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# EXACT SOLUTIONS OF BOUSSINESQ EQUATION

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**Abstract**. By considering the Boussinesq equation as a zero curvature representation of some third order linear differential equation and factorizing this linear differential equation, the hierarchy of solutions of Boussinesq equation has been obtained from the eigen spectrum of constant potentials.

## 1. Introduction

Integrable systems of nonlinear partial differential equations are among the central and fundamental problems of physics and mathematical physics, consequently they have attracted much interest both in theoretical physics and mathematics. They have numerous applications in many different branches of physics and at the same time they show a rich mathematical structure. These structures include Lax pairs, Miura maps, Bäcklund transformations, infinitely many local conservation laws and applicability of inverse scattering methods [4–6, 16].

Here we are concerned with Boussinesq equation [4, 7], which describes one-dimensional weakly nonlinear dispersive water waves [6, 10]. The Boussinesq equation has also a wide application in different branches of physics. For examples it can appear as a special limit of unelastic magnetohydrodynamic equations for modeling solar and stellar convection zones [17, 18], the electromagnetic field in dispersive nonlinear dielectrics is governed by a Boussinesq equation that has solitary solutions [26] and it can be appropriate for the propagation of near-sonic envelope electromagnetic waves in magnetized plasma [20].

The Boussinesq equation, similarly to Korteweg-de Vries (KdV) and non-linear equations have plenty of applications in plasma physics, too, where some of the important applications are: magnetosonic waves propagating in a multi-ion-species plasma perpendicular to an external magnetic field [2], the investigation of the stability of a compressible stratified shear layer by using the compressible magnetohydrodynamic (MHD) equations [23], the effects of the interaction between ion-sound solitons and resonance particles in a plasma [14], the propagation of nonlinear hydromagnetic waves in a highly conducting, self-gravitating fluid in a spherical geometry, subject to the convective forces produced by a radial temperature gradient [9], localized nonlinear structures of intense electromagnetic waves in two-electron-temperature electron-positron plasmas [21], magnetoconvection in Boussinesq fluid with nonlinear interaction between Rayleigh-Bénard convection [1].

From the mathematical points of view, the Boussinesq equation is similar to the other well-known integrable partial differential equations such as Korteweg-de Vries (KdV), modified Korteweg-de Vries (mKdV), sin-Gordon, Liouville [6, 16], the nonlinear Schrödinger equation [7] and so on. These nonlinear partial differential equations can be represented as a zero curvature condition or integrability condition of some second order linear partial differential equation [7, 19]. But, there is an exception in the case of Boussinesq equation, as it can be represented as the zero curvature condition of a third order linear partial differential equation. The close relation between the Bäcklund transformation and zero curvature condition has been shown in Boussinesq equation, too. Actually it is shown that Bäcklund transformations are related to some kind of gauge transformation. Using this fact together with the factorization of linear partial differential equation we have been able to find analytical solutions of Boussinesq equation. Actually this prescription is somehow similar to the procedure of obtaining solutions of KdV from free particle Schrödinger equation through the well known technique of supersymmetric quantum mechanics [3, 8, 11–13, 15, 22, 24, 25]. The Boussinesq equation can be obtained as the zero curvature condition for the following system of linear differential equations

$$\psi_{xxx} + \left(\frac{3}{2}w - \frac{3}{4}\right)\psi_x + u\psi = \lambda\psi\,, \tag{1}$$

$$\psi_t = -\psi_{xx} - w\psi\,. \tag{2}$$

Introducing the vector

$$F = \begin{pmatrix} \psi \\ \psi_x \\ \psi_{xx} \end{pmatrix}\,, \tag{3}$$

the above equations can be rewritten in the following form:

$$F_x = UF\,, \\ F_t = VF\,. \tag{4}$$

where the matrices $U$ and $V$ are defined as:

$$U = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ \lambda - u & \frac{3}{4} - \frac{3}{2}w & 0 \end{pmatrix}, \tag{5}$$

$$V = \begin{pmatrix} -w & 0 & -1 \\ u - w_x - \lambda & \frac{w}{2} - \frac{3}{4} & 0 \\ u_x - w_{xx} & u - \frac{w_x}{2} - \lambda & \frac{w}{2} - \frac{3}{4} \end{pmatrix}. \tag{6}$$

