of* orbital arithmetic divisors $\mathrm{Div}^{ar}\,\hat{\mathbf{X}}_\Gamma$ *is the free abelian subgroup of* $\mathrm{Div}\,\hat{\mathbf{X}}_\Gamma$ *generated by all orbital arithmetic curves on* $\hat{\mathbf{X}}_\Gamma$.

Theorem 5.1. *Let $\check{h}$ be the signature functional of the divisor functor on* $\mathbf{OrSf}$ *and $\hat{\mathbf{C}} = \widehat{\Gamma\backslash\mathbb{D}}$ the orbital arithmetic curve on $\hat{\mathbf{X}}_\Gamma$ of the K-disc $\mathbb{D} \subset \mathbb{B}$. The signature height of $\hat{\mathbf{C}}$ is the half of the Euler–Poincarè volume of a $\Gamma_\mathbb{D}$-fundamental domain on $\mathbb{D}$:*

i) $h_\tau(\hat{\mathbf{C}}) = \frac{1}{2}\,\mathrm{vol}_{EP}(\Gamma_\mathbb{D}) < 0$

ii) *The orbital signature height is* $\mathbf{h_0}(\hat{\mathbf{C}}) = \frac{1}{2v_{\hat{\mathbf{C}}}}\,\mathrm{vol}_{EP}(\Gamma_\mathbb{D}) < 0$.

Proof: From the first formula the second follows by definition of the orbital signature. For the first we dispose on a Proportionality Theorem characterizing orbital ball quotients and orbital disc quotients $\hat{\mathbf{C}}$ on them, see [12], Chapter IV, Theorem 4.9.2. In this monograph we introduced also orbital Euler heights h_e for orbital curves denoted by $\mathbf{e_f}$ there. Then the (Prop 1)-part of the theorem says in our terms that $h_e(\hat{\mathbf{C}}) = 2h_\tau(\hat{\mathbf{C}}) < 0$. Moreover, we know from [12], Prop. 4.7.4, that $h_e(\hat{\mathbf{C}}) = \mathrm{vol}_{EP}(\Gamma_\mathbb{D}) < 0$. □

Now we restrict our orbital divisor functor to the subcategory $\mathbf{OrSf}(\Lambda)$ of orbital Picard surfaces of a fixed Picard-lattice Λ. In this category we do not allow other morphisms than finite orbital coverings $\hat{\mathbf{p}}\colon \hat{\mathbf{X}}_{\Gamma'} \to \hat{\mathbf{X}}_\Gamma$ corresponding to Picard modular groups $\Gamma' \subset \Gamma$ of Λ. The main purpose for the restriction to $\mathbf{OrSf}(\Lambda)$ is to get more orbital functionals $\check{\mathbf{h}}$ defined only there, not extendable to $\mathbf{OrSf}$. We will call them *arithmetic orbital functionals*.

6. Orbital Heegner Series

Let $\mathbb{D} = \mathbb{B} \cap L$ be the K-disc with K-line L on $\mathbb{P}^2$. Each K-line is the projectivization $L_\mathfrak{a} := \mathbb{P}\mathfrak{a}^\perp$, $\mathfrak{a}^\perp$ the (indefinite) orthogonal complementary subspace of $\mathbb{C}\mathfrak{a}$ in $V_\mathbb{R}$, where $\mathfrak{a}$ belongs to V^+, see Section 5. All elements of the K-line $K\mathfrak{a}$ define the same K-line $L_\mathfrak{a}$ and the same K-disc $\mathbb{D} = \mathbb{D}_\mathfrak{a} = \mathbb{B} \cap L_\mathfrak{a}$. So we can and will assume that $\mathfrak{a} \in \Lambda^+$. We fix Λ and set for positive integers N

$$\Lambda^{(N)} := \{\lambda \in \Lambda;\ \langle \lambda, \lambda \rangle = N\}$$
$$\mathcal{D}^{(N)} := \{\mathbb{D}_\mathfrak{a}\,;\ \mathfrak{a} \in \Lambda^{(N)}\}\,.$$

Definition 6.1. *The N-th* orbital Heegner divisor *on $\hat{X}_\Gamma$, is the orbitalized reduced Weil divisor*

$$\mathbf{H}_N(\Gamma) = \mathbf{H}_N(\Lambda, \Gamma) := \sum_{\mathcal{D}^{(N)} \ni \mathbb{D} \bmod \Gamma} \widehat{\Gamma \backslash \mathbb{D}}. \tag{6.1}$$

For neat lattices Γ' it is the same to write

$$H_N(\Gamma') = \mathbf{H}_N(\Gamma') = \sum_{\Lambda^{(N)} \ni \mathfrak{a} \bmod \Gamma'} \widehat{\Gamma' \backslash \mathbb{D}_\mathfrak{a}} \tag{6.2}$$

because $\gamma(\mathfrak{a}) = c \cdot \mathfrak{a}$, $\gamma \in \Gamma$, $c \in \mathbb{C}$, implies $c = 1$. Namely, c must be a unit root in this case. But a non-trivial unit root cannot be an eigenvalue of an element of Γ by the definition of neat groups. Therefore $\mathfrak{a}$ and $\mathfrak{b}$ are Γ-equivalent iff $\mathbb{D}_\mathfrak{a}$ and $\mathbb{D}_\mathfrak{b}$ are.

For each sublattice Γ' of Γ and the corresponding finite covering $p\colon \hat{X}'_\Gamma \to \hat{X}_\Gamma$ it holds that

$$\mathbf{H}_N(\Gamma') = \mathbf{p}^\# \mathbf{H}_N(\Gamma) \tag{6.3}$$

because $\mathbf{p}^\#$ is the linear extension of $\mathbf{p}^{-1}$.

Theorem 6.1. *The orbital intersection functionals*

$$\mathbf{h}_{N,\Gamma}\colon \operatorname{Div} \hat{\mathbf{X}}_\Gamma \to \mathbb{Q}, \qquad \mathbf{A} \mapsto \int_{\mathbf{A}} \mathbf{h}_N = (\mathbf{A} \cdot \mathbf{H}_N(\Gamma))$$

form an orbital functional $\check{\mathbf{h}} = \int \mathbf{h}$ on $\mathbf{OrSf}(\Lambda)$.

Proof: As in the proof of Theorem 4.1 we use the Orbital Projection Formula (4.7) in order to check the $\mathbf{p}_\#$-compatibility for finite coverings p as above. With (6.3) one really gets

$$(\mathbf{p}_\# \hat{\mathbf{D}} \cdot \mathbf{H}_N(\Gamma)) = (\hat{\mathbf{D}} \cdot \mathbf{p}^\# \mathbf{H}_N(\Gamma)) = (\hat{\mathbf{D}} \cdot \mathbf{H}_N(\Gamma'))$$

for each $\hat{\mathbf{D}} \in \operatorname{Div} \hat{X}_{\Gamma'}$. □

The orbital functional $\check{\mathbf{h}}$ of Theorem 6.1 with the characteristic property

$$\mathbf{h}_N(\hat{\mathbf{D}}) = [\hat{\mathbf{D}} : \hat{\mathbf{C}}] \cdot \mathbf{h}_N(\hat{\mathbf{C}}). \tag{6.4}$$

for orbital curve coverings $\hat{\mathbf{D}} \to \hat{\mathbf{C}}$ is called the *orbital Heegner functional* on $\mathbf{OrSf}(\Lambda)$.

Together with the orbital signature functional $\check{\mathbf{h}}_0$ we dispose now on a well-defined sequence $\check{\mathbf{H}}$ of orbital Heegner functionals $\check{\mathbf{h}}_N$, $N = 0, 1, 2, 3, \ldots$ on each category $\mathbf{OrSf}(\Lambda)$ of orbital Picard surfaces.

Remark 6.1. *Observe that the N-th orbital Heegner divisors and functionals for $N > 0$ depend on the norm sets of K-discs on $\mathbb{B}$. These norm sets have been choosen by means of the hermitian lattice Λ we started with in Section 5. So we should write more precisely $\check{\mathbf{h}}_{N,\Lambda}$ instead of $\check{\mathbf{h}}_N$ only. But we will fix K and Λ and keep it in mind in order to simplify the notations.*

Definition 6.2. *We call*

$$\check{\mathbf{H}}(q) = \check{\mathbf{H}}^{\Lambda}(q) := \sum_{N=0}^{\infty} \check{\mathbf{h}}_N q^N = \check{\mathbf{h}}_0(\mathbf{A}) + \sum_{N=1}^{\infty} q^N \int \mathbf{h}_N$$

the orbital Heegner series *of the Picard lattice Λ. The series*

$$\mathrm{Heeg}_{\mathbf{A}}(q) = \mathbf{h}_0(\mathbf{A}) + \sum_{N=1}^{\infty} q^N \int_{\mathbf{A}} \mathbf{h}_N$$

is called the orbital Heegner series *of the orbital divisor $\mathbf{A} \in \mathrm{Div}\,\hat{\mathbf{X}}_\Gamma$.*

The orbital degree formula (3.1) for sequences of orbital functionals specializes to

$$\mathrm{Heeg}_{\hat{\mathbf{D}}}(q) = [\mathbf{D} : \mathbf{C}] \cdot \mathrm{Heeg}_{\hat{\mathbf{C}}}(q) \tag{6.5}$$

with the notations of (6.4).

Substituting $q = e^{2\pi i\tau}$, we ask for orbital divisors $\mathbf{A}$ producing holomorphic functions $\Phi_{\mathbf{A}}(\tau) = \mathrm{Heeg}_{\mathbf{A}}(e^{2\pi i\tau})$ on the upper half plane $\mathbb{H}$ and their properties. We need the most familiar cases of hermitian lattices described in the following

Definitions 6.1. *The* norm $\mathfrak{n}(\Lambda)$ *of an hermitian lattice Λ over $\mathfrak{O}_K$ is the additive subgroup of $\mathbb{Z}$ generated by all elements $\langle\lambda, \lambda\rangle$, $\lambda \in \Lambda$. A* maximal *lattice is maximal in the set of $\mathfrak{O}$-sublattices of $V = K \otimes \Lambda$ with the same norm. A* ***$\mathbb{Z}$-maximal lattice*** *is a maximal $\mathfrak{O}$-lattice Λ with norm $\mathfrak{n}(\Lambda) = \mathbb{Z}$. We say that Λ belongs to a* unimodular class *iff Λ is commensurable with a unimodular hermitian $\mathfrak{O}_K$-lattice.*

The discriminant of the quadratic extension $K/\mathbb{Q}$ is denoted by $D_{K/\mathbb{Q}}$. It defines the Dirichlet character $\chi_K := \left(\frac{D_{K/\mathbb{Q}}}{\cdot}\right)$ on $\mathbb{Z}$. The modular group $\mathbb{S}l_2(\mathbb{Z})$ is denoted by G. For $0 \neq m \in \mathbb{N}_+$ we dispose on congruence subgroups

$$G_0(m) := \{\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G;\, c \equiv 0 \mod m\}.$$

The vector space of $G_0(m)$-modular forms on $\mathbb{H}$ of wight k and Nebentypus χ_K is denoted by $\mathcal{M}_k(m, \chi_K)$. It consists of all holomorphic functions $f(\tau)$

satisfying the functional equations

$$f\left(\frac{a\tau+b}{c\tau+d}\right) = (c\tau+d)^k \chi_K(d)^k f(\tau) \tag{6.6}$$

for all elements $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in G_0(m)$.

