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At most 27 length inequalities define Maskit's fundamental domain for the modular group in genus 2

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Abstract

In recently published work Maskit constructs a fundamental domain D_g for the Teichmüller modular group of a closed surface S of genus $g \geq 2$. Maskit's technique is to demand that a certain set of $2g$ non-dividing geodesics C_{2g} on S satisfies certain shortness criteria. This gives an a priori infinite set of length inequalities that the geodesics in C_{2g} must satisfy. Maskit shows that this set of inequalities is finite and that for genus $g = 2$ there are at most 45. In this paper we improve this number to 27. Each of these inequalities: compares distances between Weierstrass points in the fundamental domain $S \setminus C_4$ for S; and is realised (as an equality) on one or other of two special surfaces.

AMS Classification 57M50; 14H55, 30F60

Keywords Fundamental domain, non-dividing geodesic, Teichmüller modular group, hyperelliptic involution, Weierstrass point

0 Introduction and preliminaries

In this paper we consider a fundamental domain defined by Maskit in [8] for the action of the Teichmüller modular group on the Teichmüller space of a closed surface of genus $g \geq 2$ in the special case of genus $g = 2$. McCarthy and Papadopoulos [9] have also defined such a fundamental domain, modelled on a Dirichlet region; for punctured surfaces there is the celebrated cell decomposition and associated fundamental domain due to Penner [10]. For genus $g = 2$ Semmler [11] has defined a fundamental domain based on locating the shortest dividing geodesic. Also for low signature surfaces the reader is referred to the papers of Keen [3] and of Maskit [7], [8].

Throughout S will denote a closed orientable surface of genus $g = 2$, with some fixed hyperbolic metric. We say that a simple closed geodesic γ on S

is: *dividing* if $S \setminus \gamma$ has two components; or *non-dividing* if $S \setminus \gamma$ has one component. By *non-dividing geodesic* we shall always mean simple closed non-dividing geodesic. We denote the length of γ with respect to the hyperbolic metric on S by $l(\gamma)$. Let $|\alpha \cap \beta|$ denote the number of intersection points of two distinct geodesics α, β.

We define a *chain* $C_n = \{\gamma_1, \ldots, \gamma_n\}$ to be an ordered set of non-dividing geodesics such that: $|\gamma_i \cap \gamma_{i+1}| = 1$ for $1 \leq i \leq n-1$ and $\gamma_i \cap \gamma_{i'} = \emptyset$ otherwise. We say that a chain C_n has *length* n, where $1 \leq n \leq 5$. Likewise we define a *bracelet* $B_n = \{\gamma_1, \ldots, \gamma_n\}$ to be an ordered set of non-dividing geodesics such that: $|\gamma_i \cap \gamma_{i+1}| = 1$ for $1 \leq i \leq n-1$; $|\gamma_n \cap \gamma_1| = 1$ and $\gamma_i \cap \gamma_{i'} = \emptyset$ otherwise. Again we say that B_n has *length* n, where $3 \leq n \leq 6$. Following Maskit, we call a bracelet of length 6 a *necklace*.

For $n \leq 4$ a chain of length n can be always be extended to a chain of length $n+1$. For $n = 4$ this extension is unique. Likewise a chain of length 5 extends uniquely to a necklace. So chains of length 4 or 5 and necklaces can be considered equivalent. We shall usually work with length 4 chains, which we call *standard*. (Maskit, for genus g, usually works with chains of length 2g+1, which he calls standard.)

As Maskit shows in [8] each surface, standard chain pair $\{S, C_4\}$ gives a canonical choice of generators for the Fuchsian group F such that $\mathbb{H}^2/F = S$ and hence a point in $DF(\pi_1(S), PSL(2,\mathbb{R}))$, the set of discrete faithful representations of $\pi_1(S)$ into $PSL(2,\mathbb{R})$. Essentially this representation corresponds to the fundamental domain $S \setminus C_4$ together with orientations for its side pairing elements. As Maskit observes, it is well known that $DF(\pi_1(S), PSL(2,\mathbb{R}))$ is real analytically equivalent to Teichmüller space. So, we define the *Teichmüller space* of closed orientable genus $g = 2$ surfaces T_2 to be the set of pairs $\{S, C_4\}$.