The integrability of the pair of equations in (4) or zero curvature condition

$$U_t - V_x + [U, V] = 0\,, \tag{7}$$

implies that the potential $w$ should satisfies the well-known Boussinesq equation

$$w_{tt} - w_{xx} + (w^2)_{xx} + \frac{w_{xxx}}{2} = 0\,. \tag{8}$$

This work is organized as follows. In Section 2, we factorize third order linear differential equations. Section 3 is devoted to isospectral deformation of these partial differential equations and the connection between the zero curvature representation and Bäcklund transformation. In Section 4, the hierarchy of third order linear differential equations are obtained and then the hierarchy of the solutions of the Boussinesq equation are found analytically.

## 2. Factorization of the Ordinary Linear Eq. (1)

Defining the differential operator $L_1$ as:

$$L_1 = \partial^3 + \left(\frac{3}{2}w_1 - \frac{3}{4}\right)\partial + u_1\,, \tag{9}$$

Eq. (1) can be considered as its eigenvalue equation, namely:

$$L_1\psi_1 = \lambda\psi_1\,. \tag{10}$$

Now let us assume that the operator $L_1$ can be factorized into the product of three first order differential operators in the following form [4]:

$$L_1 = A_1 A_2 A_3 + c\,, \tag{11}$$

where

$$A_i = \partial - v_i\,. \tag{12}$$

Therefore we have

$$L_1\psi_i = \lambda\psi_i\,, \tag{13}$$

where

$$\psi_i = A_{i-1}\psi_{i-1}\,, \qquad i = 2, 3\,, \tag{14}$$

and

$$\psi_1 = A_3\psi_3\,. \tag{15}$$

The above relations between the eigen-functions $\psi_j$, $j = 1, 2, 3$ can be represented by the following diagram

$$\psi_1 \xrightarrow{A_1} \psi_2 \xrightarrow{A_2} \psi_3 \xrightarrow{A_3} \psi_1\,. \tag{16}$$

We try to obtain the functions $v_1$, $v_2$ and $v_3$ from the corresponding wave functions. Hence choosing $\lambda = c$, Eqs (10), (13) and (15) reduce to

$$(\partial - v_i)\psi_i(c) = 0\,. \tag{17}$$

Therefore, we have

$$v_i = \frac{\partial}{\partial x}\log\psi_i(c)\,, \qquad i = 1, 2, 3\,. \tag{18}$$

We must notice that if we take $v_1 = \frac{\partial}{\partial x}\log\psi_1(c)$ and $v_2 = \frac{\partial}{\partial x}\log\psi_2(c)$, then the Eq. (18) leads to:

$$v_3 = \frac{\partial}{\partial x}\log\frac{1}{\psi_1(c)\psi_2(c)}\,. \tag{19}$$

We see that the eigen-function $\psi_2$ can be obtained from $\psi_1$ by action of the operator $A_1$, since the latter belongs to the kernel of the operator $A_1$, hence, in order to obtain $\psi_2$, we have to act the operator $A_1$ over the same linear superposition of the eigen-functions belonging to Eq. (10). Therefore, for a given linear superposition of the eigen-function of Eq. (10) denoted by $\psi_1(c)$, we choose another linear superposition of the eigenvalue Eq. (10) denoted by $\phi_1(c)$, which is not belong to the kernel of the operator $A_1$. Then we have:

$$\psi_2(c) = A_1\phi_1(c) = \frac{W\big(\psi_1(c), \phi_1(c)\big)}{\psi_1(c)}\,, \tag{20}$$

where $W(\psi_1(c), \phi_1(c))$ is the usual Wronskian of the functions $\psi_1(c)$ and $\phi_1(c)$, where the non-vanishing of Wronskian indicates their independence. Finally by substituting (20) in (19), we obtain:

$$v_3 = \frac{\partial}{\partial x} \log \frac{1}{W\Big(\psi_1(c), \phi_1(c)\Big)} \,. \tag{21}$$