Main Theorem 6.1. *Let Λ be a $\mathbb{Z}$-maximal Picard lattice over $\mathfrak{O}_K$ in a unimodular class, $\mathbf{OrSf}(\Lambda)$ the corresponding orbital category of Picard surfaces $\hat{\mathbf{X}}_\Gamma$ and $\mathbf{A} \in \mathrm{Div}^{ar}\,\hat{\mathbf{X}}_\Gamma$. Then the orbital Heegner series $\Phi_{\mathbf{A}}(\tau)$ belongs to $\mathcal{M}_3(D_{K/\mathbb{Q}}, \chi_K)$ and has $\mathbb{Q}$-rational coefficients.*

Proof: The bilinearity of orbital intersection products yields $\Phi_{\mathbf{A}_1+\mathbf{A}_2}(\tau) = \Phi_{\mathbf{A}_1}(\tau) + \Phi_{\mathbf{A}_2}(\tau)$ for $\mathbf{A}_1, \mathbf{A}_2 \in \mathrm{Div}^{ar}\,\hat{\mathbf{X}}_\Gamma$. Therefore it suffices to check that $\Phi_{\hat{\mathbf{C}}}(\tau)$ is a modular form of the announced type for all arithmetic curves $\hat{\mathbf{C}}$ on $\hat{\mathbf{X}}$. Because of the direct image compatibility (6.5) along orbital coverings $\hat{\mathbf{X}}_{\Gamma'} \to \hat{\mathbf{X}}_\Gamma$ the problem is reduced to arithmetic orbital curves $\hat{D} = \hat{\mathbf{D}} = \widehat{\Gamma'\backslash\mathbb{D}} \subset \hat{X}_{\Gamma'} = \hat{\mathbf{X}}_{\Gamma'}$ for neat congruence subgroups Γ' of Γ_1. It suffices even to prove the modular property for only one neat Picard modular group Γ' because all Picard modular groups of Λ are commensurable with each other, and a neat one exists. This will be done in Section 8 restricting to neat natural congruence subgroups. Originally, this proof is due to Cogdell [3, 5]. We give only a simplified outline of it. □

7. Leading Example

We want to illustrate the use of our geometric method to get an explicit modular Heegner series. Comparision with the analytic method yields interesting formulas for elementary arithmetic functions sitting in the Fourier coefficients. Let $\mathfrak{O} = \mathbb{Z}[\mathrm{i}]$ be the ring of Gauß integers with discriminant $\delta = 2\mathrm{i}$ and $\Lambda' = \mathfrak{O}^3$ the unimodular hermitian lattice with metric represented by the diagonal matrix $\left(\begin{smallmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$. With $\Gamma_1' := \Gamma_1(\Lambda') =: SU((2,1), \mathfrak{O}) \cong \mathbb{P}(\mathrm{Aut}\,\Lambda')$ we define the congruence subgroup $\Gamma = \Gamma_1'(1+\mathrm{i})$ with respect to the $\mathfrak{O}$-ideal $(1+\mathrm{i})$. Unfortunately, Λ' is not $\mathbb{Z}$-maximal, but has the $\mathbb{Z}$-maximal extension

$$\Lambda := \mathfrak{O}\begin{pmatrix} 0 \\ -(1+\mathrm{i})/2 \\ (1-\mathrm{i})/2 \end{pmatrix} + \mathfrak{O}\begin{pmatrix} 1 \\ 1 \\ \mathrm{i} \end{pmatrix} + \mathfrak{O}\begin{pmatrix} (1-\mathrm{i})/2 \\ 0 \\ (1+\mathrm{i})/2 \end{pmatrix} \tag{7.1}$$

with skew diagonal Gram matrix

$$\begin{pmatrix} 0 & 0 & \delta^{-1} \\ 0 & 1 & 0 \\ \bar{\delta}^{-1} & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 & -\frac{\mathrm{i}}{2} \\ 0 & 1 & 0 \\ \frac{\mathrm{i}}{2} & 0 & 0 \end{pmatrix}. \tag{7.2}$$

corresponding to the Witt basis presented in (7.1). So Γ belongs to the commensurability class of $\Gamma_1(\Lambda)$, and our theory is applicable to this ball lattice. Consider the three norm-1 vectors

$$\mathfrak{c}_0 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \qquad \mathfrak{c}_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \qquad \mathfrak{l}_1 = \begin{pmatrix} (1+\mathrm{i})/2 \\ (1-\mathrm{i})/2 \\ 0 \end{pmatrix} \in \Lambda\,.$$

The images of the corresponding discs $\mathbb{D}_{\mathfrak{c}_0}$, $\mathbb{D}_{\mathfrak{c}_1}$, $\mathbb{D}_{\mathfrak{l}_1}$ are denoted by C_0, C_1, L_1, respectively. In [14] we proved that the complex projective plane is the Baily–Borel compactification of $\Gamma\backslash\mathbb{B}$. Moreover, there are precisely three cusp points K_1, K_2, K_3. The (compactified) branch divisor of p_Γ is supported by the quadric $\hat{C}_0$ and three tangents $\hat{C}_1$, $\hat{C}_2$, $\hat{C}_3$ as drawn in the following figure.

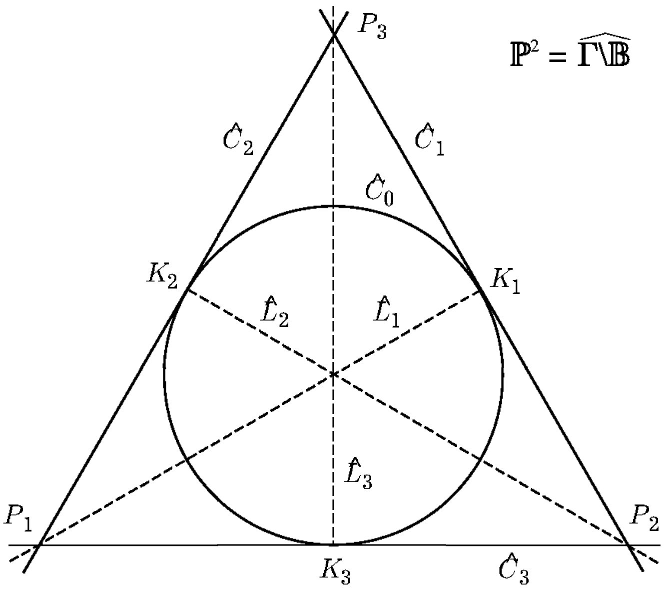

Figure 1

The quotent group $\Gamma\backslash\Gamma_1(\mathfrak{O}^3)$ is isomorphic to the symmetric group S_3. It acts in geometrically obvious manner effectively on $\mathbb{P}^2$ and on the configuration in Fig. 1.

The branch weights $v_{\hat{\mathbf{C}}_i}$ of the orbital curves $\hat{\mathbf{C}}_i$, $i = 0, 1, 2, 3$, are equal to 4. The seven irreducible curves of the configuration are all disc quotients with norm 1. Using the geometric formulas for signature heights in [12], originally in [15], we calculated

$$h_0(\hat{\mathbf{C}}_0) = h_\tau(\hat{\mathbf{C}}_0) = \frac{1}{4}[(C_0'^2) + 0 + 0] = -\frac{1}{2}, \quad \mathbf{h}_0(\hat{\mathbf{C}}_0) = -\frac{1}{8}\,.$$

Moreover, the first orbital Heegner divisor is

$$\mathbf{H}_1 = \hat{\mathbf{C}}_0 + \hat{\mathbf{C}}_1 + \hat{\mathbf{C}}_2 + \hat{\mathbf{C}}_3 + \hat{\mathbf{L}}_1 + \hat{\mathbf{L}}_2 + \hat{\mathbf{L}}_3$$

on the orbitalized projective plane $\mathbf{P}^2 = \widehat{\mathbf{\Gamma}\backslash\mathbb{B}}$. Since the orbital intersection product on $\mathbf{P}^2$ is supported by the usual intersection product for curves on the plane it is not difficult to calculate

$$\mathbf{h}_1(\hat{\mathbf{C}}_0) = (\hat{C}_0 \cdot \mathbf{H}_1) = \left(1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + 2 + 2 + 2\right) = \frac{17}{2}\,.$$

So we know the first two coefficients of the Heegner series

$$\mathrm{Heeg}_{\hat{\mathbf{C}}_0}(q) = -\frac{1}{8} + \frac{17}{2}q + \cdots$$

From Koblitz' monograph [17], IV.1 Prop. 4, we pick out the following

Proposition 7.1. *Let $\chi = \chi_K$ be the Dirichlet character of the Gauß number field $K = \mathbb{Q}(\mathrm{i})$ with discriminant $D_{K/\mathbb{Q}} = 4$. The ring of $G_0(4)$-modular forms of Nebentypus χ is generated by ϑ^2 with Jacobi theta series*

$$\vartheta := \sum_{n\in\mathbb{Z}} q^{n^2} = 1 + 2\sum_{n>0} q^{n^2}$$

and the Hecke theta series

$$\theta := \sum_{0<u \text{ odd}} \sigma(u)q^u = q \cdot \prod_{m=1}^{\infty}(1-q^{4m})^4 \prod_{n=1}^{\infty}(1+2q^n)^4$$

where $\sigma(m)$ denotes the sum of natural divisors of $m \in \mathbb{N}$.

Notice that

$$\vartheta^k = \sum_{N\in\mathbb{N}} a_k(N)q^N \quad \text{for positive integers } k$$

where $a_k(N)$ is the number of $\mathbb{Z}$-solutions of the quadratic equation $x_1^2 + x_2^2 + \cdots + x_k^2 = N$, see [2], VIII.1.

Corollary 7.1. *The space $\mathcal{M}_3(4,\chi)$ of $G_0(4)$-modular forms of weight 3 and Nebentypus χ coincides with the two-dimensional complex vector space generated by the series*

$$\begin{aligned} \vartheta^6 &= \sum_{N=0}^{\infty} a_6(N)q^N \\ &= 1 + 12q + 60q^2 + 160q^3 + 252q^4 + 312q^5 + \cdots \\ \vartheta^2\theta &= q + 4q^2 + 8q^3 + 16q^4 + 26q^5 + \cdots \\ &= \sum_{N=0}^{\infty}\left(\sum_{1=u \text{ odd}}^{N} \sigma(u)a_2(N-u)\right)q^N\,. \end{aligned} \tag{7.3}$$

It follows that each series $\sum h_N q^N \in \mathcal{M}_3(4,\chi)$ is completely determined by the first two Fourier coefficients h_0 and h_1, namely

$$\begin{aligned}\sum_{N=0}^{\infty} h_N q^N &= h_0\vartheta^6 + (h_1 - 12h_0)\vartheta^2\theta \\ &= \sum_{N=0}^{\infty}\left(h_0 a_6(N) + (h_1 - 12h_0)\sum_{1=u \text{ odd}}^{N}\sigma(u)a_2(N-u)\right)q^N.\end{aligned}$$

With $h_0 = -\frac{1}{8}$, $h_1 = -\frac{17}{2}$ we get our Heegner series explicitly with elementary arithmetic Fourier coefficients:

$$\operatorname{Heeg}_{\hat{\mathbf{C}}_0}(q) = \sum_{N=0}^{\infty}\left(-\frac{a_6(N)}{8} + 10\sum_{1=u \text{ odd}}^{N}\sigma(u)a_2(N-u)\right)q^N. \tag{7.4}$$

Cogdell (unpublished) determined also at the end of his thesis [3] the Heegner series for $\mathfrak{c}_0$ and neat principal congruence subgroups $\Gamma_1(M)$, $M > 2$. He filled stepwise the explicit Gauß lattice data in his analytic proof of the main theorem for neat congruence subgroups of ideals. The reader is invited to do this in the outline of proof given in the next section. Up to a natural scaling factor depending on M he received

$$\begin{aligned}\operatorname{Cogd}_{\mathfrak{c}_0} &:= \sum_{N=0}^{\infty}(N - \frac{1}{12})a_2(N)q^N + 2\sum_{N=1}^{\infty}\left(\sum_{m=1}^{N}\sigma(m)a_2(N-m)\right)q^N \\ &= -\frac{1}{12} + \frac{17}{3}q + \frac{65}{3}q^2 + 40q^3 + \frac{257}{3}q^4 + \frac{442}{3}q^5 + \cdots .\end{aligned}$$

We get the normalizing constant term $-\frac{1}{8}$ by multiplying Cogdell's series with $\frac{3}{2}$. Then we get another presentation of our Heegner series, namely,

$$\operatorname{Heeg}_{\hat{\mathbf{C}}_0}(q) = \sum_{N=0}^{\infty}\left(\frac{3N}{2} - \frac{1}{8}\right)a_2(N)q^N + 3\sum_{N=1}^{\infty}\sum_{m=1}^{N}\sigma(m)a_2(N-m)q^N$$

Comparing Fourier coefficients of both presentations of the Heegner series we obtain as amusing byproduct an elementary formula for the number of $\mathbb{Z}$-points on the boundary of the real six-dimensional ball with radius $\sqrt{N}$, namely

$$a_6(N) = (1 - 12N)a_2(N) + \sum_{m=1}^{N}(80\delta(m) - 24)\sigma(m)a_2(N-m)$$

with the parity symbol

$$\delta(m) := \begin{cases} 0, & m \text{ even} \\ 1, & m \text{ odd}\end{cases}$$

We also checked the formula by a computer for $0 \leq N < 100$. The author did not know these relations before. Can one prove them in elementary manner?