We say that a standard chain $C_4 = \{\gamma_1, \ldots, \gamma_4\}$ is *minimal* if for any chain $C'_m = \{\gamma_1, \ldots, \gamma_{m-1}, \alpha_m\}$ we have $l(\gamma_m) \leq l(\alpha_m)$ for $1 \leq m \leq 4$. We then define the *Maskit domain* $D_2 \subset T_2$ to be the set of surface, standard chain pairs $\{S, C_4\}$ with C_4 minimal.

For C_4 to be minimal the geodesics $\gamma_1, \ldots, \gamma_4$ must satisfy an a priori infinite set of length inequalities. For genus g, Maskit gives an algorithm using cut-and-paste to show that only a finite number N_g of length inequalities need to be satisfied. Applying his algorithm to genus $g = 2$, Maskit showed that $N_2 \leq 45$. We establish an independent proof that $N_2 \leq 27$. We could have shown that 18 of Maskit's 45 inequalities follow from the other 27. However, by tayloring all our techniques to the special case of genus 2, we are able to produce a much shorter proof.

The fact that 18 of Maskit's 45 inequalities follow from the other 27 follows from applications of Theorem 2.2 (which appeared as Theorem 1.1 in [4]) and of Corollary 2.5. The latter follows immediately from Theorem 2.4, for which we give a proof in this paper. This is a characterisation of the octahedral surface Oct (the well known genus two surface of maximal symmetry group) in terms of a finite set of length inequalities.

The 27 length inequalities have the properties that: each is realised on one or other of two special surfaces (for all but 2 this special surface is Oct); and each compares distances between Weierstrass points in the fundamental domain $S \setminus C_4$ for S.

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1 The hyperelliptic involution and the main result

It is well known that every closed genus two surface without boundary S admits a uniquely determined *hyperelliptic involution*, an isometry of order two with six fixed points, which we denote by J. The fixed points of J are known as *Weierstrass points*. Every simple closed geodesic $\gamma \subset S$ is setwise fixed by J, and the restriction of J to γ has no fixed points if γ is dividing and two fixed points if γ is non-dividing (see Haas–Susskind [2]). So every non-dividing geodesic on S passes through two Weierstrass points. It is a simple consequence that sequential geodesics in a chain intersect at Weierstrass points. We say that two non-dividing geodesics α, β *cross* if $\alpha \neq \beta$ and $\alpha \cap \beta$ contains a point that is not a Weierstrass point.

The quotient *orbifold* $O = S/J$ is a sphere with six order two cone points, endowed with a fixed hyperbolic metric. Each cone point on O is the image of a Weierstrass point under the projection $J : S \to O$ and each non-dividing geodesic on S projects to a simple geodesic between distinct cone points on O – what we shall call an *arc*. Definitions of chains, bracelets and crossing all pass naturally to the quotient.

Let C_4 be a standard chain on S, which extends to a necklace N. We number Weierstrass points on N so that $\omega_i = \gamma_{i-1} \cap \gamma_i$ for $2 \leq i \leq 6$ and $\omega_1 = \gamma_6 \cap \gamma_1$.

Choose an orientation upon S and project to the quotient orbifold $O = S/J$ – for the rest of the paper we shall work on the quotient orbifold O. We label the components of $O \setminus N$ by $H, \overline{H}$ so that $\gamma_1, \ldots, \gamma_6$ lie anticlockwise around H. Label by $\alpha^{i_1,i_2,\ldots,i_n}_{j,k}$ (respectively $\overline{\alpha^{i_1,i_2,\ldots,i_n}_{j,k}}$) the arc between the cone points ω_j, ω_k $(j < k)$ crossing the sequence of arcs $\gamma_{i_1}, \gamma_{i_2}, \ldots, \gamma_{i_n}$ and having the subarc between ω_j, γ_{i_1} lying in H (respectively $\overline{H}$).