## 3. Deformation of Functions $u_1$, $w_1$

By defining the vector field $F_1 = \begin{pmatrix} \psi_1 \\ \psi_{1x} \\ \psi_{1xx} \end{pmatrix}$ and using the relation $\psi_2 = A_1\psi_1$, we have

$$F_2 = \begin{pmatrix} \psi_2 \\ \psi_{2x} \\ \psi_{2xx} \end{pmatrix} = G_1 \begin{pmatrix} \psi_1 \\ \psi_{1x} \\ \psi_{1xx} \end{pmatrix} , \tag{22}$$

where $G_1$ is:

$$G_1 = \begin{pmatrix} -v_1 & 1 & 0 \\ -v_{1x} & -v_1 & 1 \\ \lambda - u_1 - v_{1xx} & \frac{3}{4} - \frac{3}{2}w_1 - 2v_{1x} & -v_1 \end{pmatrix} . \tag{23}$$

Assuming that $G_1$ is invertible and taking derivative with respect to $x$ on both sides of Eq. (33), we get:

$$\frac{\partial F_2}{\partial x} = (G_{1x}G_1^{-1} + G_1 U_1 G_1^{-1})F_2 \,, \tag{24}$$

similarly taking the derivative with respect to $t$ on both sides of Equation (33), we obtain:

$$\frac{\partial F_2}{\partial t} = (G_{1t}G_1^{-1} + G_1 V_1 G_1^{-1})F_2 \,, \tag{25}$$

where $U_1$ and $V_1$ can be obtained from the matrices (5) and (6) by the replacements $w \to w_1$ and $u \to u_1$. Defining matrices $U_2$ and $V_2$ associated with the vector field $F_2 = (\psi_2, \psi_{2x}, \psi_{2xx})^{\mathsf{T}}$ with $w = w_2$ and $u = u_2$ in matrices (5) and (6) , we see that they are gauge transformations of $U_1$ and $V_1$ with gauge group $G_1$, that is

$$\begin{aligned} U_2 &= G_{1x}G_1^{-1} + G_1 U_1 G_1^{-1} \,, \\ V_2 &= G_{1t}G_1^{-1} + G_1 V_1 G_1^{-1} \,. \end{aligned} \tag{26}$$

Also, it is well known that the curvature transforms homogeneously under the gauge transformations, hence the zero curvature condition is invariant under gauge transformations. Obviously for $\lambda = c$, the gauge group is not invertible, because we have $A_1\psi_1(c) = 0$ in this case.

Now substituting (23), $U_1$, $V_1$, $U_2$ and $V_2$ in the gauge transformations (26), we obtain:

$$\begin{aligned} u_2 - u_1 &= \frac{3}{2}w_{1x} + 3v_{1xx} + 3v_1v_{1x}\,, \\ w_2 &= w_1 + 2v_{1x}\,, \\ v_1(u_2 - u_1) &= -u_{1x} - v_{1xx} - \frac{3}{2}v_{1x}w_2 + \frac{3}{4}v_{1x}\,. \end{aligned} \tag{27}$$

By substituting the relations given in (27) in the second relation of (26), the time evolution of $v_1$ reduces to

$$v_{1t} + v_{1xx} + 2v_1v_{1x} = -w_{1x}. \tag{28}$$

Similarly, from the gauge transformation $\psi_3 = A_2\psi_2$ we have:

$$\begin{aligned} u_3 - u_2 &= \frac{3}{4}w_{2x} + 3v_{2xx} + 3v_2v_{2x}\,, \\ w_3 &= w_2 + 2v_{2x}\,, \\ v_2(u_3 - u_2) &= -u_{2x} - v_{2xx} - \frac{3}{2}v_{2x}w_3 + \frac{3}{4}v_{2x}\,, \\ v_{2t} + v_{2xx} + 2v_2v_{2x} &= -w_{2x}\,. \end{aligned} \tag{29}$$

Also using the gauge transformation $\psi_1 = A_3\psi_3$, we have:

$$\begin{aligned} u_1 - u_3 &= \frac{3}{4}w_{3x} + 3v_{3xx} + 3v_3v_{3x}\,, \\ w_1 &= w_3 + 2v_{3x}\,, \\ v_3(u_1 - u_3) &= -u_{3x} - v_{3xx} - \frac{3}{2}v_{3x}w_1 + \frac{3}{4}v_{3x}\,, \\ v_{3t} + v_{3xx} + 2v_3v_{3x} &= -w_{3x}\,. \end{aligned} \tag{30}$$