Remark 7.1. *The geometric way is simply applicable to any arithmetic geodesic* $\hat{\mathbf{C}}$ *on the orbital Picard plane* $\mathbf{P}^2$. *One has only to calculate the orbital signature* $\mathbf{h}_0(\hat{\mathbf{C}})$ *and the orbital intersection* $\mathbf{h}_1(\hat{\mathbf{C}}) = (\hat{\mathbf{C}} \cdot \mathbf{H}_1)$ *to get the attached Heegner series* $\mathrm{Heeg}_{\hat{\mathbf{C}}}(q)$ *via Corollary* 7.1. *The problem is to recognize more arithmetic curves. Until now we only know the seven modular curves on* $\mathbf{P}^2$ *drawn in Fig. 1. The knowledge of the Heegner series of only one of them yields a counting procedure for all, because each of them has a degree contribution in some Fourier coefficients. Our geometric method is also applicable to other orbital Picard surfaces, especially to the well-classified ones, in hopefully effective manner.*

8. The Theta Functions in the Background

The Main Theorem is an immediate consequence of the following

Decomposition Theorem 8.1. *The Heegner series* $\Phi_{\hat{C}}(\tau)$ *has the following additive decompositions*

$$\Phi_{\hat{C}}(\tau) = \Phi_{\hat{C}}^{fin}(\tau) + \Phi_{\hat{C}}^{\infty}(\tau) \tag{8.1}$$

$$\mathbb{Q}[[q]] \ni \Phi_{\hat{C}}^{fin}(\tau) = \Phi_3(\tau) - \Phi_1(\tau) \tag{8.2}$$

$$\mathbb{Q}[[q]] \ni \Phi_{\hat{C}}^{\infty}(\tau) = \Phi_3^{\infty}(\tau) + \Phi_1^{\infty}(\tau) \tag{8.3}$$

with relation

$$\Phi_1(\tau) = \Phi_1^{\infty}(\tau), \quad \textit{hence} \quad \Phi_{\hat{C}}(\tau) = \Phi_3(\tau) + \Phi_3^{\infty}(\tau) \tag{8.4}$$

and qualities:

$$\mathbb{Q}[[q]] \ni \Phi_1(\tau) = \Phi_1^{\infty}(\tau) \in \mathcal{M}_1(D_{K/\mathbb{Q}}, \chi_K) \tag{8.5}$$

$$\mathbb{C}[[q]] + \frac{1}{y}\mathbb{C}[[q]] \ni \Phi_3(\tau) \in \mathcal{M}_3^{non\text{-}hol}(D_{K/\mathbb{Q}}, \chi_K) \tag{8.6}$$

$$\mathbb{C}[[q]] + \frac{1}{y}\mathbb{C}[[q]] \ni \Phi_3^{\infty}(\tau) \in \mathcal{M}_3^{non\text{-}hol}(D_{K/\mathbb{Q}}, \chi_K)\,. \tag{8.7}$$

The latter two series define analytic functions with complex values in the two real variables $x = \Re(\tau)$, $y = \Im(\tau)$ by absolute convergence on the half plan $\mathbb{H}$, $y > 0$. The upper index $^{\text{non-hol}}$ emphasizes that the functions are not holomorphic, but the transformation laws are the same as in (6.6) for the weight $k = 3$.

The proof is a 76-year marathon through the theories of Theta and Zeta functions. The main splitting (8.1) has been well-prepared in Example 4.1. For orbital curves $\hat{C}$, $\hat{D}$ on $\hat{X}$ with proper transforms C', D' on X' we proved the relation

$$(\hat{C} \cdot \hat{D}) = (C' \cdot D') + (C^{\infty} \cdot D')$$

changing the roles of $\hat{C}$ and $\hat{D}$ with

$$T_1, \ldots, T_h \perp D' + D^{\infty} = \pi^{\#}\hat{D} \in \operatorname{Div}_{\mathbb{Q}} X'.$$

We extend it to the Heegner divisors $\hat{H}_N = H_N(\hat{X}_\Gamma)$, $N \in \mathbb{N}_+$, defined in (6.1) with proper transforms H'_N on X' and the decompositions

$$T_1, \ldots, T_h \perp H'_N + H^{\infty} = \pi^{\#}\hat{D} \in \operatorname{Div}_{\mathbb{Q}} X'.$$

The N-th coefficient of the Heegner series $\Phi_{\hat{C}}(q)$ splits into

$$(\hat{C} \cdot \hat{H}_N) = (C' \cdot H'_N) + (C^{\infty} \cdot H'_N)\,.$$

It defines our splitting (8.1)

$$\Phi_{\hat{C}}(q) = \Phi^{\text{fin}}_{\hat{C}}(q) + \Phi^{\infty}_{\hat{C}}(q)$$

setting

$$\Phi^{\text{fin}}_{\hat{C}}(q) := h(\hat{C}) + \sum_{N=1}^{\infty} (C' \cdot H'_N) q^N \in \mathbb{Q}[[q]]$$

$$\Phi^{\infty}_{\hat{C}}(q) := \sum_{N=1}^{\infty} (C^{\infty} \cdot H'_N) q^N \in q\mathbb{Q}[[q]]\,.$$

Now we split the latter series in its single cusp contributions. We let $\kappa_1, \ldots, \kappa_h$ be a complete set of representatives $\bmod \Gamma$ of the K-rational boundary set $\partial_K \mathbb{B}$, also called Γ-cusps. Writing $\kappa \bmod \Gamma$ indicates that κ runs through such a (fixed but arbitrarily choosen) set of representatives. With

$$C^{\infty} = \sum_{\kappa \bmod \Gamma} \lambda_\kappa T_\kappa = \sum_{i=1}^{n} \lambda_i T_i.$$

and

$$\Phi^{\kappa}(\tau) := \sum_{N=1}^{\infty} (T_\kappa \cdot H'_N) q^N \tag{8.8}$$

we get

$$\Phi_{\hat{C}}^{\infty}(q) = \sum_{\kappa \bmod \Gamma}^{\infty} \lambda_\kappa \Phi^\kappa(q)\,.$$

The coefficients have been already calculated in (4.1), namely

$$\lambda_\kappa = -\frac{(C' \cdot T_\kappa)}{(T_\kappa^2)} = -\frac{(C' \cdot T_i)}{(T_i^2)} \in \mathbb{Q}, \qquad \kappa = \kappa_i\,.$$

Lemma 8.1. (Cogdell [5], Lemma 2.4 (ii)) *With the above notations it holds that*

$$-\,(T_\kappa^2) = M \cdot \left|D_{K/\mathbb{Q}}\right|, \quad \textit{hence} \quad \lambda_\kappa = \frac{(C' \cdot T_\kappa)}{M \cdot \left|D_{K/\mathbb{Q}}\right|} \tag{8.9}$$

and

$$\Phi_{\hat{C}}^{\infty}(q) = \frac{1}{M \cdot \left|D_{K/\mathbb{Q}}\right|} \sum_{\kappa \bmod \Gamma} (C' \cdot T_\kappa)\Phi^\kappa(q)\,. \tag{8.10}$$

This series is closely related with the holomorphic function on $\mathbb{H}$ of theta type

$$\theta^\kappa(\tau) := \sum_{A \in \Lambda_2^\kappa} \mathrm{e}^{2\pi\langle A, A\rangle\tau}. \tag{8.11}$$

We have to explain Λ_2^κ. Cogdell proved in [3] the existence of a *Witt decomposition* of (the $\mathbb{Z}$-maximal Picard lattices) Λ *with respect to* κ. This is an orthogonal decomposition of Λ into an indefinite sublattice of rank 2 and a positive definite one together with a κ-*Witt basis* W_1, W_2, W_3 of V satisfying $KW_1 = K\kappa$,

$$\Lambda = (\mathfrak{a}^{-1}W_1 \oplus \bar{\mathfrak{a}}W_3) \textcircled{\perp} \mathfrak{a}\bar{\mathfrak{a}}^{-1}W_2 \tag{8.12}$$

W_1, W_3 isotropic, W_2 positive and $\mathfrak{a}$ the ideal defined by $\mathfrak{a}^{-1}W_1 = K \cdot W_1 \cap \Lambda$. By suitable choice of W_1 we can and will assume that $\mathfrak{a}$ is an $\mathfrak{O}_K$-ideal. Now take the positive summand of this decomposition to define

$$\Lambda_2^\kappa := \mathfrak{a}\bar{\mathfrak{a}}^{-1}W_2\,. \tag{8.13}$$

$\theta^\kappa(\tau)$ does not depend on the choice of the κ-Witt basis.

Transformation Law 8.1. (Hecke [9]) $\theta(\tau) \in \mathcal{M}_1(D_{K/\mathbb{Q}}, \chi_K)$.

The transformation law and some others below are related with congruence Zeta functions. These connections will be outlined below. By careful counting of intersection points Cogdell found

$$(T_\kappa \cdot H_N') = N \cdot M^2 \cdot \left|D_{K/\mathbb{Q}}\right| \cdot \#\Lambda_2^{\kappa,N} \tag{8.14}$$

in [5] (Lemma 6.2) with

$$\Lambda_2^{\kappa,N} := \Lambda^{(N)} \cap \Lambda_2^{\kappa}, \qquad \Lambda^{(N)} := \{\lambda \in \Lambda;\, \langle \lambda, \lambda \rangle = N\}.$$

Comparing coefficients of $\Phi^{\kappa}(\tau)$, see (8.8), with those of the derivative of $\theta^{\kappa}(\tau)$ the relations (8.14) yield

$$\Phi^{\kappa}(\tau) = \frac{M^2 \cdot |D_{K/\mathbb{Q}}|}{2\pi \mathrm{i}} \frac{\mathrm{d}}{\mathrm{d}\tau} \theta^{\kappa}(\tau).$$

From Shimura's paper [23] based on ideas of Maaß [19] we pick out the differential operator $\partial_1 := \frac{1}{2\pi \mathrm{i}}\left(\frac{1}{2\mathrm{i}y} + \frac{\partial}{\partial \tau}\right)$ in two variables and also the

Transformation Law 8.2. $\partial_1 \theta^{\kappa}(\tau) \in \mathcal{M}_3^{\textit{non-hol}}(D_{K/\mathbb{Q}}, \chi_K)$.

Moreover, we obtain the

Decomposition 8.1. $\Phi_{\hat{C}}^{\infty}(\tau) = \Phi_3^{\infty}(\tau) + \Phi_1^{\infty}(\tau)$,

which is (8.3) with

$$\Phi_3^{\infty}(\tau) = M \sum_{\kappa \bmod \Gamma} (C' \cdot T^{\kappa}) \partial_1 \theta^{\kappa}(\tau) \tag{8.15}$$

$$\Phi_1^{\infty}(\tau) := \frac{M}{4\pi y} \sum_{\kappa \bmod \Gamma} (C' \cdot T^{\kappa}) \theta^{\kappa}(\tau). \tag{8.16}$$

This follows immediately by applying $\frac{\partial}{\partial \tau} = 2\pi \mathrm{i} \partial_1 - \frac{1}{2\mathrm{i}y}$ to (8.10).

We proved (8.3) in the Decomposition Theorem 8.1 together with the qualities (8.5) and (8.7) there. Now we come to the more complicated "finite part" $\Phi_{\hat{C}}^{\text{fin}}(q)$ of (8.1).

