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Geometric Convergence of Overlapping Schwarz Methods for Obstacle Problems

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1 Obstacle Problem and its Discretization

Schwarz methods have been paid great attention in recent years. In our paper, we consider Schwarz methods for the obstacle problem of finding a u such that

$$\begin{cases} -\Delta u(x) \geq f(x), & \\ u(x) \geq 0, & x \in \Omega, \\ u(x)(-\Delta u(x) - f(x)) = 0, & \\ u(x) = g(x), & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded polyhedron convex domain in R^2 or R^3 with boundary $\partial\Omega$. f and g are given functions.

We discrete problem (1.1) as finite-dimensional linear complementarity problem by using a conforming finite element method (e.g. Lagrange linear elements):

$$\begin{cases} LU(x) \geq f(x), \quad U(x) \geq 0 \quad U^T(x)(LU(x) - f(x)) = 0. & x \in \Omega_h, \\ U(x) = g(x), & x \in \partial\Omega_h, \end{cases}$$

where Ω_h is grid set of the triangulation of Ω. Let grid function U be the restriction of $U(x)$ on Ω_h. Then we have that

$$AU \geq F, \quad U \geq 0, \quad U^T(AU - F) = 0. \tag{1.2}$$

Where A is a symmetric positive definite M-matrix if each angle of the mesh is an acute angle.

In [KNT94, KNT95] the numerical solution of linear complementarity problems with monotone operators by relaxation and Schwarz-type overlapping domain decomposition methods are considered. Monotone convergence was obtained for a special initial choice. Similar convergence was also discussed in [Sca90, Zho96]. Until now, however, we have not seen any discussion on the convergence rate except in

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[Tar96, ZZ]. In [ZZ], we analyzed convergence rate of the algorithm for solving obstacle problem if we confine the initial value to as special set, the same as [KNT94] or [KNT95]. Here, without any limitation of initial value, we prove that the iterate sequence generated by the algorithm we proposed converges to the solution of (1.2).

2 Schwarz Algorithm

Following the substructuring idea given by Dryja and Widlund (c.f. [Dry89, DW90, LSL92]), we can construct Schwarz algorithm for solving (1.2): We use the two-level triangulation of Ω given in [LSL92]. In this way we get the H-level triangulation consisting of Ω_i with diameters $H_i (i = 1, \dots, m)$ and the overlapping open subdomains $\Omega'_i \supset \Omega_i (i = 1, \dots, m)$. We assume that there is a positive constant c such that

$$\text{dist } (\partial\Omega'_i \backslash \partial\Omega, \partial\Omega_i \backslash \partial\Omega) \geq cH_i, \quad i = 1, \dots, m. \tag{2.1}$$

Let $\Omega_{ih} (i = 1, \dots, m)$ be the sets of the nodes that belong to open subdomains Ω'_i respectively, N_i be the subset of index set $\{1, 2, \dots, N\}$: $N_i = \{j \in \{1, 2, \dots, N\} : x_j \in \Omega_{ih}\}$. $A_{I,J}$ denotes the submatrix of A with elements a_{ij} $(i \in I, j \in J)$, U_I denotes the subvector of U with elements U_i $(i \in I)$. Then additive Schwarz algorithm can be stated as follows:

Additive Schwarz Algorithm

Step 1. $n := 0$.

Step 2. For $i = 1, 2, \dots, m$,

$$\begin{cases} A_{N_i,N_i} U_{N_i}^{n+1,i} \geq F^{n,i}, \\ U_{N_i}^{n+1,i} \geq 0, \\ (U_{N_i}^{n+1,i})^T (A_{N_i,N_i} U_{N_i}^{n+1,i} - F^{n,i}) = 0. \end{cases} \tag{2.2}$$

$$U_{N \backslash N_i}^{n+1,i} = U_{N \backslash N_i}^n, \tag{2.3}$$

where

$$F^{n,i} = F_{N_i} - A_{N_i, N \backslash N_i} U_{N \backslash N_i}^n. \tag{2.4}$$

Step 3. Choose ω_i satisfying

$$0 < \omega_i < 1, \quad i = 1, 2, \dots, m, \quad \sum_{i=1}^m \omega_i = 1.$$

Let

$$U^{n+1} = \sum_{i=1}^m \omega_i U^{n+1,i}. \tag{2.5}$$

Step 4. $n := n + 1$, go to step 2.

3 Geometric Convergence

Lemma 3.1(iteration error estimate) *Let* $\epsilon^{n+1,i} = U^{n+1,i} - U$, $\epsilon^n = U^n - U$ *(where* U *is the solution of (1.2)). Then for all* $n = 0, 1, \dots$ *and* $i = 1, \dots m$, *we have that*

$$(A|\epsilon^{n+1,i}|)_{N_i} \leq 0. \tag{3.1}$$

Proof. If $(U_{N_i})_k = 0$ and $(U_{N_i}^{n+1,i})_k > 0$, then

$$(A_{N_i,N_i}U_{N_i}^{n+1,i} - F^{n,i})_k = 0. \tag{3.2}$$

Let

$$F^{*,i} = F_{N_i} - A_{N_i,N\backslash N_i}U_{N\backslash N_i}.$$

We have

$$(A_{N_i,N_i}U_{N_i} - F^{*,i})_k \geq 0. \tag{3.3}$$

If we subtract (3.3) from (3.2), we get that

$$(A_{N_i,N_i}\epsilon_{N_i}^{n+1,i})_k \leq (-A_{N_i,N\backslash N_i}\epsilon_{N\backslash N_i}^{n})_k.$$