## 4. Hierarchy of Third Linear Differential Equations and the Solutions of Boussinesq Equation

Similar to the factorization of Schrödinger equation [22, 24], we consider the following series of operators

$$L_0^1, L_0^2, L_0^3 = L_1^1\,, \quad L_1^2, L_1^3 = L_2^1, \ldots, L_n^1\,, \quad L_n^2, L_n^3 = L_{n+1}^1\,, \quad \ldots\,. \tag{31}$$

According to the prescription of the Sections 2 and 3, the operators of rank $n$ can be written as:

$$\begin{aligned} L_n^1 &= A_n^3 A_n^2 A_n^1 + c_n \,, \\ L_n^2 &= A_n^1 A_n^3 A_n^2 + c_n \,, \\ L_n^3 &= A_n^2 A_n^1 A_n^3 + c_n \,, \end{aligned} \tag{32}$$

with

$$\begin{aligned} A_n^1 &= \partial - v_n^1 \,, \\ A_n^2 &= \partial - v_n^2 \,, \\ A_n^3 &= \partial - v_n^3 \,, \end{aligned} \tag{33}$$

where $v_n^1$, $v_n^2$ and $v_n^3$ are defined as:

$$\begin{aligned} v_n^1 &= \frac{\partial}{\partial x} \log \psi_n^1(c_n) \,, \\ v_n^2 &= \frac{\partial}{\partial x} \log \frac{W\Big(\psi_n^1(c_n), \phi_n^1(c_n)\Big)}{\psi_n^1(c_n)} \,, \\ v_n^3 &= \frac{\partial}{\partial x} \log \frac{1}{W\Big(\psi_n^1(c_n), \phi_n^1(c_n)\Big)} \,, \end{aligned} \tag{34}$$

where $\psi_n^1(c_n)$ and $\phi_n^1(c_n)$ are two different linear combinations of solutions of the eigenvalue equation $L_n^1 \psi_n^1 = c_n \psi_n^1$. Writing the operators $L_n^1$, $L_n^2$ and $L_{n+1}^1$ in the following form:

$$\begin{aligned} L_n^1 &= \partial^3 + \left(\frac{3}{2} w_n^1 - \frac{3}{4}\right) \partial + u_n^1 \,, \\ L_n^2 &= \partial^3 + \left(\frac{3}{2} w_n^2 - \frac{3}{4}\right) \partial + u_n^2 \,, \\ L_{n+1}^1 &= \partial^3 + \left(\frac{3}{2} w_{n+1}^1 - \frac{3}{4}\right) \partial + u_{n+1}^1 \end{aligned} \tag{35}$$

and using the prescription of the Section 3, we obtain:

$$\begin{aligned} w_n^2 &= w_n^1 + 2v_{nx}^1 \,, \\ w_{n+1}^1 &= w_n^2 + 2v_{nx}^2 \,. \end{aligned} \tag{36}$$

Now, adding the above equations we get

$$w_{n+1}^1 = w_n^1 + 2v_{nx}^1 + 2v_{nx}^2 \,, \tag{37}$$

that its iteration yields:

$$w_{n+1}^1 = w_0^1 + 2\frac{\partial}{\partial x}\sum_{i=0}^{n}(v_i^1 + v_i^2)\,. \tag{38}$$

Inserting the first two equations of (34) into (38), we obtain:

$$w_{n+1}^1 = w_0^1 + 2\frac{\partial^2}{\partial x^2}\log\prod_{i=0}^{n} W[\psi_i^1(\lambda_i), \phi_i^1(\lambda_i)]\,. \tag{39}$$

Finally, using the second equation of (36), we get:

$$w_n^2 = w_0^1 + 2\frac{\partial^2}{\partial x^2}\log\left(\psi_n^1(\lambda_n)\prod_{i=0}^{n} W[\psi_i^1(\lambda_i), \phi_i^1(\lambda_i)]\right)\,. \tag{40}$$

Obviously $\psi_i^1$ and $\phi_i^1$ are functions of $\psi_0^1$ and $\phi_0^1$ which are themselves the solution of the following eigenvalue equation

$$\psi_{0xxx}^1 + \left(\frac{3}{2}w_0^1 - \frac{3}{4}\right)\psi_{0x}^1 + u\psi_0^1 = \lambda\psi_0^1\,.$$