Our main result is then the following. (We abuse notation so that $\alpha_{1,6} = \overline{\alpha_{1,6}} = \gamma_6$ and $\alpha_{2,3} = \overline{\alpha_{2,3}} = \gamma_2$. We then have repetitions, $l(\gamma_2) \leq l(\gamma_6)$ twice, and redundancies, $l(\gamma_2) \leq l(\gamma_2)$ also twice.)

Theorem 1.1 *The standard chain C_4 is minimal if the following are satisfied:*

(1) $l(\gamma_1) \leq l(\gamma_i), i \in \{2,3,4,5\}$

(2) $l(\gamma_2) \leq l(\alpha_{i,j}), l(\overline{\alpha_{i,j}}), l(\alpha^6_{2,5}), l(\overline{\alpha^6_{2,5}}),\ i \in \{1,2\},\ j \in \{3,4,5,6\}$

(3) $l(\gamma_3) \leq l(\alpha_{3,j}), l(\overline{\alpha_{3,j}}), l(\alpha^6_{3,4}), l(\overline{\alpha^6_{3,4}}),\ j \in \{5,6\}$

(4) $l(\gamma_4) \leq l(\alpha_{4,6}), l(\overline{\alpha_{4,6}})$.

Each length $l(\gamma_i)$ or $l(\alpha_{j,k})$ (respectively $l(\overline{\alpha_{j,k}})$) is a distance between cone points in H (respectively $\overline{H}$). Likewise each length $l(\alpha^6_{j,k})$, $l(\overline{\alpha^6_{j,k}})$ is a distance between cone points in $O \setminus C_5$. So each length inequality in Theorem 1.1 compares distances between cone points in $O \setminus C_5$ (and hence distances between Weierstrass points in $S \setminus C_4$).

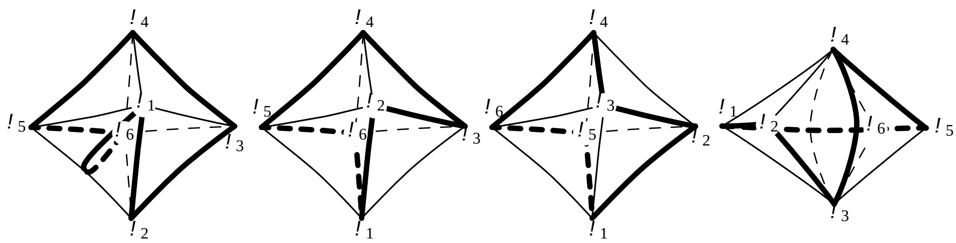

Figure 1: How the length inequalities in Theorem 1.1 are realized on Oct and E

Theorem 1.1 gives a sufficient list of inequalities. As to the necessity each inequality, we make the following observation. Each inequality is realised (as an equality) on either Oct or E – cf Theorem 1.1 in [5]. The octahedral orbifold Oct is the well known orbifold of maximal conformal symmetry group. Any minimal standard chain on Oct lies in its set of shortest arcs. This arc set has the combinatorial edge pattern of the Platonic solid. The exceptional orbifold E, which was constructed in [5], has conformal symmetry group $\mathbb{Z}_2 \times \mathbb{Z}_2$. However it is not de ned by the action of its symmetry group alone, it also requires a certain length inequality to be satis ed. Any minimal standard chain on E lies in its set of shortest and second shortest arcs.

In Figure 1 we have illustrated necklaces on Oct and E that are the extentions of minimal standard chains. As with other gures in this paper, we use wire frame diagrams to illustrate the orbifolds. Solid (respectively dashed) lines represent arcs in front (respectively behind) the gure. Thick lines represent arcs in the necklace N. The minimal standard chain on E in Figure 1 has: $l(y_1) = l(y_5)$; $l(y_2) = l(\overline{\ _{1;3}}) = l(\ _{1;4}) = l(\ _{2;4})$; $l(y_3) = l(\overline{\ ^{6}_{3;4}}) = l(\overline{\ _{3;5}}) = l(\overline{\ _{3;6}})$; $l(y_4) = l(\ _{4;6})$. Making such a list for all the orbifolds in Figure 1, together with their mirror images, we see that all the inequalities in Therem 1.1 are realised as equalities on either Oct or E.

