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An Additive Schwarz method for Elliptic Mortar Finite Element Problems in Three Dimensions

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1 Introduction

In this paper, we discuss a domain decomposition method for solving linear systems of algebraic equations arising from the discretization of elliptic problem in 3-D by the mortar element method; see [Mar90, AG93] and literature given therein. The elliptic problem is second-order with discontinuous coefficients and the Dirichlet boundary condition. Using the framework of the mortar method, the problem is approximated by the finite element method with piecewise linear functions on nonmatching meshes.

The domain decomposition method is of iterative substructuring type and is described as an additive Schwarz method (ASM) using the general framework of ASMs; see [DW95, Ben95a]. It is applied to the Schur complement of our discrete problems, i.e., interior variables of all subregions are first eliminated using a direct method.

In this paper, we consider the mortar element method in the geometrically conforming case only. The region Ω is a union of simplices Ω_i on which a coefficient ρ_i of the problem is constant. The described ASM uses a standard coarse space defined on the triangulation formed by Ω_i of diameter H, i.e., $V_0 = V^H$, a space of piecewise linear continuous functions which vanish on $\partial\Omega$.

The algorithm described is almost optimal, i.e., the number of iterations required to decrease the energy norm of the error in a conjugate gradient method is proportional to $(1 + log\frac{H}{h})^g$, where $g = 1$ or $g = \frac{3}{2}$ with the constant independent of the coarse and fine meshes (H and h) and the coefficient ρ_i. This result is proved assuming a special distribution of the ρ_i on Ω_i, called quasi-monotone (introduced in [DSW96]) and weak quasi-monotone (introduced herein). This is the main result of the paper. There are indications that this result is sharp in 3-D with respect to the distribution of ρ_i, see [Osw95] and [Xu91]. In the case of arbitrary distribution of ρ_i, the number of iterations can be bounded by $(H/h)^{\frac{1}{2}}$.

The results of this paper are generalizations of results obtained in [Ben95b] for 2-D. The idea of using a standard coarse space is taken from [Glo84], where the 2-D case

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with regular coefficients is considered. The mortar element method in the geometrically nonconforming case for problems with discontinuous coefficients is not discussed here. The reason is that it is not clear how to design and analyze ASMs for either the standard coarse space or for others; see, for example, the new coarse space used in [Ben95b] in the 2-D case.

The outline of the paper is as follows. In Section 1.2, the discrete problem obtained from the mortar element method is described. In Section 1.3, an iterative substructuring method is described in terms of an ASM for the Schur complement system. In this section, Theorem 1.3.1 is formulated as the main result of the paper. A proof of this theorem is given in Section 1.5. In Section 1.4, technical tools are given which are needed for the proof.

Some of the results of this paper have been obtained in joint work with Olof Widlund.

2 Mortar Discrete Problems

We solve the following differential problem: Find $u^* \in H_0^1(\Omega)$ such that

$$a(u^*, v) = f(v), \quad v \in H_0^1(\Omega), \tag{1}$$

where

$$a(u,v) = \sum_i \rho_i (\nabla u, \nabla v)_{L^2(\Omega_i)}, \quad f(v) = (f,v)_{L^2(\Omega)},$$

$\bar{\Omega} = \cup_{i=1}^N \bar{\Omega}_i$, and ρ_i is a positive constant in Ω_i.

Let Ω be a polygonal region in 3-D and Ω_i be tetrahedral elements. They form a coarse triangulation with a parameter H. In each Ω_i, a triangulation is introduced with tetrahedral elements $e_j^{(i)}$ and a parameter h_i. The resulting triangulation of Ω is nonmatching. We assume that the coarse triangulation and the h_i triangulation in each Ω_i are quasi-uniform, see [GPP94]. Let $X_i(\Omega_i)$ be the finite element space of piecewise linear continuous functions defined on the triangulation of Ω_i and vanishing on $\partial\Omega_i \cap \partial\Omega$. Let

$$X^h(\Omega) = X_1(\Omega_1) \times \cdots \times X_N(\Omega_N).$$

To define the mortar finite element method, we introduce some notation and spaces. Let

$$\Gamma = (\cup_i \partial\Omega_i) \backslash \partial\Omega$$

and let F_{ij}, E_{ij} denote the faces, edges of Ω_i. The union of $\bar{E}_{ij}$ forms the wire basket W_i of Ω_i. We now select open faces γ_m of Γ, called mortars (masters) such that

$$\Gamma = \cup_m \bar{\gamma}_m \ \text{ and } \ \gamma_m \cap \gamma_n = \emptyset \ \text{ if } \ m \neq n.$$