Therefore, we have:

$$\begin{aligned} \psi_i^1(\lambda_i) &= \left[\prod_{j=0}^{i-1} A_j^2(\lambda_j)A_j^1(\lambda_j)\right]\psi_0^1(\lambda_i)\,, \\ \phi_i^1(\lambda_i) &= \left[\prod_{j=0}^{i-1} A_j^2(\lambda_j)A_j^1(\lambda_j)\right]\phi_0^1(\lambda_i)\,. \end{aligned} \tag{41}$$

The relation between the eigen-solutions of rank $n$ and rank null can be represented by the following diagram:

$$\begin{aligned} &L_0^1 \mapsto L_0^2 \mapsto L_1^1 \mapsto L_1^2 \cdots L_n^1 \mapsto L_n^2 \mapsto L_{n+1}^1 \cdots \\ &\psi_0^1 \xrightarrow{A_0^1} \psi_0^2 \xrightarrow{A_0^2} \psi_1^1 \xrightarrow{A_1^1} \psi_1^2 \cdots \psi_n^1 \xrightarrow{A_n^1} \psi_n^2 \xrightarrow{A_n^2} \psi_{n+1}^1\,. \end{aligned} \tag{42}$$

The function $\psi_i^1(\lambda_i)$ in the relation (41) by using the following equality

$$\frac{W\Big(W(\phi_1, \ldots, \phi_n, f), W(\phi_1, \ldots, \phi_n, g)\Big)}{W(\phi_1, \ldots, \phi_n)} = W(\phi_1, \ldots, \phi_n, f, g)\,, \tag{43}$$

can be written as follows

$$\psi_i^1(\lambda_i) = \frac{W\Big(\psi_0^1(\lambda_0), \phi_0^1(\lambda_0), \ldots, \psi_0^1(\lambda_{i-1}), \phi_0^1(\lambda_{i-1}), \psi_0^1(\lambda_i)\Big)}{W\Big(\psi_0^1(\lambda_0), \phi_0^1(\lambda_0), \ldots, \psi_0^1(\lambda_{i-1}), \phi_0^1(\lambda_{i-1})\Big)}\,. \tag{44}$$

Also the function $\phi_i^1(\lambda_i)$ in the relation (41) can be written like to the relation (44) in which $\psi_i^1(\lambda_i)$ is replaced by $\phi_0^1(\lambda_i)$. Now substituting $\psi_i^1(\lambda_i)$ and $\phi_i^1(\lambda_i)$ into the relations (39) and (40) and using the relation (43) again we will have:

$$w_{n+1}^1 = w_0^1 + 2\frac{\partial^2}{\partial x^2}\log W\Big(\psi_0^1(\lambda_0), \phi_0^1(\lambda_0), \ldots, \psi_0^1(\lambda_n), \phi_0^1(\lambda_n)\Big)\,, \tag{45}$$

and

$$w_n^2 = w_0^1 + 2\frac{\partial^2}{\partial x^2}\log W\Big(\psi_0^1(\lambda_0), \phi_0^1(\lambda_0), \ldots, \psi_0^1(\lambda_{n-1}), \phi_0^1(\lambda_{n-1}), \psi_0^1(\lambda_n)\Big)\,. \tag{46}$$

Now, choosing $w_0^1 = 0$ and $u_0^1 = 0$, we have

$$\begin{aligned}\psi_0^1(\lambda_i) = a_{k_{i,1}}\exp\Big(k_{i,1}x + \alpha(t)\Big) + a_{k_{i,2}}\exp\Big(k_{i,2}x + \beta(t)\Big)\\ + a_{k_{i,3}}\exp\Big(k_{i,3}x + \gamma(t)\Big)\,,\end{aligned} \tag{47}$$

and

$$\begin{aligned}\phi_0^1(\lambda_i) = b_{k_{i,1}}\exp\Big(k_{i,1}x + \alpha(t)\Big) + b_{k_{i,2}}\exp\Big(k_{i,2}x + \beta(t)\Big)\\ + b_{k_{i,3}}\exp\Big(k_{i,3}x + \gamma(t)\Big)\,,\end{aligned} \tag{48}$$

where $k_{ij}$, $j = 1, 2, 3$ are the roots of the following cubic equation

$$\kappa^3 - \frac{3}{4}\kappa - \lambda_i = 0\,, \tag{49}$$

and

$$\alpha(t) = -k_{i,1}^2 t + \alpha_0\,, \quad \beta(t) = -k_{i,2}^2 t + \beta_0\,, \quad \gamma(t) = -k_{i,3}^2 t + \gamma_0\,. \tag{50}$$