2 Length inequalities for systems of arcs

In order to prove Theorem 1.1 we need a number of length inequality results for systems of arcs. Let $K_4 = \ _{0;1}; : : : ;\ _{3;0}$ denote a length 4 bracelet such that each component of $O\ n\ K_4$ contains an interior cone point. Using mod 4 addition throughout, label cone points: on K_4 by $c_k = \ _{k-1;k} \setminus \ _{k;k+1}$ for $k\ 2\ f0; : : : ;3g$; and o K_4 by c_l for $l\ 2\ f4;5g$. Label by O_l the component of $O\ n$ containing c_l and label arcs in O_l so that $\ _{k;l}$ is between $c_k;c_l$. Let $\ _k$ denote the arc between $c_4;c_5$ crossing only $\ _{k;k+1}$ K_4.

The following two results appeared as Lemma 2.3 in [5] (in Maskit's terminology this is a cut-and-paste) and as Theorem 1.1 in [4] respectively.

Lemma 2.1 (i) $2l(\ _{0;4}) < l(\ _0) + l(\ _3)$ (ii) $2l(\ _{3;0}) < l(\ _0) + l(\ _2)$.

Theorem 2.2 *If* $l(\ _{3;4})\ \ l(\ _{0;4})$, $l(\ _{3;5})\ \ l(\ _{0;5})$, $l(\ _0)\ \ l(\ _2)$ *then* $l(\ _{3;4}) = l(\ _{0;4})$, $l(\ _{3;5}) = l(\ _{0;5})$, $l(\ _0) = l(\ _2)$.

Corollary 2.3 *If* $l(\ _{3;4})\ \ l(\ _{0;4})$, $l(\ _{3;5})\ \ l(\ _{0;5})$, $l(\ _{1;4})\ \ l(\ _{2;4})$ *then* $l(\ _{1;5})\ \ l(\ _{2;5}):$

Proof of Corollary 2.3 Since $l(\ _{3;4})\ \ l(\ _{0;4})$, $l(\ _{3;5})\ \ l(\ _{0;5})$ Theorem 2.2 implies that $l(\ _0)\ \ l(\ _2)$. Moreover $l(\ _{1;4})\ \ l(\ _{2;4})$ and so again, by Theorem 2.2, $l(\ _{1;5})\ \ l(\ _{2;5})$. □

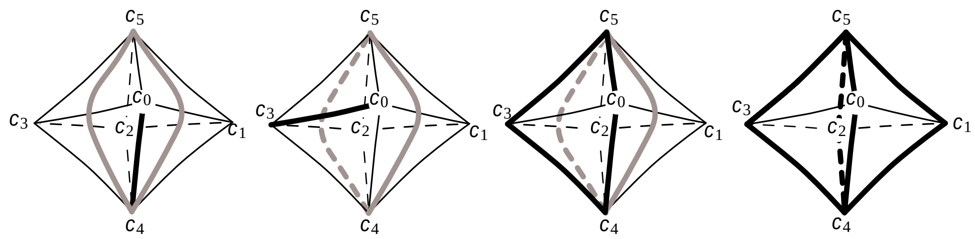

Figure 2: Arc sets for Lemma 2.1, for Theorem 2.2 and for Corollary 2.3

Theorem 2.4 *Suppose* $l(\ _{2;3})\quad l(\ _{2;l})$, $l(\ _{3;0})\quad l(\ _{1;2})\quad fl(\ _{0;l}); l(\ _{1;l})g$ *and* $l(\ _{0;1})\quad fl(\ _{0;l}); l(\ _{3;l})g$ *then* $l(\ _{k;l}) = l(\ _{k;k+1})$ *for each* $k; l$ *and* O *is the octahedral orbifold.*

Proof of Theorem 2.4 We postpone this until Section 3. □

Corollary 2.5 *Suppose* $l(\ _{2;3})\quad l(\ _{2;l})$, $l(\ _{1;2})\quad fl(\ _{0;l}); l(\ _{1;l})g$ *and* $l(\ _{0;1})\quad fl(\ _{0;l}); l(\ _{3;l})g$ *then* $l(\ _{3;0})\quad l(\ _{1;2})$.

