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ON THE REDUCTIONS AND HAMILTONIAN STRUCTURES OF N-WAVE TYPE EQUATIONS

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Abstract. The reductions of the integrable N-wave type equations solvable by the inverse scattering method with the generalized Zakharov–Shabat system L and related to some simple Lie algebra $\mathfrak{g}$ are analyzed. Special attention is paid to the $\mathbb{Z}_2$ and $\mathbb{Z}_2\times\mathbb{Z}_2$-reductions including ones that can be embedded also in the Weyl group of $\mathfrak{g}$. The consequences of these restrictions on the properties of their Hamiltonian structures are analyzed on specific examples which find applications to nonlinear optics.

1. Introduction

It is well known that the N-wave equations [1–6]

$$\mathrm{i}[J,Q_t] - \mathrm{i}[I,Q_x] + [[I,Q],[J,Q]] = 0\,, \tag{1}$$

are solvable by the inverse scattering method (ISM) [4, 5] applied to the generalized system of Zakharov–Shabat type [4, 7, 8]:

$$L(\lambda)\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}x} + [J,Q(x,t)] - \lambda J\right)\Psi(x,t,\lambda) = 0, \quad J\in\mathfrak{h}\,, \tag{2}$$

$$Q(x,t) = \sum_{\alpha\in\Delta_+}(q_\alpha(x,t)E_\alpha + p_\alpha(x,t)E_{-\alpha}) \in \mathfrak{g}/\mathfrak{h}\,, \tag{3}$$

where $\mathfrak{h}$ is the Cartan subalgebra and E_α are the root vectors of the simple Lie algebra $\mathfrak{g}$. Indeed (1) can be written in the Lax form, or in other words, it is the compatibility condition

$$[L(\lambda), M(\lambda)] = 0\,, \tag{4}$$

where

$$M(\lambda)\Psi(x,t,\lambda) = \left(\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} + [I, Q(x,t)] - \lambda I\right)\Psi(x,t,\lambda) = 0, \quad I \in \mathfrak{h}\,. \tag{5}$$

Here and below $r = \operatorname{rank}\mathfrak{g}$, Δ_+ is the set of positive roots of $\mathfrak{g}$ and $\vec{a}, \vec{b} \in \mathbb{E}^r$ are vectors corresponding to the Cartan elements $J, I \in \mathfrak{h}$. The inverse scattering problem for (2) with real valued J [1] was reduced to a Riemann–Hilbert problem for the (matrix-valued) fundamental analytic solution of (2) [4, 7]; the action-angle variables for the N-wave equations were obtained in the preprint [1] and rederived later in [9]. However, often the reduction conditions require that J be complex-valued. Then the solution of the corresponding inverse scattering problem for (2) becomes more difficult [10, 11].

The interpretation of the ISM as a generalized Fourier transform and the expansions over the “squared solutions” of (2) were derived in [8] for real J and in [11] for complex J. They were used also to prove that all N-wave type equations are Hamiltonian and possess a hierarchy of Hamiltonian structures [8, 11] $\{H^{(k)}, \Omega^{(k)}\}$, $k = 0, \pm 1, \pm 2, \ldots$. The simplest Hamiltonian formulation of (1) is given by $\{H^{(0)} = H_0 + H_{\rm int},\ \Omega^{(0)}\}$ where

$$H_0 = \frac{c_0}{2\mathrm{i}} \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \langle Q, [I, Q_x]\rangle\ , \tag{6}$$

$$H_{\rm int} = \frac{c_0}{3} \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \langle [J, Q], [Q, [I, Q]]\rangle\ , \tag{7}$$

$\langle \cdot, \cdot \rangle$ is the Killing form and the symplectic form $\Omega^{(0)}$ is equivalent to a canonical one

$$\Omega^{(0)} = \frac{\mathrm{i}c_0}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}x\, \left\langle [J, \delta Q(x,t)] \mathop{\wedge}_{,} \delta Q(x,t) \right\rangle\ . \tag{8}$$

The constant c_0 will be fixed up below. Physically each cubic term in $H_{\rm int}$ depends on a triple of positive roots such that $\alpha_i = \alpha_j + \alpha_k$ and shows how the wave of mode i decays into j-th and k-th waves. In other words we assign to each positive root α an wave with an wave number k_α and a frequency ω_α which are preserved in the elementary decays, i. e.

$$k_{\alpha_i} = k_{\alpha_j} + k_{\alpha_k}\,, \quad \omega_{\alpha_i} = \omega_{\alpha_j} + \omega_{\alpha_k}\,.$$

We shall show how one can exhibit new examples of integrable N-wave type interactions some of which have applications to physics. The integrability of a rich family of N-wave type equations and their importance as universal model of wave-wave interactions was demonstrated in [12]. Our approach allows to enrich still further this family.

Our studies are based on the reduction group G_R introduced by Mikhailov [13] and further developed in [14–16]. More recently the $\mathbb{Z}_2$ and $\mathbb{Z}_2 \otimes \mathbb{Z}_2$ reductions of the N-wave type equations were investigated [17–20]. In [18, 19] we point out that G_R can be embedded in the group of automorphisms of $\mathfrak{g}$ in several different ways which may lead to inequivalent reductions of the N-wave equations.