For $\hat{C} = \widehat{\Gamma \backslash \mathbb{D}}$, $\mathbb{D} = \mathbb{D}_{\mathfrak{c}} \subset \mathbb{B}$, $\mathfrak{c} \in \Lambda^+$, $V_1 := K\mathfrak{c}$, $V = V_1 \oplus V_0$, Cogdell introduced in [5] the hermitian sublattices

$$\Lambda_1 := V_1 \cap \Lambda \text{ (positive definite)}, \qquad \Lambda_0 := V_0 \cap \Lambda \text{ (indefinite)}$$

and

$$\Lambda' := \Lambda_1 \oplus \Lambda_0$$

of Λ. With the dual lattices in V_0, V_1, V, respectively, indicated by the upper index $^{\#}$, we get the cofinite tower

$$\Lambda_1 \oplus \Lambda_1 = \Lambda' \subseteq \Lambda \subseteq \Lambda^{\#} \subseteq \Lambda'^{\#} = \Lambda_1^{\#} \oplus \Lambda_1^{\#} \tag{8.17}$$

of hermitian lattices in V. The arithmetic group $\Gamma_{\mathbb{D}}$ coincides with the isotropy group $\Gamma_{\mathfrak{c}}$ because neat lattices do not contain any element with non-trivial unit roots as eigenvalues. It acts on the lattices of the tower (8.17) and on the orthogonal summands appearing there, hence on the finite residue class groups. The **inertia subgroup** of Λ_0 is defined as

$$\Gamma_{\mathbb{D}}^0 := \{\gamma \in \Gamma_{\mathbb{D}};\, \gamma|_{\Lambda_0^\#/\Lambda_0} = \mathrm{id}_{\Lambda_0^\#/\Lambda_0}\} \tag{8.18}$$

with obvious notations. Now we are able to define for $A_0 \in \Lambda_0^\#$ and $A_1 \in \Lambda_1^\#$ the following series of congruence theta type

$$\theta_0(\tau; A_0) := \sum_{\substack{V_0^+ \ni Y_0 \equiv A_0 (\Lambda_0) \\ Y_0 \mod \Gamma_{\mathbb{D}}^0}} \mathrm{e}^{2\pi \mathrm{i} \langle Y_0, Y_0 \rangle \tau} \tag{8.19}$$

$$\theta_1(\tau; A_1) := \sum_{V_1 \ni Y_1 \equiv A_1 (\Lambda_1)} \mathrm{e}^{2\pi \mathrm{i} \langle Y_1, Y_1 \rangle \tau}. \tag{8.20}$$

It is clear that the running vectors Y_0, Y_1 belong to $\Lambda_0^{\#+}$ or $\Lambda_1^{\#+}$, respectively. A longer counting procedure due to Cogdell summarizing suitable products of θ_0- and θ_1-functions yields the

Decomposition 8.2. ([3], [5], Proposition 5.1)

$$\Phi_{\hat{C}}^{fin}(\tau) = \sum_{\Lambda \ni Z \mod \Lambda'} \left(\delta(Z_0) h_\sigma(\hat{C}) + \frac{1}{[\Gamma_{\mathbb{D}} : \Gamma_{\mathbb{D}}^0]} \theta_0(\tau; Z_0) \theta_1(\tau; Z_1)\right)$$

with decompositions $Z = Z_0 + Z_1$, $Z_0 \in \Lambda_0^\#$, $Z_1 \in \Lambda_1^\#$, *signature height* h_σ *and* $\delta(A_0) := \begin{cases} 1, & \text{if } A_0 \in \Lambda_0 \\ 0, & \text{else}\,. \end{cases}$

Hecke proved in [9] that the θ_1-series are holomorphic on $\mathbb{H}$ satisfying the following

Transformation Law 8.3.

$$\theta_1(\tau + n; A_1) = \mathrm{e}^{2\pi \mathrm{i} \langle A_1, A_1 \rangle n} \cdot \theta_1(\tau; A_1), \qquad n \in \mathbb{Z}$$

$$\theta_1\left(-\frac{1}{\tau}; A_1\right) = \frac{-\mathrm{i}\tau}{\sqrt{[\Lambda_1^\# : \Lambda_1]}} \sum_{Y \in \Lambda_1^\# \mod \Lambda_1} \mathrm{e}^{2\pi \mathrm{i}\, \mathrm{Tr}_{K/\mathbb{Q}} \langle A_1, Y \rangle} \theta_1(\tau; Y)\,.$$

For isotropic lattices Λ_0 Cogdell [5] added to the θ_0-series two residue summands in order to get similar transformation laws. He introduced

$$E_0(\tau; A_0) := -\operatorname{Res}_0 \mathcal{Z}_0(s; A_0) - \frac{1}{y} \operatorname{Res}_{1/2} \mathcal{Z}_0(s; A_0) + \theta_0(\tau; A_0). \tag{8.21}$$

The Zeta function $\mathcal{Z}_0(s;A)$ and the reason for the modularisation effect 8.4 below will be explained and presented in the Section 9. Cogdell [5], p. 128, calculated also the explicit values of the residues, namely

$$-\operatorname{Res}_0 \mathcal{Z}_0(s;A_0) = \delta_\sigma(A_0) := \delta(A_0) h_\sigma(\widehat{\Gamma\backslash\mathbb{D}_{\mathbb{D}}^0}), \tag{8.22}$$

$$-\operatorname{Res}_{1/2} \mathcal{Z}_0(s;A_0) = \frac{\nu^\infty(A_0)}{4\pi} \tag{8.23}$$

with

$$\nu^\infty(A_0) = \nu_{\mathfrak{c}}^\infty(A_0) = \sum_{\partial_K(\mathbb{D})\ni\kappa \bmod \Gamma_{\mathbb{D}}^0} \nu^\kappa(A_0)$$

and $\nu^\kappa(A_0) = \begin{cases} 1, & \text{if } \kappa \in KA_0 + \Lambda_0 \\ 0, & \text{else.} \end{cases}$

For anisotropic lattices Λ_0 the role of $\mathcal{Z}_0$ are played by other Zeta functions, see the next Section 9. Both cases come together setting

$$E_0(\tau;A_0) := \delta_\sigma(A_0) + \frac{1}{y}\frac{\nu^\infty(A_0)}{4\pi} + \theta_0(\tau;A_0) \tag{8.24}$$

with $\frac{\nu^\infty(A_0)}{4\pi} := 0$ in the anisotropic case.

Transformation Law 8.4. ([3], Proposition 5.2)

$$E_0(\tau+n;A_0) = \mathrm{e}^{2\pi\mathrm{i}\langle A_0,A_0\rangle n}\cdot E_0(\tau;A_0), \qquad n\in\mathbb{Z}$$

$$E_0\left(-\frac{1}{\tau};A_0\right) = \frac{\tau^2}{\sqrt{[\Lambda_0^\# : \Lambda_0]}} \sum_{Y\in\Lambda_0^\# \bmod \Lambda_0} \mathrm{e}^{2\pi\mathrm{i}\operatorname{Tr}_{K/\mathbb{Q}}\langle A_0,Y\rangle} E_0(\tau;Y)\,.$$

A composition of θ_1 and E_0 is the following non-holomorphic but real analytic function

$$g(\tau;A) := \frac{1}{[\Gamma_{\mathbb{D}} : \Gamma_{\mathbb{D}}^0]}\cdot E_0(\tau;A_0)\cdot\theta_1(\tau;A_1) \tag{8.25}$$

with

$$\Lambda'^\# \ni A = A_0 + A_1,\ A_0 \in \Lambda_0^\#,\ A_1\in\Lambda_1^\#.$$

The transformation laws 8.3 and 8.4 imply immediately the

Transformation Law 8.5.

$$g(\tau+n;A) = \mathrm{e}^{2\pi\mathrm{i}\langle A,A\rangle n}\cdot g(\tau;A), \qquad n\in\mathbb{Z}$$

$$g\left(-\frac{1}{\tau};A\right) = \frac{-\mathrm{i}\tau^3}{\sqrt{[\Lambda'^\# : \Lambda']}} \sum_{Y\in\Lambda'^\# \bmod \Lambda'} \mathrm{e}^{2\pi\mathrm{i}\operatorname{Tr}_{K/\mathbb{Q}}\langle A,Y\rangle} g(\tau;Y)\,.$$

Now it is time to define

$$\Phi_3(\tau) = \Phi_{\hat{C},3}(\tau) := \sum_{\Lambda \ni Z \bmod \Lambda'} g(\tau; Z) \tag{8.26}$$

with

Transformation Law 8.6. ([5], Theorem 5.1) $\Phi^3(\tau) \in \mathcal{M}_3^{\textit{non-hol}}(D_{K/\mathbb{Q}}, \chi_K)$ *coming from* 8.5.

After substitution of (8.21) with explicit residues (8.22) and (8.23) into the $g(\tau)$-terms of $\Phi_3(\tau)$ and some counts a comparision with the θ_0, θ_1 product sum in 8.2 yields

Decomposition 8.3. $\Phi_3(\tau) = \Phi_{\hat{C}}^{fin}(\tau) + \Phi_1(\tau)$ *with the holomorphic function*

$$\Phi_1(\tau) = \Phi_{\hat{C},1}(\tau) := \sum_{\Lambda \ni Z \bmod \Lambda'} \frac{M\nu_{\mathfrak{c}}^{\infty}(Z_0)}{4\pi y} \theta_1(\tau; Z_1) \tag{8.27}$$

and $Z = Z_0 + Z_1$ *is explained in Lemma 8.2 below. For details of the calculations we refer again to* [5], *p. 130.*

Now the decompositions and the transformation laws in the Decomposition Theorem 8.1 have all been recognized. It remains to verify the relation (8.4). Comparing N-th Fourier coefficients one gets after count comparisions the following relations between the θ^{κ}'s and the θ_1's:

Lemma 8.2. *For* $\mathbb{D} = \mathbb{D}_{\mathfrak{c}}$ *let* $\kappa \in \partial_K \mathbb{D}$ *be a* $\Gamma_{\mathbb{D}}$*-cusp. With*

$$\Lambda_0 := \Lambda \cap \kappa^{\perp}, \Lambda_1 := \Lambda \cap K\mathfrak{c}, \qquad \Lambda' := \Lambda_0 \bigoplus \Lambda_1$$

and decompositions $Z = Z_0 + Z_1$ *in* $\Lambda_0^{\#} \bigoplus \Lambda_1^{\#}$ *it holds that*

$$\theta^{\kappa}(\tau) = \sum_{\Lambda \ni Z \bmod \Lambda'} \mu^{\kappa}(Z) \theta_1(\tau; Z_1)$$

where $\mu^{\kappa}(Z) = \begin{cases} 0, & \textit{if } (Z_0 + \Lambda_0) \cap K\kappa = \emptyset \\ 1, & \textit{else.} \end{cases}$

Summation over the cusps modulo Γ yields

$$\sum_{\kappa \bmod \Gamma} (C' \cdot T^{\kappa}) \theta^{\kappa}(\tau) = \sum_{\Lambda \ni Z \bmod \Lambda'} \nu_{\mathfrak{c}}(Z_0) \theta_1(\tau; Z_1)$$

which proves (8.4) using the definitions (8.16) and (8.27) of $\Phi_1^{\infty}(\tau)$ or $\Phi_1(\tau)$, respectively.

9. From Zeta Functional Equations to Theta Transformation Laws

A) The Cusp Functions

Hecke [10] introduced the congruence Zeta functions for quadratic number fields. We restrict our attention to a fixed arbitrary imaginary quadratic number field $K = \mathbb{Q}(\delta)$, with discriminant $D = D_{K/\mathbb{Q}} < 0$ and inverse $\delta = \sqrt{D}$ of the different. Let $0 \neq \mathfrak{a} \subset \mathfrak{O}_K$ be an ideal and $\rho \in \mathfrak{a}$.

$$\zeta(s) = \zeta(s; \rho, \mathfrak{a}, \sqrt{D}) := \sum_{0 \neq \mu \equiv \rho(\mathfrak{a}\sqrt{D})} \frac{1}{N(\mu)^s N(\mathfrak{a})^s} \,. \tag{9.1}$$

These series extend to meromorphic functions on $\mathbb{C}$ with at most one pole. There are at most two possible (simple) poles at $s = 0, 1$. The congruence Zeta functions belong to a class of Dirichlet L-series $L(s) = \sum_{n=1}^{\infty} \frac{a_n}{n^s}$ with functional equations reflecting $s \mapsto 1 - s$. We refer to [2], Chapter 7, §3, for more information. Setting $R(s) := (\frac{2\pi}{\lambda})^{-s}\Gamma(s)\zeta(s)$ for a suitable $\lambda \in \mathbb{R}_+$, the corresponding functional equations have the simple form

Functional Equation 9.1.

$$R(s) = \pm R(1 - s) \,.$$

There is a ring isomorphism to a class of theta-type functions on $\mathbb{H}$ with special transformation law corresponding

$$\sum_{n=1}^{\infty} \frac{a_n}{n^s} \longleftrightarrow \varphi(\tau) = \sum_{n=0}^{\infty} a_n \, \mathrm{e}^{\frac{2\pi \mathrm{i} n}{\lambda}\tau}$$

with $a_0 = \pm \frac{\lambda}{2\pi} \operatorname{Res}_1 L(s)$. Mellin's inversion formula

$$\mathrm{e}^{-\tau} = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\Gamma(\sigma + \mathrm{i}t)}{z^{\sigma + \mathrm{i}t}} \, \mathrm{d}t, \qquad s = \sigma + \mathrm{i}t$$

applied to $\varphi(t)$ yields the following analytic relation between $\varphi(\tau)$ and $L(s)$:

$$\varphi(\mathrm{i}y) = a_0 + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{R(s)}{y^s} \, \mathrm{d}t, \qquad \sigma \gg 0 \,.$$

On this way the functional equation (9.1) leads to the transformation law

$$\varphi\left(-\frac{1}{\tau}\right) = \pm(\frac{\tau}{\mathrm{i}})\varphi(\tau) \,.$$

for $\varphi(\tau)$. It is also well-known that, conversely, the transformation law implies the functional equation.