Proof of Corollary 2.5 If $l(\ _{3;0})\quad l(\ _{1;2})$ then by Theorem 2.4 $l(\ _{k;l}) = l(\ _{k;k+1})$ for each $k; l$. In particular $l(\ _{3;0}) = l(\ _{1;2})$. So $l(\ _{3;0})\quad l(\ _{1;2})$. □

3 The proofs

Proof of Theorem 1.1 Let $\ _m$ denote an arc such that $C^0_m = y_1; \dots ; y_{m-1};\ _m$ is a chain, for $1\quad m\quad 4;\ _m \not= y_m$. We will show that $l(y_m)\quad l(\ _m)$ for arcs of the form $^{i_1;i_2;\dots;i_n}_{j;k}$. The same arguments work for arcs of the form $\overline{^{i_1;i_2;\dots;i_n}_{j;k}}$. Let $X(\ ;\ )$ denote the number of crossing points of a distinct pair of arcs $\ ;\ $ { ie the number of intersection points of $\ ;\ $ that are not cone points. Let $n = 1$, if $X(y_m;\ _i) = 0$ for $i\ 2\ f1; \dots ; 6g$; otherwise, let $n = \min i\ 2\ f1; \dots ; 6g$ such that $X(y_n;\ _i) > 0$. We note that $n\quad m$.

Let $P_{m;n;p}$ be the proposition that $l(y_m)\quad l(\ _m)$ for $X(\ _m; y_n) = p$. Clearly, if $n = 1$ then $p = 0$. For $n\ 2\ f5; 6g$ it is not hard to show that $p = 1$. For $n\ 2\ f1; \dots ; 4g$ we consider $p = 1$ and $p > 1$. We order the propositions as follows: $P_{4;1;0}; \dots ; P_{1;1;0}$ which is followed by $P_{4;6;1}; P_{4;5;1}; \dots ; P_{1;6;1}; P_{1;5;1}$ followed by $P_{4;4;1}; P_{4;4;p>1}$ which is followed by $P_{3;4;1}; P_{3;4;p>1}; P_{3;3;1}; P_{3;3;p>1}$ followed by $P_{2;4;1}; P_{2;4;p>1}; \dots ; P_{2;2;1}; P_{2;2;p>1}$ followed by $P_{1;4;1}; P_{1;4;p>1}; \dots ; P_{1;1;1}; P_{1;1;p>1}$.

Suppose $n = 1;\ _m$ does not cross N. If $m > 1$ then $P_{m;1;0}$ is a hypothesis. If $m = 1$ then either $P_{1;1;0}$ is a hypothesis, $\ _1 = y_i$ for some $i\ 2\ f2; \dots ; 5g$, or $P_{1;1;0}$ follows from the hypotheses, $l(y_1) \quad l(y_i); l(y_i) \quad l(\ _1)$ for some $i\ 2\ f2;3;4g$.

Suppose $n\ 2\ f5;6g;\ \ _m$ crosses N but does not cross C_4.

For $m = 4$, by inspection, $\ _4 = \ ^6_{4;5}$. So $\ _m; y_m$ share endpoints, $n > m+1$ and we can apply the argument (i) below. So we have $P_{4;n;1}$ for $n\ 2\ f5;6g$.

In Figures 3,4,5 we illustrate applications of length inequalities results to the proof. As above we use wire frame gures of the octahedral orbifold, with the necklace N in thick black. Other arcs are in thick grey. Figures have been drawn so arcs in the application correspond to arcs in the length inequality result.