2. Preliminaries

The main idea underlying Mikhailov's reduction group [13] is to impose algebraic restrictions on the Lax operators L and M which will be automatically compatible with the corresponding equations of motion (4). Due to the purely Lie-algebraic nature of the Lax representation (4) this is most naturally done by imbedding G_R as a subgroup of $\mathrm{Aut}\,\mathfrak{g}$ — the group of automorphisms of $\mathfrak{g}$. Obviously to each reduction imposed on L and M there will correspond a reduction of the space of fundamental solutions $\mathfrak{S}_\Psi \equiv \{\Psi(x,t,\lambda)\}$ of (2) and (5).

Some of the simplest $\mathbb{Z}_2$-reductions of N-wave systems (see [2–4]) are related to outer automorphisms of $\mathfrak{g}$ and $\mathfrak{G}$, namely:

$$C_1(\Psi(x,t,\lambda)) = A_1\Psi^\dagger(x,t,\kappa_1(\lambda))A_1^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda)\,, \quad \kappa_1(\lambda) = \pm\lambda^*, \quad (9)$$

where A_1 belongs to the Cartan subgroup of the group $\mathfrak{G}$:

$$A_1 = \exp(\mathrm{i}\pi H_1)\,, \quad (10)$$

and $H_1 \in \mathfrak{h}$ is such that $\alpha(H_1) \in \mathbb{Z}$ for all roots $\alpha \in \Delta$ in the root system Δ of $\mathfrak{g}$. The reduction condition relates the fundamental solution $\Psi(x,t,\lambda) \in \mathfrak{G}$ to a fundamental solution $\tilde{\Psi}(x,t,\lambda)$ of (2) and (5) which in general differs from $\Psi(x,t,\lambda)$.

Another class of $\mathbb{Z}_2$ reductions are related to outer automorphisms, e. g.:

$$C_2\,(\Psi(x,t,\lambda)) = A_2\Psi^T(x,t,\kappa_2(\lambda))A_2^{-1} = \tilde{\Psi}^{-1}(x,t,\lambda), \quad \kappa_2(\lambda) = \pm\lambda\,, \quad (11)$$

where A_2 is again of the form (10). The best known examples of NLEE obtained with the reduction (11) are the sine-Gordon and the MKdV equations which are related to $\mathfrak{g} \simeq sl(2)$. For higher rank algebras such reductions to our knowledge have not been studied. Generically reductions of type (11) lead to degeneration of the canonical Hamiltonian structure, i. e. $\Omega^{(0)} \equiv 0$; then we need to use some of their higher Hamiltonian structures (see [8, 11]).

One may use also reductions with inner automorphisms like:

$$C_3(\Psi(x,t,\lambda)) = A_3\Psi^*(x,t,\kappa_1(\lambda))A_3^{-1} = \tilde{\Psi}(x,t,\lambda)\,, \qquad (12)$$

$$C_4(\Psi(x,t,\lambda)) = A_4\Psi(x,t,\kappa_2(\lambda))A_4^{-1} = \tilde{\Psi}(x,t,\lambda)\,. \qquad (13)$$

Since our aim is to preserve the form of the Lax pair we limit ourselves by automorphisms preserving the Cartan subalgebra $\mathfrak{h}$. This conditions is obviously fulfilled if A_k, $k = 1,\dots,4$ is in the form (10). Another possibility is to choose $A_1,\dots,A_4$ so that they correspond to a Weyl group automorphisms.

The reduction group G_R is a finite group which preserves the Lax representation (4), i. e. for each $g_k \in G_R$

$$C_k(L(\Gamma_k(\lambda))) = \eta_k L(\lambda)\,, \quad C_k(M(\Gamma_k(\lambda))) = \eta_k M(\lambda)\,. \qquad (14)$$

G_R must have two realizations: (i) $G_R \subset \operatorname{Aut}\mathfrak{g}$ and $C_k \in \operatorname{Aut}\mathfrak{g}$; (ii) $G_R \subset \operatorname{Conf}\mathbb{C}$, i. e. $\Gamma_k(\lambda)$ are conformal mappings of the complex λ-plane. Below we consider specially the cases $G_R \simeq \mathbb{Z}_2$ or $G_R \simeq \mathbb{Z}_2 \otimes \mathbb{Z}_2$.

The automorphisms C_k, $k = 1,\dots,4$ listed above lead to the following reductions for the matrix-valued functions

$$U(x,t,\lambda) = [J, Q(x,t)] - \lambda J, \quad V(x,t,\lambda) = [I, Q(x,t)] - \lambda I\,, \qquad (15)$$

of the Lax representation:

$$\begin{aligned} C_1(U^\dagger(\kappa_1(\lambda))) &= U(\lambda)\,, & C_1(V^\dagger(\kappa_1(\lambda))) &= V(\lambda)\,, \\ C_2(U^T(\kappa_2(\lambda))) &= -U(\lambda)\,, & C_2(V^T(\kappa_2(\lambda))) &= -V(\lambda)\,, \\ C_3(U^*(\kappa_1(\lambda))) &= -U(\lambda)\,, & C_3(V^*(\kappa_1(\lambda))) &= -V(\lambda)\,, \\ C_4(U(\kappa_2(\lambda))) &= U(\lambda)\,, & C_4(V(\kappa_2(\lambda))) &= V(\lambda)\,. \end{aligned} \qquad (16)$$