The theta series corresponding to the congruence Zeta functions (9.1) are

$$\vartheta(\tau;\rho,\mathfrak{a},\sqrt{D}) = \sum_{\mu\equiv\rho(\mathfrak{a}\sqrt{D})} e^{2\pi i\tau\frac{N(\mu)}{N(\mathfrak{a}\delta)}}$$

as pointed out by Hecke in [10] around formula (56) there. Hecke proved first the theta transformation law in his earlier paper [9].

We are most interested on the case $\rho = 0$ in (9.1), that means on the *ideal zeta functions*

$$\zeta(s) = \zeta(s;\mathfrak{a},\sqrt{D}) := \sum_{0\neq\mu\in\mathfrak{a}\sqrt{D})} \frac{1}{N(\mu)^s N(\mathfrak{a})^s} \tag{9.2}$$

corresponding to *ideal theta functions*

$$\vartheta(\tau;\mathfrak{a},\sqrt{D}) = \sum_{\mu\in\rho(\mathfrak{a}\sqrt{D}} e^{2\pi i\tau\frac{N(\mu)}{N(\mathfrak{a}\delta)}}$$

In this case one has $\lambda = |\delta| = \left|\sqrt{D}\right|$ in the functional equation, see [10], around formula (68).

A little bit more generally, Hecke introduced in his earlier article [9] the following zeta functions.

Definition 9.1. (Hecke [9]) *The* congruence theta function *of the integral ideal* $\mathfrak{b}$ *in* K, $\rho\in\mathfrak{b}$ *and* $Q\in\mathbb{N}$ *is the holomorphic function on* $\mathbb{H}$ *defined by*

$$\vartheta(\tau;\rho,\mathfrak{b},Q\sqrt{D} := \sum_{\mu\equiv\rho(\mathfrak{b}\delta Q} e^{2\pi i\tau\frac{N(\mu)}{N(\delta\mathfrak{b})Q}}, \qquad \tau\in\mathbb{H}$$

satisfying the following transformation law:

Proposition 9.1. ([9], Satz 7) *For* $\gamma = \left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right) \in Sl_2(\mathbb{Z}), c\equiv 0 \bmod D$, *the following transformation law holds*

$$\vartheta(\gamma\tau;\rho,\mathfrak{b},Q\sqrt{D}) = \chi_K(\delta)\, e^{2\pi i\frac{N(ab\rho)}{N(\delta\mathfrak{b})Q}}\,\vartheta(\tau;a\rho,\mathfrak{b},Q\sqrt{D})\,.$$

Restricting to $\rho = 0$ we set

$$\vartheta(\tau;\mathfrak{b},Q) := \vartheta(\tau;0,\mathfrak{b},Q\sqrt{D}) = \sum_{\mu\in\mathfrak{b}\delta Q)} e^{2\pi i\tau\frac{N(\mu)}{N(\delta\mathfrak{b})Q}}$$

and obtain the modular

Transformation Law 9.1. $\vartheta(\gamma\tau;\mathfrak{b},Q) = \chi_K(\delta)\vartheta(\tau;\mathfrak{b},Q)$.

It is clear that $\vartheta(\tau;\mathfrak{b},Q)$ depends only on the ideal class of $\mathfrak{b}$. Therefore the definition extends correctly to all fractional ideals $\mathfrak{b}$.

Proof: (transformation law 8.1) We have only to verify that the theta functions (8.11) at cusps κ are of the above Hecke type. We used a Witt decomposition (8.12) of the hermitian $\mathfrak{O}$-lattice Λ with positive orthogonal component $\Lambda_2 = \Lambda_2^\kappa = \mathfrak{a}\bar{\mathfrak{a}}^{-1}W_2$. With $Q := \langle W_2, W_2\rangle \in \mathbb{N}_+$ (without loss of generality) and $\mathfrak{b} := \mathfrak{a}\bar{\mathfrak{a}}^{-1}\delta^{-1}$ we get

$$\begin{aligned}\theta^\kappa(\tau) &= \sum_{A\in\Lambda_2^\kappa} \mathrm{e}^{2\pi\langle A,A\rangle\tau} = \sum_{\nu\in\mathfrak{a}\bar{\mathfrak{a}}^{-1}} \mathrm{e}^{2\pi N(\nu)\langle W_2,W_2\rangle\tau} \\ &= \sum_{\mu\in\mathfrak{a}\bar{\mathfrak{a}}^{-1}\delta Q} \mathrm{e}^{2\pi\tau\frac{N(\mu)}{N(\delta)Q}} = \sum_{\mu\in\mathfrak{b}\delta Q} \mathrm{e}^{2\pi\tau\frac{N(\mu)}{N(\mathfrak{b}\delta)Q}} \\ &= \vartheta(\tau;\mathfrak{b},Q)\end{aligned}$$

because $N(\mathfrak{b}) = 1$. □

B) The Zeta and Theta Functions of Modular Curves

Let $\hat{C} = \widehat{\Gamma\backslash\mathbb{D}} \subset \hat{X}_\Gamma = \Gamma\backslash\mathbb{B}$ be a modular curve. We have $\mathbb{D} = \mathbb{D}_W = \mathbb{P}W^\perp(\mathbb{R})$ for suitable $W = W_2 \in \Lambda^+$ uniquely determined by $\mathbb{D}$ up to K^*-multiplication. We set $\Lambda_0 = \Lambda_0(\mathbb{D}) := \Lambda \cap W^\perp$ and let $\Lambda_{1,1}$ be a maximal $\mathfrak{O}$-sublattice of $W^\perp$ with $\langle Y, Y\rangle \in \mathbb{Z}$ for all $Y \in \Lambda_{1,1}$ (signature $(1,1)$). $\hat{C}$ is a modular curve iff $C^\infty \neq \emptyset$, that means $W^\perp$ and also Λ_0 are isotropic. In the opposite case $C = \hat{C}$ one calls $\hat{C}$ a **Shimura curve** on $\hat{X}_\Gamma$.

There is a well-known transfer from $\mathfrak{O}$-lattices of rank 2 to $\mathbb{Z}$-lattices in $\mathrm{Mat}_2(\mathbb{Q})$ and their orders. For details we refer to Shimura's paper [22], where it is done also for Shimura curves. Fix a $\mathbb{Z}$-basis $1, \omega$ of $\mathfrak{O}$ and an $\mathfrak{O}$-basis W_1, W_3 of $W^\perp$. One corresponds to $C \in W^\perp$ with W_1, W_3-coordinates $\left(\begin{smallmatrix}\alpha\\ \gamma\end{smallmatrix}\right) = \left(\begin{smallmatrix}a+b\omega\\ c+d\omega\end{smallmatrix}\right)$ the matrix $\left(\begin{smallmatrix}a & b\\ c & d\end{smallmatrix}\right) =: \varphi(C) \in \mathrm{Mat}_2(\mathbb{Q})$.

It is clear that $\mathfrak{O}$-lattices in $W^\perp$ map onto $\mathbb{Z}$-lattices in $\mathrm{Mat}_2(\mathbb{Q})$. Each $\mathbb{Z}$-lattice L defines an order $\mathfrak{O}_L := \{g \in \mathrm{Mat}_2(\mathbb{Q}); g(L) \subseteq L\}$. Maximal $\mathfrak{O}$-lattices in $W^\perp$ correspond in this way to maximal orders of $\mathrm{Mat}_2(\mathbb{Q})$. Up to $\mathbb{G}l_2(\mathbb{Q})$-conjugation there is only one maximal order in $\mathrm{Mat}_2(\mathbb{Q})$, namely $\mathrm{Mat}_2(\mathbb{Z})$, see Eichler [6]. This is also a maximal lattice with respect to the symmetric bilinear form $(X, Y) := \mathrm{Tr}(X \cdot \mathrm{Ad}(Y))$ on $\mathrm{Mat}_2(\mathbb{Q})$, where Tr denotes the matrix trace and $\mathrm{Ad}(Y)$ is the adjoint matrix of Y. So we can arrange by suitable basis choice that $\Lambda_{1,1}$ corresponds to $\mathrm{Mat}_2(\mathbb{Z})$. Using $\mathrm{Tr}_{K/\mathbb{Q}}\langle . , .\rangle$ on $W^\perp$ the $\mathbb{Q}$-linear isomorphism φ becomes an isometry. Altogether we get a commutative isometry diagram

$$\begin{array}{rcl} \varphi: W^{\perp} & \longleftrightarrow & \mathrm{Mat}_2(\mathbb{Q}) \\ \cup & & \cup \\ \Lambda_{1,1} & \longleftrightarrow & \mathrm{Mat}_2(\mathbb{Z}) \\ \cup & & \cup \\ \Lambda_0 & \longleftrightarrow & L \\ \cap & & \cap \\ \Lambda_0^{\#} & \longleftrightarrow & L^{\#} \end{array} \tag{9.3}$$

with isometry group transfers

$$\begin{array}{rcl} \rho: SU(W^{\perp}) & \longleftrightarrow & \mathbb{S}l_2(\mathbb{Q}) \\ \cup & & \cup \\ \Gamma_{\mathbb{D}}^0 & \longleftrightarrow & \Gamma \end{array} \tag{9.4}$$

where Γ is defined by the inertia group $\Gamma_{\mathbb{D}}^0$ introduced in (8.18). Notice also that the (projective/fractional) actions on $\mathbb{D}$ of the groups on the left-hand side are transferred to actions on the upper half plane $\mathbb{H}$. For more details we refer also to the original thesis of Cogdell [3], Section 6.

Now we work with the structures on the right-hand side in the above diagrams, especially with (fixed) L, $L^{\#}$, Γ instead of Λ_0, $\Lambda_0^{\#}$ or $\Gamma_{\mathbb{D}}^0$, respectively.

Definition 9.2. (Cogdell [4], p. 181) *For* $A \in L^{\#}$ *the functions*

$$\zeta(s;A) := \sum_{\substack{L^{\#} \ni Y \equiv A(L) \\ \det Y \neq 0,\ \mathrm{mod}\ \Gamma}} \frac{1}{|\det Y|^{2s}}$$

are called modular congruence Zeta functions.