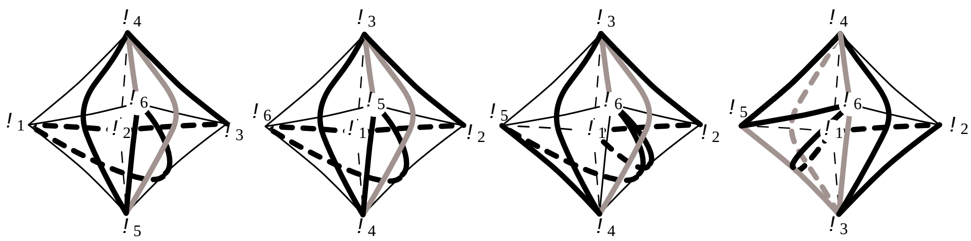

Figure 3: Application (i) for $\ _4 = \ ^6_{4;5};\ \ _3 = \ ^5_{3;4}$ and $\ ^{6;5}_{3;4}$ and of Theorem 2.2, (ii) for $\ _3 = \ ^6_{3;5}$

For $m = 3$. By inspection, $\ _3$ is one of $\ ^5_{3;4};\ ^6_{3;4};\ ^{6;5}_{3;4};\ ^6_{3;5}$. For $\ ^5_{3;4};\ ^{6;5}_{3;4};\ ^6_{3;4}$: $y_m;\ _m$ share endpoints, $n > m+1$ and so we can apply either argument (i) or (ii) below. For $\ ^6_{3;5}$ we can apply Theorem 2.2 in conjunction with argument (ii): by hypothesis $l(y_4) \quad l(\ _{4;6})$ and by argument (ii) $l(y_3) \quad l(\ ^6_{3;4})$ and so $l(\ ^6_{3;5}) \quad l(\ _{3;6})$. Again by hypothesis $l(y_3) \quad l(\ _{3;6})$ and so $l(y_3) \quad l(\ _{3;6}) \quad l(\ ^6_{3;5})$. This gives $P_{3;n;1}$ for $n\ 2\ f5;6g$.

For $m = 2;\ \ _2$ is one of $\ ^5_{2;3};\ ^6_{2;3};\ ^{6;5}_{2;3}$ or one of $\ ^5_{2;4};\ ^6_{2;4};\ ^{6;5}_{2;4};\ ^6_{2;5};\ ^5_{1;3};\ ^5_{1;4}$. By hypothesis $l(y_2) \quad l(\ ^6_{2;5})$. For $\ ^6_{2;3};\ ^{6;5}_{2;3};\ ^5_{2;3}$ we can again apply either argument (i) or (ii). For $\ ^5_{2;4};\ ^6_{2;4};\ ^{6;5}_{2;4};\ ^5_{1;3}$ we apply Theorem 2.2 in conjunction with argument (ii). We give the argument for $\ ^5_{2;4}$. By argument (ii), we have $l(y_2) < l(\ ^5_{2;3})$. Also, by hypothesis, $l(y_3) \quad l(\ _{3;5})$ and so by Theorem 2.2 $l(\ _{2;5}) < l(\ ^5_{2;4})$. Again, by hypothesis, $l(y_2) \quad l(\ _{2;5})$ and so $l(y_2) \quad l(\ _{2;5}) < l(\ ^5_{2;4})$.

For $\ _2 = \ ^5_{1;4}$ we argue as follows. By hypothesis we have $l(y_3) \quad l(\ _{3;5}); l(\overline{\ _{3;6}})$ and $l(y_2) \quad l(\ _{1;5}); l(y_6); l(\ _{2;5}); l(\overline{\ _{2;6}})$ and $l(y_1) \quad l(\ _{1;5}); l(y_6); l(y_4); l(\overline{\ _{4;6}})$. By Corollary 2.5: $l(\ ^5_{1;4}) \quad l(y_2)$. Hence $P_{2;n;1}$ for $n\ 2\ f5;6g$.