2.1. Cartan–Weyl Basis and Weyl Group

Here we fix up the notations, the normalization conditions for the Cartan–Weyl generators of $\mathfrak{g}$ and their commutation relations, see [21]:

$$\begin{aligned} &[h_k, E_\alpha] = (\alpha, e_k)E_\alpha\,, \qquad [E_\alpha, E_{-\alpha}] = H_\alpha\,, \\ &[E_\alpha, E_\beta] = \begin{cases} N_{\alpha,\beta}E_{\alpha+\beta} & \text{for } \alpha+\beta \in \Delta \\ 0 & \text{for } \alpha+\beta \notin \Delta \cup \{0\}\,. \end{cases} \end{aligned} \qquad (17)$$

If J is a regular real element in $\mathfrak{h}$ then we may use it to introduce an ordering in Δ by saying that the root $\alpha \in \Delta_+$ is positive (negative) if $(\alpha, \vec{J}) > 0$ ($(\alpha, \vec{J}) < 0$ respectively). The normalization of the basis is determined by:

$$\begin{gathered} E_{-\alpha} = E_\alpha^T\,, \qquad \langle E_{-\alpha}, E_\alpha \rangle = \frac{2}{(\alpha,\alpha)}\,, \\ N_{-\alpha,-\beta} = -N_{\alpha,\beta}\,, \qquad N_{\alpha,\beta} = \pm(p+1)\,, \end{gathered} \tag{18}$$

where the integer $p \geq 0$ is such that $\alpha + s\beta \in \Delta$ for all $s = 1, \ldots, p$ and $\alpha + (p+1)\beta \notin \Delta$. The root system Δ of $\mathfrak{g}$ is invariant with respect to the Weyl reflections S_α; on the vectors $\vec{y} \in \mathbb{E}^r$ they act as

$$S_\alpha \vec{y} = \vec{y} - \frac{2(\alpha, \vec{y})}{(\alpha,\alpha)}\alpha, \qquad \alpha \in \Delta\,. \tag{19}$$

S_α generate the Weyl group $W_\mathfrak{g}$ and act on the Cartan–Weyl basis by:

$$\begin{aligned} S_\alpha(H_\beta) &\equiv A_\alpha H_\beta A_\alpha^{-1} = H_{S_\alpha \beta}\,, \\ S_\alpha(E_\beta) &\equiv A_\alpha E_\beta A_\alpha^{-1} = n_{\alpha,\beta} E_{S_\alpha\beta}\,, \qquad n_{\alpha,\beta} = \pm 1\,. \end{aligned} \tag{20}$$

In fact $W_\mathfrak{g}$ is the group of inner automorphisms of $\mathfrak{g}$ preserving the Cartan subalgebra $\mathfrak{h}$. The same property is possessed also by $\mathrm{Ad}_\mathfrak{h}$ automorphisms: choosing $C = \exp(\mathrm{i}\pi H_{\vec{c}})$ we get from (17):

$$C H_\alpha C^{-1} = H_\alpha\,, \qquad C E_\alpha C^{-1} = e^{2\pi \mathrm{i}(\alpha,\vec{c})/2} E_\alpha\,, \tag{21}$$

where $\vec{c} \in \mathbb{E}^r$ is the vector corresponding to $H_{\vec{c}} \in \mathfrak{h}$. Then the condition $C^2 = \mathbb{1}$ means that $(\alpha, \vec{c}) \in \mathbb{Z}$ for all $\alpha \in \Delta$.

3. Scattering Data and the $\mathbb{Z}_2$-reductions

In order to determine the scattering data of the Lax operator (2) we start with the Jost solutions

$$\lim_{x\to\infty} \psi(x,\lambda) e^{\mathrm{i}\lambda J x} = \mathbb{1}\,, \qquad \lim_{x\to-\infty} \phi(x,\lambda) e^{\mathrm{i}\lambda J x} = \mathbb{1}\,, \tag{22}$$

and the scattering matrix

$$T(\lambda) = (\psi(x,\lambda))^{-1} \phi(x,\lambda)\,. \tag{23}$$

Here we limit ourselves with the simplest nontrivial case when J has real and pair-wise different eigenvalues, i. e. when $(a, \alpha_j) > 0$ for $j = 1, \ldots, r$, see [8]. Since the classical papers of Zakharov and Shabat [7, 22] the most efficient way to solve the inverse scattering problem for $L(\lambda)$ is to construct the **fundamental analytic solutions** (FAS) $\chi^\pm(x,\lambda)$ of (2) and then to make use of the equivalent Riemann–Hilbert problem (RHP). To do this we have to use the Gauss decomposition of $T(\lambda)$:

$$T(\lambda) = T^{-}(\lambda)D^{+}(\lambda)\hat{S}^{+}(\lambda) = T^{+}(\lambda)D^{-}(\lambda)\hat{S}^{-}(\lambda)\,, \tag{24}$$

where 'hat' above denotes the inverse matrix $\hat{S} \equiv S^{-1}$ and

$$S^{\pm}(\lambda) = \exp \sum_{\alpha\in\Delta_+} s_{\pm}^{\pm\alpha}(\lambda)E_{\pm\alpha}\,, \quad T^{\pm}(\lambda) = \exp \sum_{\alpha\in\Delta_+} t_{\pm}^{\pm\alpha}(\lambda)E_{\pm\alpha}\,, \tag{25}$$

