ISSN 1464-8997 (on line) 1464-8989 (printed)

Geometry & Topology Monographs
Volume 3: Invitation to higher local fields
Part II, section 9, pages 293–298

9. Local reciprocity cycles

Ivan Fesenko

In this section we introduce a description of totally ramified Galois extensions of a local field with finite residue field (extensions have to satisfy certain arithmetical restrictions if they are infinite) in terms of subquotients of formal power series $\mathbb{F}_p^{\text{sep}}[[X]]^*$. This description can be viewed as a non-commutative local reciprocity map (which is not in general a homomorphism but a cocycle) which directly describes the Galois group in terms of certain objects related to the ground field. Abelian class field theory as well as metabelian theory of Koch and de Shalit [K], [KdS] (see subsection 9.4) are partial cases of this theory.

9.1. Group $U^{\diamond}_{N(\widehat{L/F})}$

Let F be a local field with finite residue field. Denote by $\varphi \in G_F$ a lifting of the Frobenius automorphism of F_{ur}/F.

Let F^{φ} be the fixed field of φ. The extension F^{φ}/F is totally ramified.

Lemma ([KdS, Lemma 0.2])**.** *There is a unique norm compatible sequence of prime elements π_E in finite subextensions E/F of F^{φ}/F.*

Proof. Uniqueness follows from abelian local class field theory, existence follows from the compactness of the group of units. □

In what follows we fix F^{φ} and consider Galois subextensions L/F of F^{φ}/F. Assume that L/F is *arithmetically profinite*, ie for every x the ramification group $\text{Gal}(L/F)^x$ is open in $\text{Gal}(L/F)$ (see also subsection 6.3 of Part II). For instance, a totally ramified p-adic Lie extension is arithmetically profinite.

For an arithmetically profinite extension L/F define its Hasse–Herbrand function $h_{L/F}\colon [0,\infty) \to [0,\infty)$ as $h_{L/F}(x) = \lim h_{M/F}(x)$ where M/F runs over finite subextensions of L/F (cf. [FV, Ch. III §5]).

If L/F is infinite let $N(L/F)$ be the field of norms of L/F. It can be identified with $k_F((\Pi))$ where Π corresponds to the norm compatible sequence π_E (see subsection 6.3 of Part II, [W], [FV, Ch.III §5]).

Denote by φ the automorphism of $N(L/F)_{\mathrm{ur}}$ and of its completion $\widehat{N(L/F)}$ corresponding to the Frobenius automorphism of F_{ur}/F.

Definition. Denote by $U^{\diamond}_{\widehat{N(L/F)}}$ the subgroup of the group $U_{\widehat{N(L/F)}}$ of those elements whose $\widehat{F}$-component belongs to U_F. An element of $U^{\diamond}_{\widehat{N(L/F)}}$ such that its $\widehat{F}$-component is $\varepsilon \in U_F$ will be called a lifting of ε.

The group $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$ is a direct product of a quotient group of the group of multiplicative representatives of the residue field k_F of F, a cyclic group $\mathbb{Z}/p^a$ and a free topological $\mathbb{Z}_p$-module. The Galois group $\mathrm{Gal}(L/F)$ acts naturally on $U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}$.

9.2. Reciprocity map $\mathcal{N}_{L/F}$

To motivate the next definition we interpret the map $\Upsilon_{L/F}$ (defined in 10.1 and 16.1) for a finite Galois totally ramified extension L/F in the following way. Since in this case both π_Σ and π_L are prime elements of L_{ur}, there is $\varepsilon \in U_{L_{\mathrm{ur}}}$ such that $\pi_\Sigma = \pi_L\varepsilon$. We can take $\tilde\sigma = \sigma\varphi$. Then $\pi_L^{\sigma-1} = \varepsilon^{1-\sigma\varphi}$. Let $\eta \in U_{\widehat{L}}$ be such that $\eta^{\varphi-1} = \varepsilon$. Since $(\eta^{\sigma\varphi-1}\varepsilon^{-1})^{\varphi-1} = (\eta^{(\sigma-1)\varphi})^{\varphi-1}$, we deduce that $\varepsilon = \eta^{\sigma\varphi-1}\eta^{(1-\sigma)\varphi}\rho$ with $\rho \in U_L$. Thus, for $\xi = \eta^{\sigma\varphi-1}$

$$\Upsilon_{L/F}(\sigma) \equiv N_{\Sigma/F}\pi_\Sigma \equiv N_{\widehat{L}/\widehat{F}}\xi \bmod N_{L/F}L^*, \qquad \xi^{1-\varphi} = \pi_L^{\sigma-1}.$$