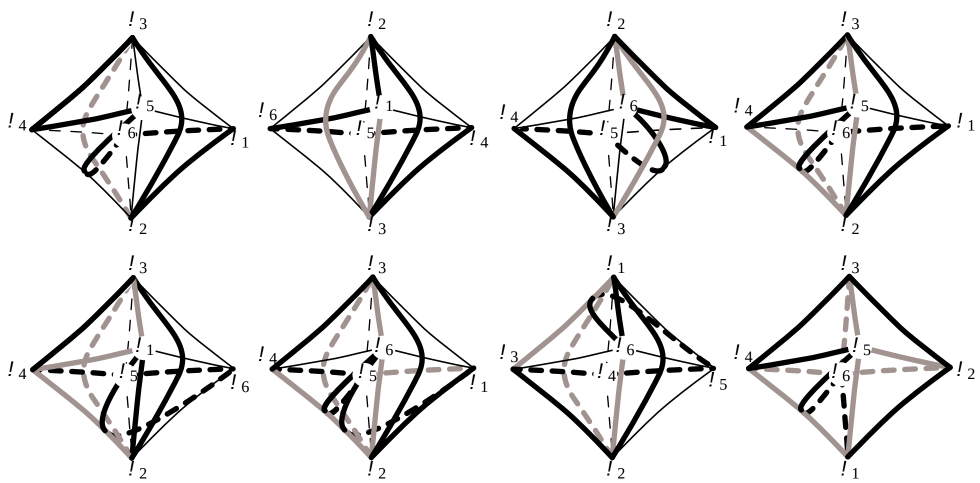

Figure 4: Applications of (i) or (ii) for ${}_2 = {}^{5}_{2,3}, {}^{6}_{2,3}$ and ${}^{6,5}_{2,3}$; of Theorem 2.2, (ii) for ${}_2 = {}^{5}_{2,4}, {}^{6}_{2,4}, {}^{6,5}_{2,4}$ and ${}^{5}_{1,3}$; and of Corollary 2.5 for ${}_2 = {}^{5}_{1,4}$

For $m = 1$. If $\{j,k\} \neq \{1,2\}$ or $\{j,k\} \neq \{5,6\}$ then $l(\gamma_1) \quad l(\gamma_i), l(\gamma_i) \quad l({}_1)$ are hypotheses, or preceding propositions, for some $i \in \{2,3,4\}$. If $\{j,k\} = \{1,2\}$ then, by inspection, ${}_1 = {}^{5}_{1,2}$ we can again apply argument (i). By inspection there is no such ${}_1$ for $\{j,k\} = \{5,6\}$. This completes $P_{m,n,1}$ for $n \in \{5,6\}$.

We now give the arguments for: ${}_m, \gamma_m$ share endpoints and $n > m+1$. The arc set $\Gamma := {}_m \cup \gamma_m$ divides O into two components. Either: (i) Γ divides one cone point (c) from three; or (ii) Γ divides two cone points from two. For (i) we let O_c, O'_c denote the components of $O \setminus \Gamma$ so that $c \in O_c$ and we let ${}'_m$ (respectively ${}''_m$) denote the arc between ω_m, c (respectively between ω_{m+1}, c) in O_c.

First $m = 4$, (i), $n = 6$. None of $\gamma_1, \gamma_2, \gamma_3$ crosses $\Gamma = {}_4 \cup \gamma_4$, so $C_3 = \gamma_1, \gamma_2, \gamma_3$ lies in one or other component of $O \setminus \Gamma$. Now C_3 contains three cone points disjoint from Γ, so $C_3 \quad O'_c$. So $c = \omega_6$ and $C'_4 = \gamma_1, \gamma_2, \gamma_3, {}'_4$ is a chain. We observe – see Figure 3 – that ${}'_4 = {}_{4,6}$ and hence $l(\gamma_4) \quad l({}'_4)$ is a hypothesis. By Lemma 2.1(i): $2l({}'_4) < l(\gamma_4) + l({}_4)$ and so $l(\gamma_4) \quad l({}'_4) < l({}_4)$.

Second $m = 3$, (i), $n \in \{5,6\}$. Neither γ_1 nor γ_2 crosses $\Gamma = {}_3 \cup \gamma_3$, so $C_2 = \gamma_1, \gamma_2$ lies in one or other component of $O \setminus \Gamma$. Now C_2 contains two cone points disjoint from Γ, so $C_2 \quad O'_c$; $c = \omega_5$ or ω_6 and $C'_3 = \gamma_1, \gamma_2, {}'_3$ is a chain. We observe – see Figure 3 – that ${}'_3 = {}_{3,5}$ or ${}'_3 = {}_{3,6}$ and hence $l(\gamma_3) \quad l({}'_3)$ is hypothesis. Again, by Lemma 2.1(i): $2l({}'_3) < l(\gamma_3) + l({}_3)$ and so $l(y_3)\quad l({}^{0}_{3}) < l({}_{3})$. For (ii) we have that ${}_{3} = {}^{6}_{3;4}$ and $l(y_3)\quad l({}^{6}_{3;4})$ is a hypothesis.