As an example we consider $\lambda_0 = 0.1$ and $\lambda_1 = -0.2$ with

$$\begin{aligned}
a_{k_{0,1}} &= 10\,, & a_{k_{0,2}} &= -20\,, & a_{k_{0,3}} &= 30\,,\\
b_{k_{0,1}} &= 0\,, & b_{k_{0,2}} &= -5\,, & b_{k_{0,3}} &= -25\,,\\
a_{k_{1,1}} &= 15\,, & a_{k_{1,2}} &= -12\,, & a_{k_{1,3}} &= 0\,,\\
b_{k_{1,1}} &= 0\,, & b_{k_{1,2}} &= -30\,, & b_{k_{1,3}} &= 10\,.
\end{aligned}$$

In Figure 1 is shown the diagram of solution of Boussinesq equation in third step, that is $w_2^1$ versus $x$ has been given at three different values of $t$. According to the relation (45), the function $w_2^1$ is given as follows:

$$w_2^1 = 2\frac{\partial^2}{\partial x^2}\log W\Big(\psi_0^1(\lambda_0), \phi_0^1(\lambda_0), \psi_0^1(\lambda_1)\Big)\,. \tag{51}$$

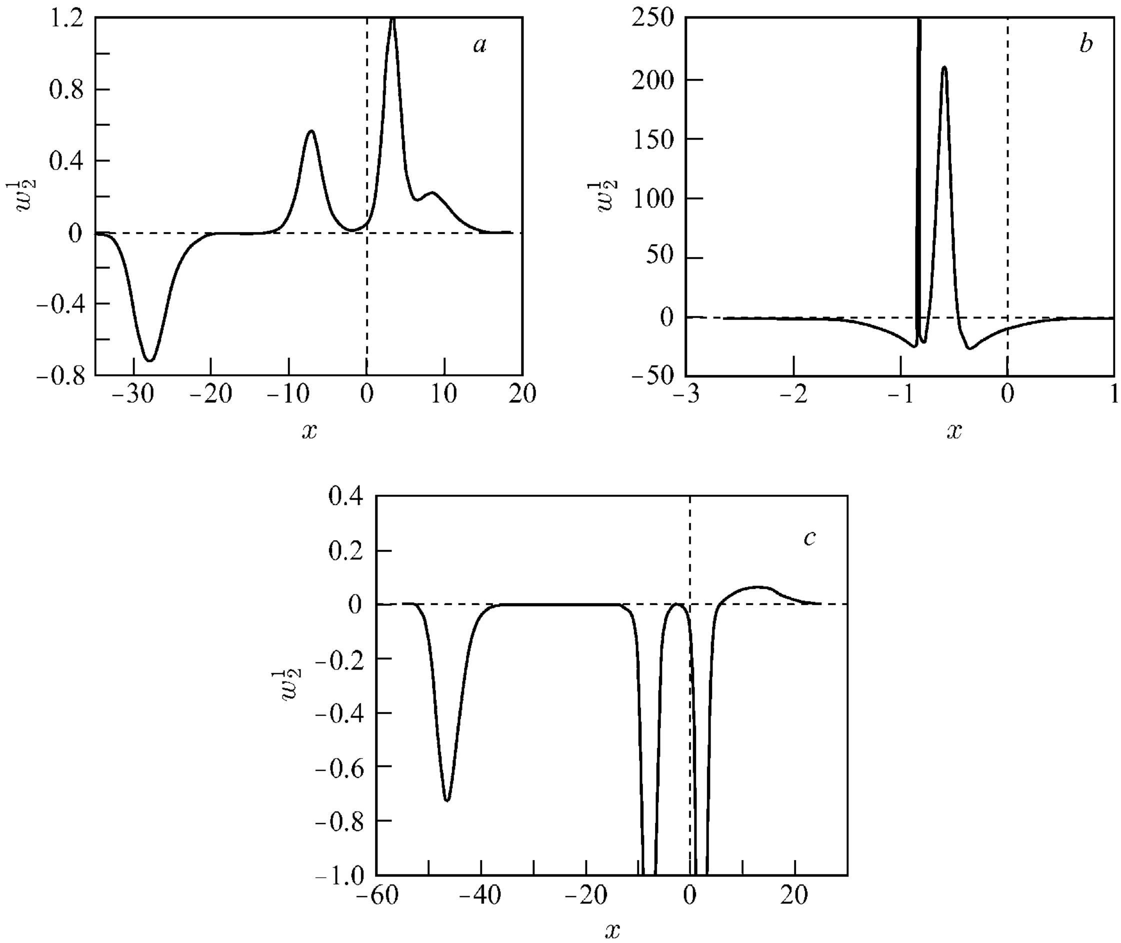


**Figure 1.** Diagram of solutions of Boussinesq equation $w_2^1$
The eigenvalues are $\lambda_0 = 0.1$ and $\lambda_1 = -0.2$ together with the choice of $a_{0,1} = 10$, $a_{0,2} = -20$, $a_{0,3} = 30$, $b_{0,1} = 0$, $b_{0,2} = -5$, $b_{0,3} = -25$, $a_{1,1} = 15$, $a_{1,2} = -12$, $a_{1,3} = 0, b_{1,1} = 0$, $b_{1,2} = -30$, $b_{1,3} = 10$ and corresponding to different times: *a*) four soliton solutions at $t = -10$; *b*) soliton solution together with a collapse solution at $t = 0$; and *c*) single soliton solution together with two collapse solutions at $t = 10$

## References

[1] Bekki N. and Karakisawa T., Physics of Plasmas **2**(8) (1995) 2945.

[2] Boldyrev S., Physics of Plasmas **5**(5) (1998) 1315.

[3] Cooper F., Khare A. and Sukhatme U., Phys. Rep. **251c** (1995) 267.

[4] Dickey L., *Soliton Equations and Hamiltonian Systems*, World Scientific, Singapore 1991 p. 60.