By $\gamma_{m(i)}$ we denote a face of Ω_i. Let $\gamma_{m(i)}$ be a face common to Ω_i and Ω_j. As a face of Ω_j it is denoted by $\delta_{m(j)}$ and it is called nonmortar (slave). The rule for selecting $\gamma_{m(i)} = F_{ij}$, a common face to Ω_i and Ω_j, as mortar is that $\rho_i \geq \rho_j$. Let $W^{h_i}(F_{ij})$ be the restriction of $X_i(\Omega_i)$ to F_{ij}. Note that on $F_{ij} = \gamma_{m(i)} = \delta_{m(j)}$, the common face to Ω_i and Ω_j, we have two triangulation, denoted in terms of h_i and h_j and two different spaces $W^{h_i}(\gamma_{m(i)})$ and $W^{h_j}(\delta_{m(j)})$.

Let $M^{h_j}(\delta_{m(j)})$ denote a subspace of $W^{h_j}(\delta_{m(j)})$ defined as follows: Let $v \in W^{h_j}(\delta_{m(j)})$. A function $\tilde{v} \in M^{h_j}(\delta_{m(j)})$ has the same values as v at the interior nodal points of $\delta_{m(j)}$. The value of $\tilde{v}$ at a nodal point $x_k \in \partial\delta_{m(j)}$, the boundary of $\delta_{m(j)}$, is equal to

$$\tilde{v}(x_k) = \sum_{i=1}^{n_k} \alpha_i v(x_{i(k)}) \quad \sum_{i=1}^{n_k} \alpha_i = 1,$$

where $\alpha_i \geq 0$ and the sum is taken over interior nodal points $x_{i(k)}$ of $\delta_{m(j)}$ such that an interval $(x_k, x_{i(k)})$ is a side of the triangulation and its number is equal to n_k, for details see [AG93].

We say that $u_{i(m)}$ and $u_{j(m)}$, the restrictions of $u_i \in X_i(\Omega_i)$ and $u_j \in X_j(\Omega_j)$ to δ_m, a common face to Ω_i and Ω_j, satisfy the mortar condition if

$$\int_{\delta_m} (u_{i(m)} - u_{j(m)})\Psi ds = 0, \quad \Psi \in M^{h_j}(\delta_m). \tag{2}$$

This condition can be rewritten as follows. Let $\Pi_m(u_{i(m)}, v_{j(m)})$ denote a projection from $L^2(\delta_m)$ on $W^{h_j}(\delta_m)$ defined by

$$\int_{\delta_m} \Pi_m(u_{i(m)}, v_{j(m)})\Psi ds = \int_{\delta_m} u_{i(m)}\Psi ds, \quad \Psi \in M^{h_j}(\delta_m) \tag{3}$$

and

$$\Pi_m(u_{i(m)}, v_{j(m)})_{|\partial\delta_m} = v_{j(m)}. \tag{4}$$

Thus $u_{j(m)} = \Pi_m(u_{i(m)}, v_{j(m)})$ if $v_{j(m)} = u_{j(m)}$ on $\partial\delta_m$.

By V^h we denote a space of $v \in X^h$ which satisfies the mortar condition for each $\delta_m \subset \Gamma$. The discrete problem for (1) in V^h is defined as follows: Find $u_h^* \in V^h$ such that

$$b(u_h^*, v_h) = f(v_h), \quad v_h \in V^h, \tag{5}$$

where

$$b(u_h, v_h) = \sum_{i=1}^{N} a_i(u_{ih}, v_{ih}) = \sum_{i=1}^{N} \rho_i(\nabla u_{ih}, \nabla v_{ih})_{L^2(\Omega_i)}$$

and $v_h = \{v_{ih}\}_{i=1}^N \in V^h$. It is known that V^h is a Hilbert space with an inner product defined by $b(u, v)$. This problem has an unique solution and an estimate of the error is known, see [AG93].