Definition. For a $\sigma \in \mathrm{Gal}(L/F)$ let $U_\sigma \in U_{\widehat{N(L/F)}}$ be a solution of the equation

$$\boxed{U^{1-\varphi} = \Pi^{\sigma-1}}$$

(recall that $\mathrm{id} - \varphi\colon U_{\widehat{N(L/F)}} \to U_{\widehat{N(L/F)}}$ is surjective). Put

$$\mathcal{N}_{L/F}\colon \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)}, \quad \mathcal{N}_{L/F}(\sigma) = U_\sigma \mod U_{N(L/F)}.$$

Remark. Compare the definition with Fontaine-Herr's complex defined in subsection 6.4 of Part II.

Properties.

(1) $\mathcal{N}_{L/F} \in Z^1(\mathrm{Gal}(L/F), U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)})$ is injective.

(2) For a finite extension L/F the $\widehat{F}$-component of $\mathcal{N}_{L/F}(\sigma)$ is equal to the value $\Upsilon_{L/F}(\sigma)$ of the abelian reciprocity map $\Upsilon_{L/F}$ (see the beginning of 9.2).

(3) Let M/F be a Galois subextension of L/F and E/F be a finite subextension of L/F. Then the following diagrams of *maps* are commutative:

$$\begin{array}{ccc} \mathrm{Gal}(L/E) & \xrightarrow{\mathcal{N}_{L/E}} & U^{\diamond}_{\widehat{N(L/E)}}/U_{N(L/E)} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(L/F) & \xrightarrow{\mathcal{N}_{L/F}} & U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)} \end{array} \qquad \begin{array}{ccc} \mathrm{Gal}(L/F) & \xrightarrow{\mathcal{N}_{L/F}} & U^{\diamond}_{\widehat{N(L/F)}}/U_{N(L/F)} \\ \downarrow & & \downarrow \\ \mathrm{Gal}(M/F) & \xrightarrow{\mathcal{N}_{M/F}} & U^{\diamond}_{\widehat{N(M/F)}}/U_{N(M/F)}. \end{array}$$

(4) Let $U^{\diamond}_{n,\widehat{N(L/F)}}$ be the filtration induced from the filtration $U_{n,\widehat{N(L/F)}}$ on the field of norms. For an infinite arithmetically profinite extension L/F with the Hasse–Herbrand function $h_{L/F}$ put $\mathrm{Gal}(L/F)_n = \mathrm{Gal}(L/F)^{h^{-1}_{L/F}(n)}$. Then $\mathcal{N}_{L/F}$ maps $\mathrm{Gal}(L/F)_n \setminus \mathrm{Gal}(L/F)_{n+1}$ into $U^{\diamond}_{n,\widehat{N(L/F)}}U_{N(L/F)} \setminus U^{\diamond}_{n+1,\widehat{N(L/F)}}U_{N(L/F)}$.

(6) The set $\mathrm{im}(\mathcal{N}_{L/F})$ is not closed in general with respect to multiplication in the group
$U_{\widehat{N(L/F)}}/U_{N(L/F)}$. Endow $\mathrm{im}(\mathcal{N}_{L/F})$ with a new group structure given by $x \star y = x\mathcal{N}^{-1}_{L/F}(x)(y)$. Then clearly $\mathrm{im}(\mathcal{N}_{L/F})$ is a group isomorphic to $\mathrm{Gal}(L/F)$.

Problem. What is $\mathrm{im}(\mathcal{N}_{L/F})$?

One method to solve the problem is described below.

9.3. Reciprocity map $\mathcal{H}_{L/F}$

Definition. Fix a tower of subfields $F = E_0 - E_1 - E_2 - \dots$, such that $L = \cup E_i$, E_i/F is a Galois extension, and E_i/E_{i-1} is cyclic of prime degree. We can assume that $|E_{i+1} : E_i| = p$ for all $i \geqslant i_0$ and $|E_{i_0} : E_0|$ is relatively prime to p.

Let σ_i be a generator of $\mathrm{Gal}(E_i/E_{i-1})$. Denote

$$X_i = U^{\sigma_i - 1}_{\widehat{E}_i}.$$

The group X_i is a $\mathbb{Z}_p$-submodule of $U_{1,\widehat{E}_i}$. It is the direct sum of a cyclic torsion group of order p^{n_i}, $n_i \geqslant 0$, generated by, say, α_i ($\alpha_i = 1$ if $n_i = 0$) and a free topological $\mathbb{Z}_p$-module Y_i.