$$D^{+}(\lambda) = \exp\sum_{j=1}^{r} \frac{2d_j^{+}(\lambda)}{(\alpha_j,\alpha_j)}H_j\,, \quad D^{-}(\lambda) = \exp\sum_{j=1}^{r} \frac{2d_j^{-}(\lambda)}{(\alpha_j,\alpha_j)}H_j^{-}\,, \tag{26}$$

$$H_j \equiv H_{\alpha_j}\,, \quad H_j^{-} = w_0(H_j)\,.$$

Here the superscript $+$ (or $-$) in $D^{\pm}(\lambda)$ shows that $D_j^{+}(\lambda)$ (or $D_j^{-}(\lambda)$) are analytic functions of λ for $\operatorname{Im}\lambda > 0$ (or $\operatorname{Im}\lambda < 0$) respectively and w_0 is the Weyl reflection that maps the highest weight ω_j^{+} in $R(\omega_j^{+})$ into the lowest weight ω_j^{-} of $R(\omega_j^{+})$ (see [21] for details). Then we can prove that

$$\chi^{\pm}(x,\lambda) = \phi(x,\lambda)S^{\pm}(\lambda) = \psi(x,\lambda)T^{\mp}(\lambda)D^{\pm}(\lambda) \tag{27}$$

are fundamental analytic solutions (FAS) of (2) for $\operatorname{Im}\lambda \gtrless 0$. On the real axis $\chi^{+}(x,\lambda)$ and $\chi^{-}(x,\lambda)$ are linearly related by

$$\chi^{+}(x,\lambda) = \chi^{-}(x,\lambda)G_0(\lambda)\,, \quad G_0(\lambda) = S^{+}(\lambda)\hat{S}^{-}(\lambda)\,, \tag{28}$$

and the sewing function $G_0(\lambda)$ may be considered as a minimal set of scattering data provided the Lax operator (2) has no discrete eigenvalues. The presence of discrete eigenvalues $\lambda_k^{\pm}$ means that some of the functions

$$D_j^{\pm}(\lambda) = \langle\omega_j^{\pm} \mid D^{\pm}(\lambda) \mid \omega_j^{\pm}\rangle = \exp\left(d_j^{\pm}(\lambda)\right)\,,$$

where ω_j^{+} are the fundamental weights of $\mathfrak{g}$ and $\omega_j^{-} = w_0(\omega_j^{+})$, will have zeroes and poles at $\lambda_k^{\pm}$, for more details see [23, 19]. Equation (28) can be easily rewritten in the form:

$$\xi^{+}(x,\lambda) = \xi^{-}(x,\lambda)G(x,\lambda)\,, \quad G(x,\lambda) = \mathrm{e}^{-\mathrm{i}\lambda Jx}G_0(\lambda)\,\mathrm{e}^{\mathrm{i}\lambda Jx}\,. \tag{29}$$

Then (29) together with

$$\lim_{\lambda\to\infty} \xi^{\pm}(x,\lambda) = \mathbb{1} \tag{30}$$

can be considered as a RHP with canonical normalization condition.

The solution $\xi^{+}(x,\lambda)$, $\xi^{-}(x,\lambda)$ to (29), (30) is called regular if $\xi^{+}(x,\lambda)$ and $\xi^{-}(x,\lambda)$ are nondegenerate and non-singular functions of λ for all $\operatorname{Im}\lambda > 0$

and $\operatorname{Im}\lambda < 0$ respectively. To the class of regular solutions of RHP there correspond Lax operators (2) without discrete eigenvalues. The presence of discrete eigenvalues $\lambda_k^\pm$ leads to singular solutions of the RHP; their explicit construction can be done by the Zakharov–Shabat dressing method [22], for the case of orthogonal algebras see also [19].

If the potential $Q(x,t)$ of the Lax operator (2) satisfies the N-wave equation (1) then $S^\pm(t,\lambda)$ and $T^\pm(t,\lambda)$ satisfy the linear evolution equations

$$\mathrm{i}\frac{\mathrm{d}S^\pm}{\mathrm{d}t} - \lambda[I, S^\pm(t,\lambda)] = 0\,, \qquad \mathrm{i}\frac{\mathrm{d}T^\pm}{\mathrm{d}t} - \lambda[I, T^\pm(t,\lambda)] = 0\,, \tag{31}$$

while the functions $D^\pm(\lambda)$ are time-independent. In other words $D_j^\pm(\lambda)$ can be considered as the generating functions of the integrals of motion of (1).

Each reduction on L imposes restriction also on the scattering data. If L satisfies (14) then the scattering matrix will satisfy

$$C_k\left(T(\Gamma_k(\lambda)) = T(\lambda)\,, \qquad \lambda \in \mathbb{R}\,. \tag{32}$$