3 Additive Schwarz Method

In this section, we describe an additive Schwarz method for (5). It will be given for the Schur complement system. For that we first eliminate all interior unknowns of Ω_i using for $u_i \in X_i(\Omega_i)$ (here and below we drop the index h for functions)

$$u_i = Pu_i + Hu_i, \tag{6}$$

where Hu_i is discrete harmonic in Ω_i in the sense of $(\nabla u, \nabla v)_{L^2(\Omega_i)}$ with $Hu_i = u_i$ on $\partial\Omega_i$. Using this, we get

$$s(u^*, v) = f(v), \quad v \in V^h(\Omega), \tag{7}$$

where here and below V^h denotes a space of discrete harmonic functions in each Ω_i and

$$s(u, v) = b(u, v), \quad u, v \in V^h(\Omega).$$

An additive Schwarz method (ASM) for (7) is designed and analyzed using the general ASM framework, see [Ben95a]. Using this framework, the method is designed in terms of a decomposition of V^h, certain bilinear forms given on these subspaces, and the projections onto these subspaces in the sense of these bilinear forms.

The decomposition of V^h is taken as

$$V^h(\Omega) = V_0(\Omega) + \sum_{\gamma_m \subset \Gamma} V_m^{(F)}(\Omega) + \sum_{i=1}^{N} \sum_{x_k \in W_{ih}} V_k^{(W_i)}(\Omega). \tag{8}$$

Here $V_0 = V^H$ is a space of piecewise linear continuous functions, on the coarse triangulation, which vanish on $\partial\Omega$. The space $V_m^{(F)}(\Omega)$ is a subspace of V^h associated with the master face γ_m. It is the restriction of V^h to γ_m and δ_m ($\gamma_m = \delta_m$), and the zero on $\partial\gamma_m$ and $\partial\delta_m$, the remaining master and slave faces, and on $\partial\Omega$. W_{ih} is the set of nodal points of W_i. $V_k^{(W_i)}$ is an one-dimensional space associated with $x_k \in W_{ih}$ and spanned by Φ_k. The function Φ_k is discrete harmonic with data on the boundary of the substructures defined as follows: Let x_k be a nodal point of $\partial\gamma_{m(i)}$, the boundary of the mortar face $\gamma_{m(i)}$ of Ω_i. We set $\Phi = \varphi_k(x)$ on $\gamma_{m(i)}$, where $\varphi_k(x)$ is a nodal basis function associated with x_k. Let $\delta_{m(j)} = \gamma_{m(i)} = F_{ij}$ be the face common to Ω_i and Ω_j. Φ_k is equal to $\Pi_m(\varphi_k, 0)$ on $\delta_{m(j)}$; see (3) and (4). Φ_k is defined on the remaining mortar faces of Ω_i in the same way if x_k is a nodal point of their boundaries. Φ_k is zero on the remaining mortar and nonmortar faces of Γ. Let x_k be a nodal point of $\partial\delta_{m(i)}$, the boundary of a nonmortar face of Ω_i. $\Phi_k(x)$ is equal to $\Pi_m(0, \varphi_k)$ on $\delta_{m(i)}$. This means that $\Phi_k = 0$ on the mortar face $\gamma_{m(j)} = \delta_{m(i)}$. Φ_k is defined on the remaining nonmortar faces of Ω_i in the same way if x_k is a nodal point of their boundaries. Φ_k is zero on the mortar and nonmortar faces belonging to remaining substructures. If x_k is a nodal point common to the boundaries of mortar or nonmortar faces, Φ_k is defined on these faces as above.

Let us now introduce bilinear forms defined on the introduced spaces. $b_m^{(F)}$ associated with $V_m^{(F)} \times V_m^{(F)} \to R$ is of the form

$$b_m^{(F)}(u_{m(i)}, v_{m(i)}) = (\rho_i + \rho_j)(\nabla u_{m(i)}, \nabla v_{m(i)})_{L^2(\Omega_i)}, \tag{9}$$

where $u_{m(i)}$ is the discrete harmonic function in Ω_i with data $u_{m(i)}$ on the mortar face $\gamma_{i(m)}$ of Ω_i which is common to Ω_j and zero on the remaining faces of Ω_i.