The same notation should be used for its Γ-function modification

$$\mathcal{Z}(s;A) := (2\pi)^{-2s}\Gamma(2s)\zeta(s;A)\,.$$

These Zeta functions define via estimations ad hoc holomorphic functions on the complex half plane $\Re(s) > 1$. There is an integral representation

$$\frac{\pi}{4s-2}\mathcal{Z}(s;A) = \int\limits_{\mathbb{G}l_2(\mathbb{R})^+/\Gamma} \vartheta_1(g;A)|g|^{2s}\,\mathrm{d}g \tag{9.5}$$

with

$$\vartheta_1(g;A) := \sum_{O \neq Y \equiv A(L)} \mathrm{e}^{-\pi Q(Y)}$$

where Q is the quadratic form $X \mapsto Q(X) := \mathrm{Tr}({}^tX \cdot X)$ on $\mathrm{Mat}_2(\mathbb{R})$. Using Poisson sums Cogdell proved the following

Transformation Law 9.2. $\vartheta(g;A) = \frac{(\det g)^{-2}}{\mathrm{Vol}(L)} \sum_{B\in L^\#/L} \mathrm{e}^{2\pi \mathrm{i}(A,B)} \vartheta(\check{g};B)$ *with*

$$\mathrm{Vol}(L) := \int\limits_{\mathrm{Mat}_2(\mathbb{R})/L} d\,Y = \sqrt{[L^\# : L]}, \qquad \check{g} := (\mathrm{Ad}(g))^{-1} = \frac{g}{\det g}\,.$$

Cogdell also introduces

$$\zeta_0(s;A) := \sum_{\substack{L^\# \ni Y \equiv A(L) \\ \det Y>0,\ \mathrm{mod}\ \Gamma}} \frac{1}{|\det Y|^{2s}}$$

because the difference series $\zeta_2(s;A)$ defined by

$$2\zeta_0(s;A) = \zeta(s;A) + \zeta_2(s;A)$$

comes with alternating signs at the summands and has therefore a holomorphic extension to $\mathbb{C}$. With

$$\mathcal{Z}_0(s;A) := (2\pi)^{-2s}\Gamma(2s)\zeta_0(s;A), \qquad \mathcal{Z}_2(s;A) := (2\pi)^{-2s}\Gamma(2s)\zeta_2(s;A)$$

we obtain the relation

$$2\mathcal{Z}_0(s;A) = \mathcal{Z}(s;A) + \mathcal{Z}_2(s;A)\,. \tag{9.6}$$

Cogdell's central result is the following:

Theorem 9.1. ([4]) *The Zeta functions $\mathcal{Z}(s;A)$ and $\mathcal{Z}_0(s;A)$ have meromorphic extensions to $\mathbb{C}$ with (at most) three simple poles at $s = 0, 1, \frac{1}{2}$ with residues*

$$\mathrm{Res}_0\, \mathcal{Z}(s;A) = \frac{\delta(A)}{2\pi}\, \mathrm{Vol}(\mathbb{S}l_2(\mathbb{R})/\Gamma) = \frac{1}{2}\, \mathrm{Res}_0\, \mathcal{Z}_0(s;A)$$

$$\mathrm{Res}_1\, \mathcal{Z}(s;A) = \frac{\mathrm{Vol}(\mathbb{S}l_2(\mathbb{R})/\Gamma)}{2\pi\, \mathrm{Vol}(\Gamma)} = \frac{1}{2}\, \mathrm{Res}_1\, \mathcal{Z}_0(s;A)$$

$$\mathrm{Res}_{1/2}\, \mathcal{Z}(s;A) = -\frac{\nu^\infty(A_0)}{4\pi} = \frac{1}{2}\, \mathrm{Res}_{1/2}\, \mathcal{Z}_0(s;A), \ \textit{see}\ (8.23)$$

with $\delta(A) = \begin{cases} 0, & \textit{if } A \in L \\ 1, & \textit{if } A \notin L \end{cases}$ *and they satisfy the Functional Equations*

$$\mathcal{Z}(1-s;A) = -\frac{1}{\sqrt{[L^\# : L]}} \sum_{B\in L^\#/L} \mathrm{e}^{2\pi \mathrm{i}(A,B)} \mathcal{Z}(s;B)$$

$$\mathcal{Z}_0(1-s;A) = -\frac{1}{\sqrt{[L^\# : L]}} \sum_{B\in L^\#/L} \mathrm{e}^{2\pi \mathrm{i}(A,B)} \mathcal{Z}_0(s;B)\,.$$

It turns out that the *inverse Mellin transform*

$$f(y) = \frac{1}{2\pi \mathrm{i}} \int\limits_{\sigma-\mathrm{i}\infty}^{\sigma+\mathrm{i}\infty} \mathcal{Z}_0(s;A) y^{-s}\,\mathrm{d}s, \qquad y \in \mathbb{R}, \quad \sigma = \Re(s)$$

of $\mathcal{Z}_0(s;A)$ coincides with

$$\theta_0(\mathrm{i}y;A) = \sum_{\substack{X \equiv A(L) \\ \det X > 0,\ \mathrm{mod}\ \Gamma}} \mathrm{e}^{-2\pi y \det X}.$$

But this is the restriction to the positive part of imaginary axes of

$$\theta_0(\tau;A) = \sum_{\substack{X \equiv A(L) \\ \det X > 0,\ \mathrm{mod}\ \Gamma}} \mathrm{e}^{-2\pi \mathrm{i}\tau \det X}, \qquad y = \Im(\tau)$$

introduced in (8.19) in original Λ_0-terms. For the translation we have to use the diagrams (9.3) and (9.4).

Unfortunately, $\theta_0(\tau;A)$ is not a modular form. As in the case of classical Dirichlet series one has to add residue terms, but two instead of one, to get the modular transformation law we look for. This has the price to leave holomorphic functions but not the class of real analytic functions of $\tau = x + \mathrm{i}y \in \mathbb{H}$ in two variables x, y. Cogdell introduces

$$E_0(\tau;A) := -\operatorname{Res}_0 \mathcal{Z}_0(s;A) - \frac{1}{y}\operatorname{Res}_{1/2} \mathcal{Z}_0(s;A) + \theta_0(s;A)\,.$$

With a result of Maaß and the $\mathcal{Z}_0$-functional equation he proves the transformation law

$$E_0\left(-\frac{1}{\tau};A\right) = \frac{1}{\tau^2\sqrt{[L^\# : L]}} \sum_{B \in L^\#/L} \mathrm{e}^{2\pi \mathrm{i}(A,B)} E_0(\tau;B)$$

which is the difficult part of 8.4 for the function E_0 in Λ_0-terms.

C) The Zeta and Theta functions of Shimura curves

Following ideas of Hecke Schoeneberg introduced in [21] congruence Zeta functions of indefinite quaternion skew fields $\mathcal{S}$ over (the center) $\mathbb{Q}$. Let $\mathfrak{d}$ be the different ideal of $\mathcal{S}$, $\mathcal{I}$ be a maximal order in $\mathcal{S}$, $\mathcal{I}'$ its conjugate, $\mathfrak{a}$ a right ideal in $\mathcal{I}$, $\rho \in \mathfrak{a}$ and Q a positive integer.

Definition 9.3. (Schoeneberg) *The* congruence Zeta functions of $\mathcal{S}$ *are defined as*

$$\zeta(s;\mathfrak{a}Q\mathfrak{d},\rho) := N(\mathfrak{a}Q\mathfrak{d},\rho)^s \sum_{\substack{\mu \equiv \rho(\mathfrak{a}Q\mathfrak{d}) \\ \mathrm{mod}\ ^\times(\mathfrak{a}Q\mathfrak{d})'_1}}{}' \frac{1}{|N(\mu)|^s}$$

where $N = n^2$ *denotes the absolute norm on* S, n *denotes the norm and* $(\mathfrak{a}Q\mathfrak{d})_1'$ *is the group of units of* I' *congruent* $1 \bmod \mathfrak{a}Q\mathfrak{d}$.

These series define ad hoc holomorphic functions for $\Re(s) > 1$ extendable to meromorphic functions on $\mathbb{C}$ with at most two (simple) poles at $0, 1$. Now let $\mathfrak{q}$ be a positive quadratic form on $\mathbb{Q}^4$ represented by the symmetric matrix $\mathfrak{Q} = (q_{ij}) \in \mathbb{G}l_4(\mathbb{Q})$ with respect to the canonical basis and ω_k, $k = 1, \ldots, 4$, a $\mathbb{Z}$-basis of $\mathcal{I}$. Both together define the quadratic form

$$f_\mathfrak{q} \colon \mathcal{I} \to \mathbb{Q},\ \omega = \sum u_k \omega_k \mapsto \sum_{i,j} q_{ij} u_i u_j =: f_\mathfrak{q}(\omega)$$

on $\mathcal{I}$ and the **congruence theta functions**

$$\vartheta(\omega; \mathfrak{a}Q\mathfrak{d}, \rho, \mathfrak{q}) := \sum_{\mu \equiv (\mathfrak{a}Q\mathfrak{d})} \mathrm{e}^{\pi f_\mathfrak{q}(\omega\mu)/|n(\mathfrak{a})|Q\sqrt[4]{\det(\mathfrak{Q})}N(\mathfrak{d})}.$$

As in the modular case one gets via a $\mathrm{Mat}_2(\mathbb{R})$-integration functions Φ connecting congruence Zeta and Theta functions, namely:

$$\Phi(s; \mathfrak{a}Q\mathfrak{d}, \rho, \mathfrak{Q}) = \int_{\mathfrak{F}_1} [\vartheta(\omega; \mathfrak{a}Q\mathfrak{d}, \rho, \mathfrak{Q}) - \delta_{\rho,0}] \cdot |N(\omega)|^{s-1}\, \mathrm{d}U$$

$$\Phi(s; \mathfrak{a}Q\mathfrak{d}, \rho, \mathfrak{Q}) = \frac{\pi^{-2s}(\det(\mathfrak{Q}))^{s/2}}{(Q^2|n(\mathfrak{d})|)^s} \Gamma(s; \mathfrak{Q}) \zeta(s; \mathfrak{a}Q\mathfrak{d}, \rho)$$

where $\mathfrak{F}_1$ is a fundamental domain with respect to our unit group $(\mathfrak{a}Q\mathfrak{d})_1'$, and the Gamma-function factor is

$$\Gamma(s; \mathfrak{Q}) = \det(\mathfrak{Q})^{-s/2} |\Delta|^{(s-1)/2} \pi^{3/2} \Gamma(s) \Gamma\left(s - \frac{1}{2}\right)$$

with discriminant Δ of $\mathcal{I}$.

On this way Schoeneberg uses transformation laws for the Theta functions to get the functional equation for the congruence Zeta functions

Functional Equation 9.2.

$$\zeta(1 - s; \mathfrak{a}Q\mathfrak{d}, \rho) = \frac{(2\pi)^{2(1-2s)}}{(Q^2 n(\mathfrak{d}))^{2s}} \frac{\Gamma(2s)}{\Gamma(2 - 2s)} \times \sum_{\mathfrak{a} \ni \alpha \bmod \mathfrak{a}Q\mathfrak{d}} \mathrm{e}^{2\pi s\left(\frac{\alpha\rho'}{|n(\mathfrak{a})|Qn(\mathfrak{d})}\right)} \zeta(s; \mathfrak{a}Q\mathfrak{d}, \alpha).$$

Shimura transferred in [22] hermitian spaces of signature $(1, 1)$ over imaginary quadratic number fields to indefinite quaternion spaces over $\mathbb{Q}$ together with the transfer of unitary group action to unit group actions of quaternion orders with diagrams similar to (9.3), (9.4) but with S instead of $\mathrm{Mat}_2(\mathbb{Q})$ and the automorphism group of a maximal order instead of $\mathbb{S}l_2(\mathbb{Q})$. Along this way Kudla translated in [18] Schoeneberg's quaternionic congruence Zeta and Theta series to hermitian ones as described in 9.2, Definition (8.19). He observed that the related function $E_0(\tau; A_0)$ defined in (8.24) is the Mellin transform of $\theta_0(iy; A_0)$. Then Kudla discovered in the cocompact unitary case the important composed function $g(\tau; A)$, see (8.25) with nice transformation law 8.5. For more details we refer to [18], Section 8.

Appendix A. Signature Heights of Orbital Curves

For our geometric constructions and applications we need explicit formulas for signature heights of orbital curves and the proof of the height property (2.7). We extend Section 2 using immediately the notations there. There are two types of finite orbital points: Let

$$\hat{U}_S \to \hat{V}_R = \hat{U}_S / G_S$$

be a representative local finite uniformization of $\mathbf{R}$, $R \in X = \hat{X} \setminus X^\infty$. If the finite isotropy group G_S is abelian, then we call $\mathbf{R}$ an *abelian (orbital) point*. Otherwise we say that $\mathbf{R}$ is *non-abelian*. From the classification of finite orbital points in [12] respecting the singularity type of $R \in X$ and the local branch situation around we know that the definitions are correct. This means that both cases are well-distinguished. Each abelian point is supported by a **cyclic singularity**. This is a surface singularity locally isomorphic to $(\mathbb{C}^2, 0)/Z$, $Z \neq 1$ a finite cyclic subgroup of $\mathbb{G}l_2(\mathbb{C})$, such that 1 is not an eigenvalue of any non-trivial element of Z. Sometimes it is convenient to include also a smooth point (with $Z = 1$) in our terminology. Then we use the notion of *cyclic point* for both cyclic cases, the non-trivial and the trivial, together.