We shall need a sufficiently "nice" injective map from characteristic zero or p to characteristic p

$$f_i \colon U^{\sigma_i - 1}_{\widehat{E}_i} \to U_{\widehat{N(L/E_i)}} \to U_{N(L/F)}.$$

If F is a local field of characteristic zero containing a non-trivial pth root ζ and f_i is a homomorphism, then ζ is doomed to go to 1. Still, from certain injective maps (not homomorphisms) f_i specifically defined below we can obtain a subgroup $\prod f_i(U_{\widehat{E_i}}^{\sigma_i - 1})$ of $U^{\diamond}_{\widehat{N(L/F)}}$.

Definition. If $n_i = 0$, set $A^{(i)} \in U_{\widehat{N(L/E_i)}}$ to be equal to 1.
If $n_i > 0$, let $A^{(i)} \in U_{\widehat{N(L/E_i)}}$ be a lifting of α_i with the following restriction: $A^{(i)}_{\widehat{E_{i+1}}}$ is not a root of unity of order a power of p (this condition can always be satisfied, since the kernel of the norm map is uncountable).

Lemma ([F]). *If* $A^{(i)} \neq 1$, *then* $\beta_{i+1} = {A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i}}$ *belongs to* X_{i+1}.

Note that every β_{i+1} when it is defined doesn't belong to X_{i+1}^p. Indeed, otherwise we would have ${A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i}} = \gamma^p$ for some $\gamma \in X_{i+1}$ and then ${A^{(i)}_{\widehat{E_{i+1}}}}^{p^{n_i - 1}} = \gamma\zeta$ for a root ζ of order p or 1. Taking the norm down to $\widehat{E_i}$ we get $\alpha_i^{p^{n_i - 1}} = N_{\widehat{E_{i+1}}/\widehat{E_i}}\gamma = 1$, which contradicts the definition of α_i.

Definition. Let $\beta_{i,j}$, $j \geqslant 1$ be free topological generators of Y_i which include β_i whenever β_i is defined. Let $B^{(i,j)} \in U_{\widehat{N(L/E_i)}}$ be a lifting of $\beta_{i,j}$ (i.e. ${B^{(i,j)}}_{\widehat{E_i}} = \beta_{i,j}$), such that if $\beta_{i,j} = \beta_i$, then $B^{(i,j)}_{\widehat{E_k}} = B^{(i)}_{\widehat{E_k}} = {A^{(i-1)}_{\widehat{E_k}}}^{p^{n_i - 1}}$ for $k \geqslant i$.

Define a map $X_i \to U_{\widehat{N(L/E_i)}}$ by sending a convergent product $\alpha_i^c \prod_j \beta_{i,j}^{c_j}$, where $0 \leqslant c \leqslant n_i - 1$, $c_j \in \mathbb{Z}_p$, to ${A^{(i)}}^c \prod_j {B^{(i,j)}}^{c_j}$ (the latter converges). Hence we get a map

$$f_i \colon U_{\widehat{E_i}}^{\sigma_i - 1} \to U_{\widehat{N(L/E_i)}} \to U_{\widehat{N(L/F)}}$$

which depends on the choice of lifting. Note that $f_i(\alpha)_{\widehat{E_i}} = \alpha$.

Denote by Z_i the image of f_i. Let

$$Z_{L/F} = Z_{L/F}(\{E_i, f_i\}) = \left\{ \prod_i z^{(i)} : z^{(i)} \in Z_i \right\},$$

$$Y_{L/F} = \{ y \in U_{\widehat{N(L/F)}} : y^{1-\varphi} \in Z_{L/F} \}.$$

Lemma. *The product of* $z^{(i)}$ *in the definition of* $Z_{L/F}$ *converges.* $Z_{L/F}$ *is a subgroup of* $U^{\diamond}_{\widehat{N(L/F)}}$. *The subgroup* $Y_{L/F}$ *contains* $U_{N(L/F)}$.

Theorem ([F]). *For every* $(u_{\widehat{E_i}}) \in U^{\diamond}_{\widehat{N(L/F)}}$ *there is a unique automorphism* τ *in the group* $\mathrm{Gal}(L/F)$ *satisfying*

$$(u_{\widehat{E_i}})^{1-\varphi} \equiv \Pi^{\tau-1} \mod Z_{L/F}.$$

If $(u_{\widehat{E_i}}) \in Y_{L/F}$, *then* $\tau = 1$.