We set $b_k^{(W_i)} : V_k^{(W_i)} \times V_k^{(W_i)} \to R$ and $b_0 : V_0 \times V_0 \to R$ equal to $b(u, v)$.

Let us now introduce operators $T_m^{(F)}$, $T_k^{(W_i)}$, and T_0 by the bilinear forms $b_m^{(F)}$, $b_k^{(W_i)}$, and b_0, respectively, in the standard way. For example, $T_m^{(F)} : V^h \to V_m^{(F)}$ is the solution of

$$b_m^{(F)}(T_m^{(F)} u, v) = b(u, v), \quad v \in V_m^{(F)}. \tag{10}$$

Let

$$T = T_0 + \sum_{\gamma_m \subset \Gamma} T_m^{(F)} + \sum_{i=1}^{N} \sum_{x_k \in W_{ih}} T_k^{(W_i)}.$$

The method described is almost optimal assuming a special distribution of the coefficients ρ_i, called quasi-monotone, introduced in [DSW96]. The quasi-monotone distribution on substructures with common vertex x_k requires a monotone path from each substructure to the substructure having the largest coefficient, traversing through faces of substructures only. If the vertex $x_k \in \partial\Omega$, we additionally assume that $\partial\Omega_i \cap \partial\Omega$ contains a face of the substructure Ω_i with the largest coefficient ρ_i. This is a local condition. The distribution ρ_i in Ω is quasi-monotone if it is quasi-monotone at each vertex of the substructures; for details see [DSW96]. We also introduce the concept of a weak quasi-monotone distribution of ρ_i for which the traversing path is also allowed to go through edges. In this case, for a vertex $x_k \in \partial\Omega$, we assume that $\partial\Omega_i \cap \partial\Omega$ contains the face or the edge of Ω_i for which ρ_i is the largest in Ω_i.

There are indications that the estimates given below are sharp; see [Osw95] and [Xu91].

Theorem 3.1 *For all $u \in V^h$*

$$C_0(1 + log\frac{H}{h})^{-2}\delta^{-1}a(u,u) \le a(Tu,u) \le C_1 a(u,u), \tag{11}$$

where C_i are positive constants independent of H, h_i and ρ_i, $h = inf_i h_i$ and

$$\delta = \begin{cases} 1 & \textit{when } \rho_i \textit{ is quasi-monotone} \\ (1 + \log\frac{H}{h}) & \textit{when } \rho_i \textit{ is weakly quasi-monotone} \\ \frac{H}{h} & \textit{when } \rho_i \textit{ is not even weakly quasi-monotone} \end{cases} \tag{12}$$

4 Technical Tools

In this section, we formulate some auxiliary results that we need to prove Theorem 3.1.

Lemma 4.1 *Let $\gamma_{i(m)} = \delta_{j(m)}$ be a face common to Ω_i and Ω_j, and let $u_{i(m)}$ and $u_{j(m)}$ be the restrictions of $u_i \in X_i(\Omega_i)$ and $u_j \in X_j(\Omega_j)$ to $\gamma_{i(m)}$ and $\delta_{j(m)}$, respectively. If $u_{i(m)}$ and $u_{j(m)}$ satisfy the mortar condition (2) on $\delta_{j(m)}$ and $u_{j(m)}$ vanishes on $\partial\delta_{j(m)}$, then*

$$||u_{j(m)}||^2_{L^2(\delta_{j(m)})} \le C||u_{i(m)}||^2_{L^2(\gamma_{i(m)})}, \tag{13}$$

where C is independent of h_i and h_j.

This lemma follows from Lemma 2.1 in [AG93].

Lemma 4.2 *Let the assumptions of Lemma 4.1 be satisfied and additionally $u_{i(m)}$ vanishes on $\partial\delta_{i(m)}$. Then,*

$$||u_{j(m)}||^2_{H_{00}^{\frac{1}{2}}(\delta_{j(m)})} \le C||u_{i(m)}||^2_{H_{00}^{\frac{1}{2}}(\gamma_{i(m)})}, \tag{14}$$

where C is independent of h_i and h_j.