[5] Dos A., *Integrable Models*, World Scientific, Singapore 1989 p. 46.

[6] Drazin P. and Johnson R., *Solitons: An Introduction*, Cambridge Univ. Press, Cambridge 1989 p. 109.

[7] Faddeev L. and Takhtadjan L., *Hamiltonian Methods in The Theory of Solitons*, Springer, Berlin 1980 p. 20.
[8] Gront A. and Rosner J., J. Math. Phys. **35** (1994) 2142.
[9] Hamabata H., Physics of Plasmas **1**(8) (1994) 2574.
[10] Hirota R., J. Math. Phys. **14** (1973) 810.
[11] Jafarizadeh M. and Fakhri H., Phys. Lett. A **230** (1997) 164.
[12] Jafarizadeh M. and Fakhri H., Ann. Phys. **262** (1998) 260.
[13] Jafarizadeh M., Esfandyari A. and Panahi-Talemi H., J. Math. Phys. **41** (2000) 675.
[14] Karpman V., Soviet Physics (JETP) **50**(4) (1979) 701.
[15] Keung W.-Y., Wange Q., Sukhatme U., Wange Q. and Imbo T., J. Phys. A **22** (1989) 987.
[16] Lamb G., *Elements of Soliton Theory*, Wiley, New York 1980 p. 84.
[17] Lantz S. and Fan Y., The Astrophysical Journal, Suppl. Series **121**(1) (1999) 247.
[18] Steven R. and Lantz R., The Astrophysical Journal **441**(2) (1995) 925.
[19] Matveev V. and Salle M., *Darboux Transformations*, Springer, Berlin 1991 p. 43.
[20] Rao N., Phys. Rev. A **37** (1988) 4846.
[21] Shatashvili N. and Rao N., Physics of Plasmas **6**(1) (1999) 66.
[22] Sukhmar C., J. Phys. A: Math. Gen. **19** (1986) 2297.
[23] Wang Z. and Pritchett P., Physics of Fluids B: Plasma Physics **1**(9) (1989) 1767.
[24] Wange Q., Sukhatme U., Keung W.-Y. and Imbo T., Mod. Phys. Lett. A **5** (1990) 524.
[25] Witten E., Nucl. Phys. B **188** (1981) 513.
[26] Xu L., Auston D. and Hasegawa A., Phys. Rev. **45**(5) (1992) 3184.