Equation (32) is valid only for real values of λ. If the reduction is of the form (9), (11) and (12) then for the FAS and for the Gauss factors $S^\pm(\lambda)$, $T^\pm(\lambda)$ and $D^\pm(\lambda)$ we will get:

$$\begin{aligned} S^+(\lambda) &= A_1\left(\hat{S}^-(\lambda^*)\right)^\dagger A_1^{-1}\,, & T^+(\lambda) &= A_1\left(\hat{T}^-(\lambda^*)\right)^{\dagger\dagger} A_1^{-1}\,, \\ D^+(\lambda) &= A_1\left(\hat{D}^-(\lambda^*)\right)^\dagger A_1^{-1}\,, & F(\lambda) &= A_1\left(F(\lambda^*)\right)^\dagger A_1^{-1}\,, \end{aligned} \tag{33}$$

$$\begin{aligned} S^+(\lambda) &= A_2 S^-(-\lambda) A_2^{-1}\,, & T^+(\lambda) &= A_2 T^-(-\lambda) A_2^{-1}\,, \\ D^+(\lambda) &= A_2 D^-(-\lambda) A_2^{-1}\,, & F(\lambda) &= A_2 F(-\lambda) A_2^{-1}\,, \end{aligned} \tag{34}$$

$$\begin{aligned} S^\pm(\lambda) &= A_3\left(S^\pm(-\lambda^*)\right)^* A_3^{-1}\,, & T^\pm(\lambda) &= A_3\left(T^\pm(-\lambda^*)\right)^T A_3^{-1}\,, \\ D^\pm(\lambda) &= A_3\left(D^\pm(-\lambda^*)\right)^* A_3^{-1}\,, & F(\lambda) &= A_3\left(F(-\lambda^*)\right)^* A_3^{-1}\,, \end{aligned} \tag{35}$$

where A_1 and A_3 are assumed to be elements of the Cartan subgroup of $\mathfrak{G}$ while A_2 corresponds to the w_0 element in the Weyl group.

We will also make use of the integral representations for $d_j^\pm(\lambda)$ allowing one to reconstruct them as analytic functions in their regions of analyticity $\mathbb{C}_\pm$. In the case of absence of discrete eigenvalues we have [8, 11]:

$$\mathcal{D}_j(\lambda) = \frac{\mathrm{i}}{2\pi}\int\limits_{-\infty}^{\infty} \frac{\mathrm{d}\mu}{\mu-\lambda} \ln\langle\omega_j^+|\hat{T}^+(\mu)T^-(\mu)|\omega_j^+\rangle\,, \tag{36}$$

where $|\omega_j^+\rangle$ is the highest weight vector in the corresponding fundamental representation $R(\omega_j^+)$ of $\mathfrak{g}$. The function $\mathcal{D}_j(\lambda)$ as a fraction-analytic function of

λ is equal to:

$$\mathcal{D}_j(\lambda) = \begin{cases} d_j^+(\lambda), & \text{for } \lambda \in \mathbb{C}_+ \\ (d_j^+(\lambda) - d_{j'}^-(\lambda))/2, & \text{for } \lambda \in \mathbb{R}, \\ -d_{j'}^-(\lambda), & \text{for } \lambda \in \mathbb{C}_- , \end{cases} \tag{37}$$

where $d_j^\pm(\lambda)$ were introduced in (26) and the index j' is related to j by $w_0(\alpha_j) = -\alpha_{j'}$. The functions $\mathcal{D}_j(\lambda)$ can be viewed also as generating functions of the integrals of motion. Indeed, if we expand

$$\mathcal{D}_j(\lambda) = \sum_{k=1}^{\infty} \mathcal{D}_{j,k}\lambda^{-k} , \tag{38}$$

and take into account that $D^\pm(\lambda)$ are time independent we find that $\mathrm{d}\mathcal{D}_{j,k}/\mathrm{d}t = 0$ for all $k = 1, \dots, \infty$ and $j = 1, \dots, r$. Moreover it can be checked that $\mathcal{D}_{j,k}$ expressed as functionals of $q(x,t)$ has kernel that is local in q, i. e. depends only on q and its derivatives with respect to x.

From (36) and (33–35) we easily obtain the effect of the reductions on the set of integrals of motion:

$$\mathcal{D}_j(\lambda) = -\mathcal{D}_j^*(\lambda^*) , \quad \text{i. e.} \quad \mathcal{D}_{j,k} = -\mathcal{D}_{j,k}^* , \tag{39}$$

$$\mathcal{D}_j(\lambda) = -\mathcal{D}_j(-\lambda) , \quad \text{i. e.} \quad \mathcal{D}_{j,k} = (-1)^{k+1}\mathcal{D}_{j,k} , \tag{40}$$

$$\mathcal{D}_j(\lambda) = \mathcal{D}_j^*(-\lambda^*) , \quad \text{i. e.} \quad \mathcal{D}_{j,k} = (-1)^k\mathcal{D}_{j,k}^* . \tag{41}$$

for the reductions (33), (34) and (35) respectively.

In particular from (40) it follows that al integrals of motion with even k become degenerate, i. e. $\mathcal{D}_{j,2k} = 0$. The reduction (39) means that the integrals $\mathcal{D}_{j,k}$ become purely imaginary. Finally, if we have chosen the reduction (35) from (41) it follows that $\mathcal{D}_{j,2k}$ are real while $\mathcal{D}_{j,2k+1}$ are purely imaginary.

We finish this section with a few comments on the simplest local integrals of motion. To this end we write down the first two types of integrals of motion $\mathcal{D}_{j,1}$ and $\mathcal{D}_{j,2}$ as functionals of the potential Q of (2). Skipping the details (see [8]) we get:

$$\mathcal{D}_{j,1} = -\frac{\mathrm{i}}{4} \int_{-\infty}^{\infty} \mathrm{d}x \, \langle [J,Q], [H_j^\vee, Q] \rangle , \tag{42}$$

$$\mathcal{D}_{j,2} = -\frac{1}{2} \int_{-\infty}^{\infty} \mathrm{d}x \, \langle Q, [H_j^\vee, Q_x] \rangle - \frac{\mathrm{i}}{3} \int_{-\infty}^{\infty} \mathrm{d}x \, \langle [J,Q], [Q, [H_j^\vee, Q]] \rangle , \tag{43}$$

where $H_j^\vee = 2H_{\omega_j}/(\alpha_j, \alpha_j)$.