A proof of this lemma follows from Lemma 4.1 and properties of the standard L^2 projection on $W^{h_j}(\delta_{j(m)}) \cap H_0^1(\delta_{j(m)})$. In the 2-D case, Lemma 4.2 is a particular case of Lemma 1 in [Pes72]. Alternative proofs of this result, also in the 2-D case, are given in [Glo84] and [Ben95b].

Lemma 4.3 *Let Φ_k be a function defined in Section 1.3 and associated with a nodal point $x_k \in W_i$. Then*

$$b(\Phi_k, \Phi_k) \le C\rho_i h_i, \tag{15}$$

where C is independent of h_i and ρ_i.

A proof of this lemma follows from Lemma 4.1 and the definition of Φ_k.

Let $R(x_k)$ be a union of the substructures Ω_i with a common vertex x_k.

Lemma 4.4 *For $u \in V^h$*

$$\inf_{\alpha \in R} ||u - \alpha||^2_{L^2(R(x_k))} \le C \sum_{\Omega_i \subset R(x_k)} H^2 |u|^2_{H^1(\Omega_i)}, \tag{16}$$

where C is a positive constant independent of h_i and H.

A proof of this lemma in the 2-D case is given in [Glo84]. An alternative proof follows from

$$||u - \alpha||^2_{L^2(R(x_k))} \le 2 \sum_{i=1}^{n_k} \left(||u - \bar{u}_i||^2_{L^2(\Omega_i)} + ||\bar{u}_i - \alpha||^2_{L^2(\Omega_i)} \right). \tag{17}$$

Here the Ω_i with a common x_k are ordered from $i = 1, \ldots, n_k$ in such a way that Ω_i and Ω_{i+1} have a common face $F_{i,i+1}$ and $\bar{u}_i$ is the average value of u_i over $F_{i,i+1}$. Using now Poincare's inequality, we get (16).

Let Q_ρ^H denote the L_ρ^2 projection from V^h to $V_0 = V^H$ in the weighted inner product.

Lemma 4.5 *For $u \in V^h$*

$$b(Q_\rho^H u, Q_\rho^H u) \le C\delta b(u, u) \tag{18}$$

and

$$||u - Q_\rho^H u||^2_{L_\rho^2(\Omega)} \le CH^2 \delta b(u, u), \tag{19}$$

where δ is given by (12) and C is constant independent of H, h_i and ρ_i.

A proof of this lemma is a slighted modification of the proof of Lemma 9 in [DSW96].

5 Proof of Theorem 1.3.1

Using the general theorem of ASMs, we need to check three key assumptions; see [DW95] and [Ben95a].

Assumption (ii) It is shown that $\rho(\varepsilon) \le C$ in view of Lemma 4.3.

Assumption (iii) Of course, $\omega = 1$ for $b_0(u,u),\ u \in V_0$ and $b_k^{(W_i)}(u,u),\ u \in V_k^{(W_i)}$. We now show that for $u \in V_m^{(F)}$

$$b(u,u) \le C b_m^{(F)}(u,u). \tag{20}$$

Let $\gamma_{i(m)} = \delta_{j(m)}$ be the mortar and nonmortar sides of Ω_i and Ω_j, respectively. We have for $u \in V_m^{(F)}$

$$b(u,u) = a_i(u_i,u_i) + a_j(u_j,u_j) \le C\big(\rho_i |u_i|^2_{H_{00}^{\frac{1}{2}}(\gamma_{i(m)})} + \rho_j |u_j|^2_{H_{00}^{\frac{1}{2}}(\delta_{j(m)})}\big).$$

Using now Lemma 4.2, we get (20) with $w = C$.