Hint. Step by step, passing from $\widehat{E_i}$ to $\widehat{E_{i+1}}$. □

Remark. This theorem can be viewed as a non-commutative generalization for finite k of exact sequence $(*)$ of 16.2.

Corollary. *Thus, there is map*

$$\mathcal{H}_{L/F}: U^{\diamond}_{\widehat{N(L/F)}} \to \mathrm{Gal}(L/F), \quad \mathcal{H}_{L/F}((u_{\widehat{E_i}})) = \tau.$$

The composite of $\mathcal{N}_{L/F}$ *and* $\mathcal{H}_{L/F}$ *is the identity map of* $\mathrm{Gal}(L/F)$.

9.4. Main Theorem

Theorem ([F]). *Put*

$$\mathcal{H}_{L/F}: U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F} \to \mathrm{Gal}(L/F), \quad \mathcal{H}_{L/F}((u_{\widehat{E}})) = \tau$$

where τ *is the unique automorphism satisfying* $(u_{\widehat{E}})^{1-\varphi} \equiv \Pi^{\tau-1} \mod Z_{L/F}$. *The injective map* $\mathcal{H}_{L/F}$ *is a bijection. The bijection*

$$\mathcal{N}_{L/F}: \mathrm{Gal}(L/F) \to U^{\diamond}_{\widehat{N(L/F)}}/Y_{L/F}$$

induced by $\mathcal{N}_{L/F}$ *defined in 9.2 is a 1-cocycle.*

Corollary. Denote by q the cardinality of the residue field of F. Koch and de Shalit [K], [KdS] constructed a sort of metabelian local class field theory which in particular describes totally ramified metabelian extensions of F (the commutator group of the commutator group is trivial) in terms of the group

$$\mathfrak{n}(F) = \left\{(u \in U_F, \xi(X) \in \mathbb{F}_p^{\mathrm{sep}}[[X]]^*) : \xi(X)^{\varphi-1} = \{u\}(X)/X\right\}$$

with a certain group structure. Here $\{u\}(X)$ is the residue series in $\mathbb{F}_p^{\mathrm{sep}}[[X]]^*$ of the endomorphism $[u](X) \in O_F[[X]]$ of the formal Lubin–Tate group corresponding to π_F, q, u.

Let M/F be the maximal totally ramified metabelian subextension of F_φ, then M/F is arithmetically profinite. Let R/F be the maximal abelian subextension of M/F. Every coset of $U^{\diamond}_{\widehat{N(M/F)}}$ modulo $Y_{M/F}$ has a unique representative in

$\mathrm{im}(\mathcal{N}_{M/F})$. Send a coset with a representative $(u_{\widehat{Q}}) \in U^{\diamond}_{\widehat{N(M/F)}}$ ($F \subset Q \subset M$, $|Q : F| < \infty$) satisfying $(u_{\widehat{Q}})^{1-\varphi} = (\pi_Q)^{\tau-1}$ with $\tau \in \mathrm{Gal}(M/F)$ to

$$\left(u_{\widehat{F}}^{-1}, (u_{\widehat{E}}) \in U^{\diamond}_{\widehat{N(R/F)}}\right) \quad (F \subset E \subset R, |E : F| < \infty).$$

It belongs to $\mathfrak{n}(F)$, so we get a map

$$g\colon U^{\diamond}_{\widehat{N(M/F)}}/Y_{M/F} \to \mathfrak{n}(F).$$

This map is a bijection [F] which makes Koch–de Shalit's theory a corollary of the main results of this section.

References

[F] I. Fesenko, Nonabelian local reciprocity maps, to appear in Class Field Theory: Its Centenary and Prospect, ed. K. Miyake, Advanced Studies in Pure Mathematics, Math. Soc. Japan, Tokyo 2001.

[FV] I. Fesenko and S. Vostokov, Local Fields and Their Extensions, AMS, Providence, R.I., 1993.

[K] H. Koch, Local class field theory for metabelian extensions, In Proceed. 2nd Gauss Symposium. Conf. A: Mathematics and Theor. Physics (Munich, 1993), de Gruyter, Berlin, 1995, 287–300.

[KdS] H. Koch and E. de Shalit, Metabelian local class field theory, J. reine angew. Math. 478(1996), 85–106.

[W] J.-P. Wintenberger, Le corps des normes de certaines extensions infinies des corps locaux; applications, Ann. Sci. E.N.S., 4 série 16(1983), 59–89.

Department of Mathematics University of Nottingham
Nottingham NG7 2RD England
E-mail: ibf@maths.nott.ac.uk