We are able to remove cusp points and non-abelian points substituting them by elliptic curves or projective lines. For this purpose consider a global uniformization $\hat{Y} \to \hat{\mathbf{X}}$, and blow up the cusp singularies of $\hat{Y}$ to elliptic cusp curves as described in diagram 2.1. The p'_G-images of these curves on X'/G are elliptic again or isomorphic to $\mathbb{P}^1$. Now we blow up all points $S \in Y$ with non-abelian isotropy group G_S to exceptional lines L_S. Then one gets a model Y^0 of Y' and birational morphisms $Y^0 \to Y' \to \hat{Y}$. Going down to the G-quotient surfaces we obtain birational morphisms $X' \to Y'/G \to \hat{X}$. Now it is clear that

- X' has only cyclic singularities
- the exceptional (Weil) divisor of $\varphi' \colon X' \to \hat{X}$ is a disjoint sum of projective lines and, perhaps, some elliptic curves.

The orbital surface $\mathbf{X}'$ defined by the branch divisor of the quotient morphism $Y^0 \to X' = Y^0/G$ has only abelian orbital points. It is uniquely determined by $\hat{\mathbf{X}}$, more precisely by its non-abelian and infinite orbital points. We call it the $\hat{\mathbf{B}}^0$-*model* of $\hat{X}$.

- the proper transform C_i' on X' of the component $\hat{C}_i$ of $\hat{B}^1$ has at most double points as singularities
- the curve singularities of C_i' are precisely the (non-resolved) singularities of $\hat{C}$ sitting at abelian points of $\hat{\mathbf{X}}$.

Namely (see [12]), the (smooth) tangent directions of germes at $S \in Y$ of the ramification locus of the Galois covering $Y \to X$ are eigenlines of reflections of G_S. They are separated after blowing up S. If G_S is non-abelian, there are at most three G_S-inequivalent reflection lines. The image germes through L_S/G_S are smooth and separated. For abelian G_S there are at most two G_S-inequivalent reflection lines in the tangent plane.
Now consider an orbital curve $\hat{\mathbf{C}}$ on the orbital surface $\hat{\mathbf{X}}$.

Definition A.1. *Critical points of $\hat{C}$ are the singularities of $\hat{C}$, the cusp points and the non-abelian orbital points of $\hat{\mathbf{X}}$ supported by $\hat{C}$-points.*

Lemma-Definition A.1. *There is a unique model $X^0_{\hat{C}}$ of $\hat{X}$ and a birational morphism $\varphi'_{\hat{C}}\colon X^0_{\hat{C}} \to \hat{X}$ factorizing through X' with the following*

Conditions A.1.

- *The proper transform C^0 of $\hat{C}$ on $X^0_{\hat{C}}$ is a smooth curve*
- *the exceptional (Weil-) divisor $E(\varphi'_{\hat{C}})$ is a disjoint sum of elliptic or smooth rational curves*
- *$E(\varphi'_{\hat{C}})$ is contracted by $\varphi'_{\hat{C}}$ onto the critical points of $\hat{C}$*
- *C^0 crosses the components of $E(\varphi'_{\hat{C}})$ at each common point.*

We call $X^0_{\hat{C}}$ the C^0-*model* of $\hat{X}$. Thereby we say that C' *crosses* a curve L at a regular surface point, if the intersection is transversal there. If the intersection point is a cyclic singularity, say locally isomorphic to $(\mathbb{C}^2, O)/Z$, then crossing means: locally isomorphic to the intersection of the images of two different Z-eigenlines on $\mathbb{C}^2$. This property can be equivalently defined via minimal resolution of the cyclic singularity, see below.

Proof: (Lemma A.1) We work with a $\hat{C}$-*uniformization* $\hat{Y}$ of $\hat{\mathbf{X}}$, diagrams 2.1, 2.4 and the notations around. Moreover, we dispose already on the models Y^0, X' after blowing up points with non-abelian G-isotropy groups. The conditions 2.2 for D' are preserved for the proper transform D^0 of D' on Y^0 and the exceptional divisor $E(\psi^0)$ of $\psi^0\colon Y^0 \to Y' \to \hat{Y}$. The set of honest G-cross points of D^0 lies outside of $E(\psi^0)$. Now we have only to blow up these G-cross points to get a model $Y^0_{\hat{C}}$. Going down to the G-quotient surfaces we see that

$$X^0_{\hat{C}} := Y^0_{\hat{C}}/G \to X' = Y^0/G$$

resolves the singularities of $D'/G \subset X'$ by substituting each of these points by a projective line. Moreover, the conditions of A.1 are satisfied for $C^0 = D^0/G \subset X^0_{\hat{C}}$.

For the proof of uniqueness we have to look carefully to the resolution $X^0_{\hat{C}} \to X'$ of the D'/G-singularities supported by cyclic surface points R. Let S be a (honest) G-cross point of D' over R and $L = L_S \subset Y^0_{\hat{C}}$ the exceptional line over S. The abelian isotropy group G_S acts on L. Let S_1, S_2 be the points on L corresponding to two different eigenlines of G_S in the tangential representation $G_S \subset \mathbb{G}l(T_S)$. Their image points R_1, R_2 on $L_{S/G} \subset X^0_{\hat{C}}$ are the only possible surface singularities on $L/G_S \subset X^0_{\hat{C}}$. The curve germs of GD' at R correspond to different fibres in the normal bundle N of L in $Y^0_{\hat{C}}$, which is a G_S-bundle. On this way we get bijective correspondences

$$\begin{aligned}\{GD'\text{-components at } S\}/G_S \ &\Leftrightarrow \ \{\text{ branches of } D'/G \text{ at } R\} \\ &\Leftrightarrow \ \{\text{ branches of } C^0 \text{ through } L_S/G_S\}\,. \end{aligned} \qquad \text{(A.1)}$$

The (smooth) branches of C^0 through $L_R = L_S/G_S$ are separated because C^0 is smooth. Each of them crosses the exceptional line L_R. On L_R there are at most two (cyclic) surface singularities R_1, R_2. Now take a minimal (surface) singularity resolution of R_1 and R_2. Altogether we get a linear singularity resolution of R with the marked component L'_R, the proper transform of L_R. Linear means, that the resolving curve consists of a linear tree of (at most) transversally intersecting projective lines. Because of crossing intersections the transformed curve germes intersect transversally the marked line or one of the two external lines of the tree but not at a singularity of the tree. Since we have at least two C^0-curve germes through the tree, it is easy to see that a contraction of the tree to one projective component L_j (producing at most two cyclic surface points on it) with the crossing condition for the C^0-germes is only possible for $L_j = L_R$. □

Altogether we have for each $\hat{\mathbf{X}}$-orbital curve $\hat{C}$ on $\hat{X}$ a commutative diagram

$$\begin{array}{ccccccc} Y^0_{\hat{C}} & \longrightarrow & Y^0 & \longrightarrow & Y' & \longrightarrow & \hat{Y} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{X}^0_{\hat{\mathbf{C}}} & \longrightarrow & \mathbf{X}' & \longrightarrow & \mathbf{Y}'/\mathbf{G} & \longrightarrow & \hat{\mathbf{X}} \end{array} \qquad \text{(A.2)}$$

with corresponding orbital curve coverings

$$\begin{array}{ccccccc} D^0 & \xrightarrow{\sim} & D^0 & \xrightarrow{\sim} & D' & \longrightarrow & \hat{D} \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathbf{C}^0 & \longrightarrow & \mathbf{C}' & \longrightarrow & \mathbf{D}'/\mathbf{G} & \longrightarrow & \hat{\mathbf{C}} \end{array} \tag{A.3}$$

Let γ be a smooth curve germ through a point R on the surface $\hat{X}$ supporting the orbital surface $\hat{\mathbf{X}}$. We only need curve germs localizing global curves. If R is not a cusp point we consider a (finite) local uniformization $S \to \mathbf{R}$ with Galois group G_S and a preimage germ δ of γ through S. Then γ is called an *eigen germ* at $\mathbf{R}$, if the tangent line of δ is an eigenline of an element of G_S not belonging to the *symmetry subgroup* ZG_S of G_S defined as

$$ZG_S := \rho^{-1}(\rho(G_S) \cap \mathbb{Z}l_2(\mathbb{C}))$$

where $\mathbb{Z}l_2(\mathbb{C})$ denotes the center of $\mathbb{G}l_2(\mathbb{C})$ and ρ the tangential plane representation of G_S at S. We say that G_S is *symmetric* iff $ZG_S = G_S$ and call $\mathbf{R}$ a *symmetry point* in this case. The blowing up of the smooth surface point S to the exceptional line L_S defines the modification $X'(\mathbf{R}) \to \hat{X}$ plugging in the quotient line $L_{\mathbf{R}} = L_S/G_S$. The ineffective kernel of G_S with respect to the action on L_S is ZG_S. A *honest* G_S *eigenline* in the tangent plane T_S is an eigenline of a non-central element of G_S. The honest eigenlines are in bijective correspondence with all isolated fixed points on L_S of elements of G_S. If G_S is non-symmetric, these fixed points go down to two (if G_S is abelian) or three (if G_S is non-abelian) marked (branch) points $P_1, P_2, (P_3)$ on $L_{\mathbf{R}}$, which are cyclic surface points. This marking is uniquely determined by $\mathbf{R}$ only. It is easy to check that

Proposition A.1. *γ is an eigen-germe at $\mathbf{R}$ iff its proper transform γ' on $X'(\mathbf{R})$ crosses $L_{\mathbf{R}}$ at one of these marked two or three points.*

This fact allows to define eigen-germes purely geometrically on $\hat{\mathbf{X}}$.

Definition A.2. *The* weight of an orbital point $\mathbf{R}$ *on* $\hat{\mathbf{X}}$ *is defined as*

$$v_{\mathbf{R}} := \begin{cases} 1, & \textit{abelian point} \\ \infty, & \textit{cusp point} \\ v_{L_{\mathbf{R}}}, & \textit{non-abelian} \end{cases}$$

where the latter number is the ramification index of L_S (or $L_{\mathbf{R}}$) with respect to the Galois-covering $L_S \to \mathbf{L}_{\mathbf{R}}$ of orbital curves.

Let $\hat{\mathbf{C}}$ be an orbital curve on $\hat{\mathbf{X}}$, R a critical point of its supporting curve $\hat{C}$ and L_R the exceptional line on $X^0_{\hat{C}}$ over R. Remember that the proper transform C^0 on $X^0_{\hat{C}}$ crosses L_R at each common point. Especially, the L_R-intersecting curve germes of C^0 are smooth and separated. Its number is denoted by $Br_R(\hat{C})$. It is clear that

$$Br_R(\hat{C}) = \text{ number of curve branches of } \hat{C} \text{ at } R.$$

Lemma-Definition A.2. *If $\hat{C}$ has a branch at R, which is an eigen germe, then also the germes of $\hat{C}$ at R are eigen germes. In this case we say that $\hat{C}$* has eigen branches at R *or, equivalently,* $\mathbf{R}$ *is an* eigen point *of* $\hat{\mathbf{C}}$.

Proof: Assume that there are at least two $\hat{C}$-branches at R, which means that R is a curve singularity of $\hat{C}$. Looking at a $\hat{C}$-uniformization $Y' \to Y'/G$ with curve covering $D' \to D'/G$, see diagram (2.4), we know from (2.5) that the branches come from G-equivalent points on D', which are not $N_G(D')$-equivalent. But if the branch of D' at S (over R) has eigen vector direction with respect to G_S, then the G-equivalent branches must have the same property. □

Example A.1. *All orbital points of any component $\hat{\mathbf{C}}_i$ of the basic orbital divisor are eigen points, because the covering $\hat{D}_i$ on a finite uniformization is pointwise fixed by at least one non-trivial group element.*

Corollary A.1. *If $\hat{C}$ has eigen branches at the orbital point $\mathbf{R} \in \mathbf{X}$, then for the number of branches of $\hat{C}$ at R it holds that*

- $Br_R(\hat{C}) \in \{1, 2, 3\}$, *if* $\mathbf{R}$ *is non-abelian*
- $Br_R(\hat{C}) = 2$, *if* $\mathbf{R}$ *is abelian and R is a singularity of $\hat{C}$.*

So we have only triple or double singular points as curve singularities at eigen points of $\hat{C}$.