The fact that $\mathcal{D}_{j,1}$ are integrals of motion for $j = 1, \dots, r$, can be considered as natural analog of the Manley–Rowe relations [1, 3]. In the case when the reduction is of the type (9), i. e. $p_\alpha = s_\alpha q_\alpha^*$ then (42) is equivalent to

$$\sum_{\alpha>0} \frac{2(\vec{a}, \alpha)(\omega_j, \alpha)}{(\alpha, \alpha)} \int\limits_{-\infty}^{\infty} \mathrm{d}x \, s_\alpha |q_\alpha(x)|^2 = \mathrm{const}\,, \tag{44}$$

and can be interpreted as relations between the densities $|q_\alpha|^2$ of the 'particles' of type α. For the other types of reductions such interpretation is not so obvious. The integrals of motion $\mathcal{D}_{j,2}$ are related directly to the Hamiltonian of the N-wave equations (1), namely:

$$H_{N-w} = -\sum_{j=1}^{r} \frac{2(\alpha_j, \vec{b})}{(\alpha_j, \alpha_j)} \mathcal{D}_{j,2} = \frac{1}{2\mathrm{i}} \left\langle\left\langle \dot{\mathcal{D}}(\lambda)\,,\, F(\lambda) \right\rangle\right\rangle_0 , \tag{45}$$

where $\dot{\mathcal{D}}(\lambda) = \mathrm{d}\mathcal{D}/\mathrm{d}\lambda$ and $F(\lambda) = \lambda I$ is the dispersion law of the N-wave equation (1). In (45) we used just one of the hierarchy of scalar products in the Kac–Moody algebra (see [24]) $\widehat{\mathfrak{g}} \equiv \mathfrak{g} \otimes \mathbb{C}[\lambda, \lambda^{-1}]$:

$$\langle\langle X(\lambda), Y(\lambda) \rangle\rangle_k = \mathrm{Res}\, \lambda^{k+1} \left\langle \hat{D}^+(\lambda) X(\lambda), Y(\lambda) \right\rangle, \quad X(\lambda), Y(\lambda) \in \widehat{\mathfrak{g}}. \tag{46}$$

4. Example: N-wave Systems Related to B_2-algebra

Let us illustrate these general results by an example related to the B_2 algebra. This algebra has two simple roots $\alpha_1 = e_1 - e_2$, $\alpha_2 = e_2$, and two more positive roots: $\alpha_1 + \alpha_2 = e_1$ and $\alpha_1 + 2\alpha_2 = e_1 + e_2 = \alpha_{\max}$. When they come as indices, e. g. in q_α, we will replace them by sequences of two integers: $\alpha \to kn$ if $\alpha = k\alpha_1 + n\alpha_2$; if $\alpha = -(k\alpha_1 + n\alpha_2)$ we will use $\overline{kn}$.
The reduction $KU^\dagger(\lambda^*)K^{-1} = U(\lambda)$ where K is an element of the Cartan subgroup with $K = \mathrm{diag}(s_1, s_2, 1, s_2, s_1)$ and $s_k = \pm 1$, $k = 1, 2$, extracts the real forms of $B_2 \simeq so(5)$. So $a_i = a_i^*$, $i = 1, 2$ and q_α must satisfy:

$$p_{10} = -s_2 s_1 q_{10}^*\,, \quad p_{01} = -s_2 q_{01}^*\,, \quad p_{11} = -s_1 q_{11}^*\,, \quad p_{12} = -s_1 s_2 q_{12}^*\,. \tag{47}$$

Thus we get 4-wave system which is described by the Hamiltonian $H = H_0 + H_{\mathrm{int}}$ with:

$$\begin{aligned} H_0 = \frac{\mathrm{i}}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}x \, \Big[&(b_1 - b_2)(q_{10} q_{10,x}^* - q_{10,x} q_{10}^*) + 2b_2 (q_{01} q_{01,x}^* - q_{01,x} q_{01}^*) \\ &+ 2b_1 (q_{11} q_{11,x}^* - q_{11,x} q_{11}^*) + (b_1 + b_2)(q_{12} q_{12,x}^* - q_{12,x} q_{12}^*) \Big] \end{aligned} \tag{48}$$

$$H_{\rm int} = 2\kappa s_1 \int\limits_{-\infty}^{\infty} {\rm d}x\, \left[s_2 (q_{12} q_{11}^* q_{01}^* + q_{12}^* q_{11} q_{01}) + (q_{11} q_{01}^* q_{10}^* + q_{11}^* q_{01} q_{10}) \right] ,$$

where $\kappa = a_1 b_2 - a_2 b_1$, and the symplectic 2-form:

$$\begin{aligned} \Omega^{(0)} = {\rm i} \int\limits_{-\infty}^{\infty} {\rm d}x\, [(a_1 - a_2) \delta q_{10} \wedge \delta q_{10}^* + 2 a_2 \delta q_{01} \wedge \delta q_{01}^* \\ + 2 a_1 \delta q_{11} \wedge \delta q_{11}^* + (a_1 + a_2) \delta q_{12} \wedge \delta q_{12}^*] , \end{aligned} \tag{49}$$

The corresponding wave-decay diagram is shown in Fig. 1.