Since $A_{kj} \leq 0$ for $k \neq j$ and (2.3), we have

$$(A_{N_i,N_i}|\epsilon_{N_i}^{n+1,i}|)_k \leq (-A_{N,N\backslash N_i}|\epsilon_{N\backslash N_i}^{n+1,i}|)_k.$$

i.e.

$$((A|\epsilon^{n+1,i}|)_{N_i})_k \leq 0.$$

If $(U_{N_i})_k > 0$ and $(U_{N_i}^{n+1,i})_k > 0$, then

$$(A_{N_i,N_i}U_k^{n+1,i} - F^{n,i})_k = 0,$$

and

$$(A_{N_i,N_i}U_{N_i} - F^{*,i})_k = 0.$$

Hence

$$(A_{N_i,N_i}\epsilon_{N_i}^{n+1,i})_k = (-A_{N_i,N\backslash N_i}\epsilon_{N\backslash N_i}^{n})_k.$$

Therefore,

$$(A_{N_i,N_i}|\epsilon_{N_i}^{n+1,i}|)_k \leq (-A_{N,N\backslash N_i}|\epsilon_{N\backslash N_i}^{n}|)_k = (-A_{N_i,N\backslash N_i}|\epsilon_{N\backslash N_i}^{n+1,i}|)_k.$$

If $(U_{N_i})_k > 0$ and $(U_{N_i}^{n+1,i})_k = 0$, then

$$(A_{N_i,N_i}U_k^{n+1,i} - F^{n,i})_k \geq 0,$$

and

$$(A_{N_i,N_i}U_{N_i} - F^{*,i})_k = 0.$$

Then

$$(A_{N_i,N_i}\epsilon_{N_i}^{n+1,i})_k \geq (-A_{N_i,N\backslash N_i}\epsilon_{N\backslash N_i}^{n})_k.$$

Since $(\epsilon_{N_i}^{n+1,i})_k < 0$, we also have

$$(A_{N_i,N_i}|\epsilon_{N_i}^{n+1,i}|)_k \leq (-A_{N,N\backslash N_i}|\epsilon_{N\backslash N_i}^{n}|)_k = (-A_{N_i,N\backslash N_i}|\epsilon_{N\backslash N_i}^{n+1,i}|)_k.$$

That is (3.1) holds. Thus we complete the proof.

In order to prove the geometric convergence, we also need to establish the discrete strong maximum principle as follows:

Lemma 3.2 *For finite element discretization, if each angle of the mesh is an acute angle and the boundary of every subdomain has a common part with the boundary of the whole domain. Then, for any $V \geq 0$ satisfying $(AV)_{N_i} \leq 0$, there exists a constant $k_i \in (0,1)$ such that*

$$\max_{j \in N_i} V_j \leq k_i \max_{j \notin N_i} V_j.$$

Remark Under the conditions of lemma 3.2, matrix A has the following properties: (i) $a_{ii} > 0$, $a_{ij} \leq 0 (j \neq i)$; (ii) A is irreducible and weak diagonal dominant; (iii) $A_{N_i N_i}$ are irreducible, $i = 1, \dots, m$; (iv) for every $i = 1, \dots, m$, there exists $l_i \in N_i$ such that $\sum\limits_{j=1}^{N} a_{l_i j} > 0$. Therefore, lemma 3.1 can be proved by reduction to absurdity.

By lemma 3.1 and lemma 3.2 we get the following geometric convergence result:

Theorem 3.1 *Under the conditions of lemma 3.2, we have that*

$$\|\epsilon^{n+1}\|_\infty \leq \max_{1 \leq i \leq m} (\omega_i k_i + \sum_{j \neq i} \omega_j) \|\epsilon^n\|_\infty = k \|\epsilon^n\|_\infty, \tag{3.4}$$

where $k = \max\limits_{1 \leq i \leq m} (\omega_i k_i + \sum\limits_{j \neq i} \omega_j) \in (0,1)$.

Proof $\forall k \in \{1, 2, \dots, N\}$ and $i \in \{1, 2, \dots, m\}$. If $k \notin N_i$, then $|\epsilon^{n+1,i}|_k = |\epsilon^n|_k \leq \|\epsilon^n\|_\infty$; If $k \in N_i$, since $(A|\epsilon^{n+1,i}|)_{N_i} \leq 0$. By use of lemma 3.1 and lemma 3.2, we have that

$$|\epsilon^{n+1,i}|_k \leq k_i \max_{j \notin N_i} |\epsilon^{n+1,i}|_j = k_i \max_{j \notin N_i} |\epsilon^n|_j \leq k_i \|\epsilon^n\|_\infty.$$

Since $\{1, 2, \dots, N\} = \bigcup\limits_1^m N_i$, we get (3.4).

4 Concluding Remarks

The goal of the paper is to give the convergence proof of additive Schwarz algorithm for the algebraic obstacle problems by establishing discrete maximum principle. The convergence theory can be found useful for many resons. First, Our results are suitable for obstacle problems with nonsymmetric operators. As we know, many discussions about Schwarz methods for obstacle problems require the problems have self-adjoint and positive definite operators (c.f. [Lio88, Lio89, Sca90, Xu92, Zho96]). Second, we can get geometric convergent rate by establishing corresponding discrete maximum principle. Especially, we can estimating h-independent convergence of the additive Schwarz algorithm by estimating k_i in lemma 3.2 (see [ZZ] for details). Finally, we notice that the convergence theory of this paper can be extended to multiplicative Schwarz algorithm or other obstacle problems as well. For example, using a similar analysis, we could prove geometric convergence of additive or multiplicative algorithms applied to solving bilateral discrete obstacle problems. These extensions will be studies in the forthcoming publications.

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