Proof: This follows immediately from Proposition A.1. □

Now we need minimal resolutions of cyclic singularities. A cyclic singularity is of type $\langle d, e\rangle$, if it is isomorphic to $(\mathbb{C}^2, O)/\operatorname{diag}(\zeta, \zeta^e)$, with a primitive d-th unit root ζ and $e \in \mathbb{N}$, relatively prime to d, $0 \leq e < d$. From the isomorphy classification of singularities it is well-known that two singularities of types $\langle d, e\rangle$ and $\langle d', e'\rangle$ are isomorphic iff $d = d'$ and $e = e'$ or $ee' \equiv 1 \mod d$. So $d_P := d$ is well-defined. In order to distinguish them we introduced in [12] an orientation related with a curve germe γ on a surface crossing another one at the cyclic singularity P of type $\langle d, e\rangle$. The minimal resolution of P is a linear tree of projective lines L_i, $i = 1, \ldots, r$, with selfintersections $-b_i \leq -2$, calculable by the continued fractions

$$d/e = b_1 - \cfrac{1}{b_2 - \cdots - \cfrac{1}{b_{r-1} - \cfrac{1}{b_r}}}$$

or

$$d/e' = b_r - \cfrac{1}{b_{r-1} - \cdots - \cfrac{1}{b_2 - \cfrac{1}{b_1}}}.$$

The crossing property is equivalenly reflected by one of the following two possibilities for the proper transform γ' of γ on the resolving surface: either γ' crosses L_1 or it crosses L_r (outside of L_2 respectively L_{r-1}, if $r \geq 1$). In the first case we set $e_P(\gamma) = e$, in the latter case $e_P(\gamma) = e'$. The pair $\langle d_P, e_P(\gamma)\rangle$ is called the *γ-oriented type* of the cyclic singularity P. We set

$$\Sigma^0 = \Sigma^0(C^0, X^0) := \{\text{cyclic surface singularities } P \in C^0 \subset X^0\};$$

$$h_P(\hat{C}) = \frac{e_P(C^0)}{v d_P}$$

$$\Sigma' = \Sigma(C') := \{\text{singular points } P' \text{ of } C'\}$$

$$h_{P'}(\hat{C}) := \begin{cases} \frac{2}{v_{\hat{C}} d_{P'}} & P' \text{ eigen point of } C' \\ Br_{P'}(C') & \text{else} \end{cases}$$

$$\Sigma^{na} = \Sigma^{na}(\hat{C}, \hat{\mathbf{B}}^0) := \{\text{non-abelian orbital points } \mathbf{Q} \in \mathbf{X}\}$$

$$h_{\mathbf{Q}}(\hat{C}) := \begin{cases} \frac{Br_Q(\hat{C})}{d_Q(\hat{C}) v_{\mathbf{Q}}} & Q \text{ eigen point of } \hat{C} \\ Br_Q(\hat{C}) & \text{else.} \end{cases}$$

These rational numbers are called *local signature heights* of $\hat{C}$ at the points P, P', Q, respectively.

Thereby $d_Q(\hat{C}) := d_{P_i}(C')$, $i \in \{1, 2, 3\}$, where P_i is a (marked) cross point of L_Q and C'. If C' goes through two of them, then their preimages on the blown up surface Y' of a finite uniformization $\hat{Y}$ of $\hat{\mathbf{X}}$, as described around diagram 2.1, are G-equivalent. Otherwise the point P_i would be uniquely determined. Therefore the singularity type of P_i , hence also d_{P_i}, does not depend on the choice of $P_i \in C'$.

Definition A.3. *With the above notations and the minimal singularity resolution* $\tilde{X} \to X^0_{\hat{C}}$ *with proper transform* $\tilde{C}$ *of* C^0 *we call*

$$h(\hat{\mathbf{C}}) := \frac{1}{v_{\hat{\mathbf{C}}}}(\tilde{C}^2) + \sum_{P\in\Sigma^0} h(P) + \sum_{P\in\Sigma'} h(P') + \sum_{\mathbf{Q}\in\Sigma^{na}} h(Q) \tag{A.4}$$

the signature height *of* $\hat{\mathbf{C}}$.

Let $\hat{\mathbf{p}}: \hat{Y} \to \hat{\mathbf{X}}$ and $\hat{\mathbf{q}}: \hat{Y} \to \hat{\mathbf{Z}}$ be finite uniformizations with the same covering surface $\hat{Y}$. If the supporting Galois covering $\hat{p}$ factors though $\hat{q}$, then we call the induced orbital morphism $\hat{\mathbf{Z}} \to \hat{\mathbf{X}}$ a *finite orbital surface covering*. Its restriction $\hat{\mathbf{D}} \to \hat{\mathbf{C}}$ to two orbital curves is a *finite orbital curve covering* . For the general definitions of orbital morphisms we refer to [12]. We are now able to generalize the degree formula in [12] for orbital branch curves to arbitrary orbital curves.

Theorem A.1. *For finite orbital curve coverings as above it holds that*

$$h(\hat{\mathbf{D}}) = [\hat{D} : \hat{C}]h(\hat{\mathbf{C}})\,. \tag{A.5}$$

Proof: Since the degree is multiplicative for compositions of curve coverings, it suffices to prove the degree formula (A.5) for the case $\hat{\mathbf{Z}} = \hat{\mathbf{Y}} := (\hat{Y}, Y^\infty)$. For smooth open orbital curves we defined signature heights h_τ in [12]. If there are moreover only abelian points on the underlying orbital surface, the signature height depends only on selfintersection of the supporting (compact) curve and all cyclic singularities on it. This happens in the case $\mathbf{C}^0 \subset \mathbf{X}^0_{\hat{C}}$. Its signature height is

$$h_\tau(\mathbf{C}^0) = \frac{1}{v}(\tilde{C}^2) + \sum_{P\in X^0_{\hat{C}}} \frac{e_P(C^0)}{v d_P}, \qquad v = v_{\hat{\mathbf{C}}}\,.$$

Since $e_P(C^0) = 0$ for smooth surface points, the sum runs only over the cyclic singularities, which form the finite set of all singularities. Comparing with (A.4) we see that $h_\tau(\mathbf{C}^0)$ is nothing else but $h(\mathbf{C}^0)$. Especially, this is true also for the orbital curve D'^0 on $Y^0_{\hat{C}}$, where we have simply

$$h_\tau(D'^0) = h(D'^0) = (D'^0 \cdot D'^0)$$

because of absence of singularities. The degree formula for h_τ has been already proved in [12] for this situation. Therefore it holds that

$$(D'^0 \cdot D'^0) = h(D'^0) = [\hat{D} : \hat{C}]h(\mathbf{C}^0)\,. \tag{A.6}$$

Our strategy is to shift this formula stepwise from the left side to the coverings on the right hand side in the diagrams A.2 and A.3. We have first to count the number $\#'$ of blown up points of the morphism $Y^0_{\hat{C}} \to Y^0$ because

$$h(D^0) = (D^0 \cdot D^0) = (D'^0 \cdot D'^0) + \#' = h(D'^0) + \#'. \tag{A.7}$$

These points ly over the points $P' \in \Sigma' = \Sigma(C') \subset X$. For fixed P' we can count the points $S \in D^0$ over P'. This number $Bl_{P'}(D^0)$ is equal to $Br_{P'}(C') \cdot \deg D^0 \cdot \#G_{D^0,S}$ by (A.1), where

$$G_D := N_G(D)/Z_G(D)$$

with

$$N_G(D) := \{g \in G;\, g(D) = D\}, \qquad Z_G(D) := \{g \in G;\, g|_D = \mathrm{id}_D\}\,.$$

for a finite group G acting effectively on a surface supporting the curve D. If $\mathbf{P}'$ is not an eigen point of $\mathbf{C}'$, we have $G_{D^0,S} = 1$. If $\mathbf{P}'$ is an eigen point, then G_S is abelian of order $v^2 d_{P'}$, hence $\#G_{D^0,S} = vd_{P'}$, and $Br_{P'}(C') = 2$. In any case, by the definitions before (A.4), we get $Bl_{P'}(D^0) = [\hat{D} : \hat{C}]h_{P'}(\hat{C})$. The sum over all $P' \in \Sigma'$ and A.7 yield

$$h(D^0) = h(D'^0) + [\hat{D} : \hat{C}](\sum h_{P'}(\hat{C}))\,.$$

Together with A.6 and Definition A.3 applied to $\mathbf{C}'$ we get

$$h(D^0) = [\hat{D} : \hat{C}](h(\mathbf{C}^0) + \sum h_{P'}(\hat{C})) = [\hat{D} : \hat{C}]h(\mathbf{C}')\,.$$

In the same style one shifts the formula further to the covering $D' \to \mathbf{D}'/\mathbf{G}$ to get the resulting formula (A.5). The check of more details is left to the reader. □

References

[1] Barthel G., Hirzebruch F. and Höfer Th., *Geradenkonfigurationen und algebraische Flächen*, Aspects of Mathematics D **4**, Vieweg, Braunschweig-Wiesbaden 1986.

[2] Busam R. and Freitag E., *Funktionentheorie*, Springer-Lehrbuch, Berlin 1993.

[3] Cogdell J., *Arithmetic Cycles on Quotients of the Complex 2-Ball and Modular Forms of Nebentypus*, Thesis, Rutgers University, 1982.

[4] Cogdell J., *Congruence Zeta Functions for $M_2(\mathbb{Q})$ and their Associated Modular Forms*, Math. Ann. **266** (1983) 141–198.

[5] Cogdell J., *Arithmetic Cycles on Picard Modular Surfaces and Modular Forms of Nebentypus*, Journ. f. Math. **357** (1984) 115–137.

[6] Eichler M., *Lectures on Modular Correspondences*, Tata Institute, Bombay, 1957.

[7] Feustel J.-M., *Spiegelungs- und Spitzenkontributionen zum Arithmetischen Geschlecht Picardscher Modulflächen*, Habilitation, Karl-Weierstraß-Institut, Akad. d. Wiss. d. DDR, 1990.

[8] Fulton W., *Intersection Theory*, Springer 1984.

[9] Hecke E., *Zur Theorie der elliptischen Modulfunktionen*, Math. Ann. **97** (1926) 210–242.

[10] Hecke E., *Über die Bestimmung Dirichletscher reihen durch ihre Funktionalgleichung*, Math. Ann. **112** (1936) 664–699.

[11] Hirzebruch F. and Zagier D., *Intersection Numbers of Curves on Hilbert Modular Surfaces and Modular Forms of Nebentypus*, Inv. Math. **36** (1976) 57–113.

[12] Holzapfel R.-P., *Ball and Surface Arithmetics*, Aspects of Mathematics E **29**, Vieweg, Braunschweig-Wiesbaden 1998.

[13] Holzapfel R.-P., *Jacobi Theta Embedding of a Hyperbolic 4-Space with Cusps*, In: Geometry, Integrability and Quantization, III. Mladenov and G. Naber (Eds), Coral Press, Sofia 2001, pp 11–63.

[14] Holzapfel R.-P. and Vladov N., *Picard-Einstein Metrics on* $\mathbb{P}^2$ *Degenerated Along Projective Plane Quadrics*, Preprint-Ser., Inst. Math. Humboldt-Univ. **2000-4**, Berlin 2000. Also in: Mitt. d. Berliner Math. Ges., 2002.

[15] Holzapfel R.-P., Pineiro and Vladov N., *Picard-Einstein Metrics and Class Fields Connected with Apollonius Cycle*, Preprint-Ser., Inst. Math. Humboldt-Univ. **2000-4**, Berlin 2000.

[16] Ivinskis K., *Normale Flächen und die Miyaoka-Kobayashi-Ungleichung*, Diplomarbeit, Bonn, 1985.

[17] Koblitz N., *Introduction to Elliptic Curves and Modular Forms*, Springer, New York-Berlin-Heidelberg-Tokyo 1984.

[18] Kudla S., *Intersection Numbers for Quotients of the Complex 2-Ball and Hilbert Modular Forms*, Inv. Math. **47** (1978) 189–515.

[19] Maass H., *Über eine neue Art von nichtanalytischen automorphen Funktionen und die Bestimmung Dirichletscher Reihen durch Funktionalgleichungen*, Math. Ann. **121** (1949) 141–183.

[20] Mumford D., *The Topology of Normal Singularities of an Algebraic Surface and a Criterion for Simplicity*, Publ. Math. I.H.E.S. **9** (1961) 5–22.

[21] Schoeneberg B., *Indefinite Quaternionen und Modulfunktionen*, Math. Ann. **113** (1936) 380–391.

[22] Shimura G., *Arithmetics of Unitary Groups*, Ann. Math. **79** (1964) 369–409.

[23] Shimura G., *On Some Arithmetic Properties of Modular Forms of One and Several Variables*, Ann. Math. **102** (1975) 491–515.