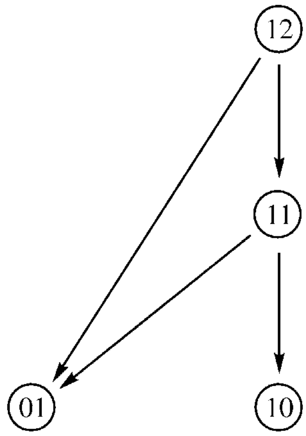

Figure 1. Wave-decay diagram for the $so(5)$ algebra
To each positive root of the algebra $\underline{kn} \equiv k\alpha_1 + n\alpha_2$ we put in correspondence a wave of type $\underline{kn}$. If the positive root $\underline{kn} = \underline{k'n'} + \underline{k''n''}$ can be represented as a sum of two other positive roots, we say that the wave $\underline{kn}$ decays into the waves $\underline{k'n'}$ and $\underline{k''n''}$.

The particular case $s_1 = s_2 = 1$ leads to N-wave equations on the compact real form $so(5,0) \simeq so(5,\mathbb{R})$ of the B_2-algebra, see also [19, 25]. The choices $s_1 = -s_2 = -1$ and $s_1 = s_2 = -1$ lead to N-wave equations on the noncompact real forms $so(2,3)$ and $so(1,4)$ respectively.

Let us apply a second $\mathbb{Z}_2$-reduction to the already reduced system of the previous subsection. We take it in the form $w_0(U(-\lambda)) = U(\lambda)$ which gives $a_i = a_i^*$, $b_i = b_i^*$ and:

$$q_{10}^* = -s_1 s_2 q_{10} , \quad q_{01}^* = -s_2 q_{01} , \quad q_{11}^* = -s_1 q_{11} , \quad q_{12}^* = -s_1 s_2 q_{12} . \tag{50}$$

This gives the following 4-wave system for 4 real-valued functions:

$$\begin{aligned} {\rm i}(a_1 - a_2) q_{10,t} - {\rm i}(b_1 - b_2) q_{10,x} + 2\kappa q_{11} q_{01} = 0 , \\ {\rm i} a_2 q_{01,t} - {\rm i} b_2 q_{01,x} + \kappa (q_{11} q_{12} + q_{11} q_{10}) = 0 , \\ {\rm i} a_1 q_{11,t} - {\rm i} b_1 q_{11,x} + \kappa (q_{12} q_{01} - q_{10} q_{01}) = 0 , \\ {\rm i}(a_1 + a_2) q_{12,t} - {\rm i}(b_1 + b_2) q_{12,x} - 2\kappa q_{11} q_{01} = 0 . \end{aligned} \tag{51}$$

Since $w_0(J) = -J$ the Hamiltonian structure $\{H^{(0)}, \Omega^{(0)}\}$ becomes degenerated and we must consider the next Hamiltonian structure in the hierarchy.

It is known that the j-type discrete eigenvalues of L are located at the zeroes $\lambda_k^\pm \in \mathbb{C}_\pm$ of the functions $D_j^\pm(\lambda)$ [8, 19]. If we assume that L has only two eigenvalues $\lambda_1^\pm$, of type j then we can write

$$D_j^+(\lambda) = \frac{\lambda - \lambda_1^+}{\lambda - \lambda_1^-}\tilde{D}_j^+(\lambda), \qquad D_j^-(\lambda) = \frac{\lambda - \lambda_1^-}{\lambda - \lambda_1^+}\tilde{D}_j^-(\lambda), \tag{52}$$

where $\tilde{D}_j^\pm(\lambda)$ have no zeroes in $\mathbb{C}_\pm$. Then the first reduction which is of the type (33) ensures that the eigenvalues must be pair-wise complex conjugate, i. e. $\lambda_1^- = (\lambda_1^+)^*$. The second reduction of the type (34) leads to $\lambda_1^- = -\lambda_1^+$. Therefore if L has only two eigenvalues of type j and both reductions are imposed this means that $\lambda_1^\pm = \pm \mathrm{i}\zeta_1$ where $\zeta_1 > 0$ is a positive real number. However, if L has two pairs of eigenvalues $\lambda_k^\pm$, $k = 1, 2$ there is another nontrivial way to satisfy both reductions simultaneously:

$$\lambda_1^\pm = \mu_1 \pm \mathrm{i}\zeta_1 , \qquad \lambda_2^\pm = -\mu_1 \pm \mathrm{i}\zeta_1 ,$$

where μ_1, ζ_1 are real positive numbers. Therefore when both reductions are effective the operator L may have two different types of eigenvalue configurations and to each such configuration there corresponds a reflectionless potential for L and soliton solution for the N-wave system.

Such configurations have been well known for the sine-Gordon equation [4, 5] where we have: (i) topological solitons related to the purely imaginary eigenvalues $\pm\mathrm{i}\zeta_k$ and (ii) the breathers related to the quadruplets of eigenvalues.