Assumption (i) We show that for $u \in V^h$, there exists a decomposition

$$u = u_0 + \sum_{\gamma_m \subset \Gamma} u_m^{(F)} + \sum_{i=1}^{N} \sum_{x_k \in W_{ih}} u_k^{(W_i)}, \tag{21}$$

where $u_0 \in V_0$, $u_m^{(F)} \in V_m^{(F)}$ and $u_k^{(W_i)} \in V_k^{(W_i)}$, such that

$$b_0(u_0,u_0) + \sum_{\gamma_m \subset \Gamma} b_m^{(F)}(u_m^{(F)}, u_m^{(F)}) + \sum_{i=1}^{N} \sum_{x_k \in W_{ih}} b_k^{(W_i)}(u_k^{(W_i)}, u_k^{(W_i)})$$
$$\le C\delta(1 + log\frac{H}{h})^2 b(u,u). \tag{22}$$

Let $u_0 = Q_\rho^H u$, $w = u - u_0$, and w_i be the restriction of w to $\bar{\Omega}_i$. It is decomposed on $\partial\Omega_i$ as

$$w_i = \sum_{F_{ij} \subset \partial\Omega_{ih}} w_i^{(F_{ij})}(x) + w_i^{(W_i)}, \quad w_i^{(W_i)} = \sum_{x_k \in W_{ih}} w_i(x)\Phi_k, \tag{23}$$

where $w_i^{(F_{ij})}(x)$ is the restriction of $w_i - w_i^{(W_i)}$ to F_{ij}, the face of Ω_i, and zero on $\partial\Omega_i \backslash F_{ij}$.

To define $u_m^{(F)}$ let $F_{ij} = \gamma_{i(m)} = \delta_{j(m)}$ be a face common to Ω_i and Ω_j. We set

$$u_m^{(F)} = \{w_i^{(F_{ij})} \text{ on } \partial\Omega_i \text{ and } w_j^{(F_{ij})} \text{ on } \partial\Omega_j\}$$

and zero at the remaining nodal points of Γ. The function $u_k^{(W_i)}$ is defined as

$$u_k^{(W_i)} = w_i(x_k)\Phi_k(x). \tag{24}$$

It is easy to see that these functions satisfy (21).

To prove (22) note first that

$$b_0(u_0,u_0) \le C\delta b(u,u) \tag{25}$$

by Lemma 4.5.

Let us now consider the estimate for $u_m^{(F)} \in V_M^{(F)}$ when $\gamma_{m(i)} = \delta_{m(i)} = F_{ij}$, a face common to Ω_i and Ω_j. It is known that

$$b_m^{(F)}(u_m^{(F)}, u_m^{(F)}) \leq C(\rho_i + \rho_j)||w_i^{(F_{ij})}||^2_{H_{00}^{\frac{1}{2}}(\gamma_{i(m)})}$$
$$\leq C\rho_i(1 + log\frac{H}{h_i})^2||u_i - u_0||^2_{H^1(\Omega_i)};$$

see, for example, [DW95]. We have used here also the fact that $\rho_i \geq \rho_j$. Summing with respect to γ_m and using Lemma 4.5, we get

$$\sum_{\gamma_m \subset \Gamma} b_m^{(F)}(u_m^{(F)}, u_m^{(F)}) \leq C\delta(1 + log\frac{H}{h})^2 b(u,u). \tag{26}$$

We now prove that

$$\sum_{i=1}^{N} \sum_{x_k \in W_{ih}} b_k^{(W_i)}(u_k^{(W_i)}, u_k^{(W_i)}) \leq C\delta(1 + log\frac{H}{h}) b(u,u). \tag{27}$$

For that note first that, see (24),

$$b_k^{(W_i)}(u_k^{(W_i)}, u_k^{(W_i)}) \leq Cw_i^2(x_k)b(\Phi_k, \Phi_k) \leq C\rho_i h_i w_i^2(x_k)$$

in view of Lemma 4.3. Summing with respect to $x_k \in W_{ih}$, we get

$$\sum_{x_k \in W_{ih}} b_k^{(W_i)}(u_k^{(W_i)}, u_k^{(W_i)}) \leq C\rho_i||w_i||^2_{L^2(W_i)} \leq C\rho_i(1 + log\frac{H}{h_i})||w_i||^2_{H^1(\Omega_i)}.$$

Summing now with respect to i and using Lemma 4.5, we get (27).

To get (22), we add the inequalities (25), (26), and (27). The proof of Theorem 3.1 is complete.

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