5. Hierarchy of Hamiltonian Structures of N-wave Equations and Reductions

The generic N-wave interactions (i. e., prior to any reductions) possess a hierarchy of Hamiltonian structures which is generated by the so-called generating (or recursion) operator $\Lambda = (\Lambda_+ + \Lambda_-)/2$ [8]:

$$\begin{aligned} \Lambda_\pm Z(x) = \mathrm{ad}_J^{-1} \Bigg(\mathrm{i}\frac{\mathrm{d}Z}{\mathrm{d}x} &+ P_0 \cdot ([q(x), Z(x)]) \\ &+ \mathrm{i}\,[q(x), I_\pm\,(\mathbb{1} - P_0)\,[q(y), Z(y)]]\Bigg), \end{aligned}$$

$$P_0 S \equiv \mathrm{ad}_J^{-1} \cdot \mathrm{ad}_J \cdot S\,, \quad (I_\pm S)(x) \equiv \int_{\pm\infty}^{x} \mathrm{d}y\, S(y)\,, \tag{53}$$

where $q(x,t) = [J, Q(x,t)]$. The hierarchy of symplectic forms is given by:

$$\Omega^{(k)} = \frac{\mathrm{i}}{2} \int\limits_{-\infty}^{\infty} \mathrm{d}x \left\langle [J, \delta Q(x,t)] \mathop{\wedge}_{,} \Lambda^k \delta Q(x,t) \right\rangle , \tag{54}$$

Using the completeness relation for the "squared" solutions which is directly related to the spectral decomposition of Λ we can recalculate $\Omega^{(k)}$ in terms of the scattering data of L with the result [8]:

$$\begin{aligned} \Omega^{(k)} &= \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \mathrm{d}\lambda \, \lambda^k \left(\Omega_0^+(\lambda) - \Omega_0^-(\lambda) \right) , \\ \Omega_0^\pm(\lambda) &= \left\langle \hat{D}^\pm(\lambda) \hat{T}^\mp(\lambda) \delta T^\mp(\lambda) D^\pm(\lambda) \mathop{\wedge}_{,} \hat{S}^\pm(\lambda) \delta S^\pm(\lambda) \right\rangle . \end{aligned} \tag{55}$$

Therefore the kernels of $\Omega^{(k)}$ differs only by the factor λ^k; i. e., all of them can be cast into canonical form simultaneously. This is quite compatible with the results of [1, 2, 9] for the action-angle variables.

Again it is not difficult to find how the reductions influence $\Omega^{(k)}$. Using the invariance of the Killing form, from (55) and (33–35) we get:

$$\Omega_0^+(\lambda) = \left(\Omega_0^-(\lambda^*) \right)^* , \tag{56}$$

$$\Omega_0^+(\lambda) = \Omega_0^-(-\lambda), \tag{57}$$

$$\Omega_0^\pm(\lambda) = \left(\Omega_0^\pm(-\lambda^*) \right)^* . \tag{58}$$

Then for $\Omega^{(k)}$ from (33), (34) and (35) we obtain:

$$\Omega^{(k)} = -\left(\Omega^{(k)} \right)^* , \tag{59}$$

$$\Omega^{(k)} = (-1)^{k+1} \Omega^{(k)} , \tag{60}$$

$$\Omega^{(k)} = (-1)^k \left(\Omega^{(k)} \right)^* . \tag{61}$$

respectively. Like for the integrals $\mathcal{D}_{j,k}$ we find that the reductions (33) and (35) mean that each $\Omega^{(k)}$ can be made real with a proper choice of the constant c_0 in (8).

Let us now briefly analyze the reduction (34) which may lead to degeneracies. We already mentioned that $\mathcal{D}_{j,2k} = 0$, see (40); in addition from (60) it follows that $\Omega^{(2k)} \equiv 0$. In particular this means that the canonical 2-form $\Omega^{(0)}$ is also degenerated, so the N-wave equations with the reduction (34) do not allow

Hamiltonian formulation with canonical Poisson brackets. However they still possess a hierarchy of Hamiltonian structures:

$$\Omega^{(k)}\left(\frac{\mathrm{d}q}{\mathrm{d}t}, \cdot\right) = \nabla_q H^{(k+1)}, \qquad (62)$$

where $\nabla_q H^{(k+1)} = \Lambda \nabla_q H^{(k)}$; by definition $\nabla_q H = (\delta H)/(\delta q^T(x,t))$. Thus we find that while the choices $\{\Omega^{(2k)}, H^{(2k)}\}$ for the N-wave equations are degenerated, the choices $\{\Omega^{(2k+1)}, H^{(2k+1)}\}$ provide us with correct nondegenerated (though non-canonical) Hamiltonian structures, see [8, 11, 13].

6. Conclusion

Here we have analyzed how can be imposed one or two $\mathbb{Z}_2$-reductions on the N-wave type equations related to the simple Lie algebras and what will be the consequences of these reductions to the Hamiltonian structures and to the structure of their soliton solutions. A list of all nontrivial $\mathbb{Z}_2$-reductions for the low-rank simple Lie algebras (rank less than 4) can be found in [18]. The reductions that lead to a real forms of $\mathfrak{g}$ are discussed in [20]. The classification of the $\mathbb{Z}_2 \times \mathbb{Z}_2$-reductions is under investigation. We note also that the explicit construction of the dressing factors for the symplectic and orthogonal algebras requires modifications of the Zakharov–Shabat dressing method [19]. This leads to new types of reflectionless potentials and soliton solutions.

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