Next $m = 2$, (i), $n\ 2\ f4;5;6g$. The arc y_1 does not cross $\Gamma = {}_{2}\ [\ y_2$, so $y_1\quad O^{0}_{c}$ and $c\ 2\ f\,!_4;\,!_5;\,!_6g$ (respectively $y_1\quad O_c$ and $c = !_1$). For $n\ 2\ f5;6g$ { see Figure 4 { we have that ${}^{0}_{2} = {}_{2;6}$ (respectively ${}^{00}_{2} = {}_{1;3}$). For $n = 4$ { see Figure 5 { we have that ${}^{0}_{2} = {}_{2;4}$ or ${}_{2;5}$ (respectively there is no such ${}_{2}$). So $l(y_2)\quad l({}^{0}_{2})$ (respectively $l(y_2)\quad l({}^{00}_{2})$) is a hypothesis. By Lemma 2.1(i): $2l({}^{0}_{2})$ or $2l({}^{00}_{2}) < l(y_2) + l({}_{2})$ and so $l(y_2)\quad l({}^{0}_{2}) < l({}_{2})$ (respectively $l(y_2)\quad l({}^{00}_{2}) < l({}_{2})$).

For (ii), again, y_1 lies in one component of $O\,n\,\Gamma$. Let ${}^{000}_{2}$ denote the unique arc disjoint from Γ in this component of $O\,n\,\Gamma$. For $n\ 2\ f5;6g$ { again see Figure 4{ we have that ${}^{000}_{2} = y_6$. For $n = 4$ { again see Figure 5 { we have ${}^{000}_{2} = {}_{1;4}$ or ${}_{1;5}$. So $l(y_2)\quad l({}^{000}_{2})$ is a hypothesis. By Lemma 2.1(ii): $2l({}^{000}_{2}) < l(y_2) + l({}_{2})$ and so $l(y_2)\quad l({}^{000}_{2}) < l({}_{2})$.

Finally, $m = 1$, (i), $n\ 2\ f3;:::;6g$. For $n\ 2\ f5;6g$: ${}^{0}_{1} = \overline{{}_{2;6}}$ and $l(y_2)\quad l({}^{0}_{1})$ is a hypothesis. For $n\ 2\ f3;4g$: $l(y_2)\quad l({}^{0}_{1})$ is a proceeding proposition. Since $l(y_1)\quad l(y_2)$ is a hypothesis, we have that $l(y_1)\quad l(y_2)\quad l({}^{0}_{1})$. By Lemma 2.1(i): $2l({}^{0}_{1}) < l(y_1) + l({}_{1})$ and so $l(y_1)\quad l({}^{0}_{1}) < l({}_{1})$.

For (ii), $n\ 2\ f5;6g$, there is no such ${}_{1}$. For $n\ 2\ f3;4g$, we let ${}^{0}_{3}$ denote the unique arc disjoint from Γ in the same component of $O\,n\,\Gamma$ as y_2. Here $C^{0}_{3} = y_1;y_2;\ {}^{0}_{3}$ is a chain and so $l(y_3)\quad l({}^{0}_{3})$ is a proceeding proposition. Since $l(y_1)\quad l(y_3)$ is a hypothesis, we have that $l(y_1)\quad l(y_3)\quad l({}^{0}_{3})$. By Lemma 2.1(ii): $2l({}^{0}_{3}) < l(y_1) + l({}_{1})$ and so $l(y_1)\quad l({}^{0}_{3}) < l({}_{1})$.

Now suppose $n\ 2\ f1;:::;4g;\ {}_{m}$ crosses C_4.

Lemma 3.1 *Suppose that either $X({}_{m};y_n) > 1$ or ${}_{m};y_n$ share an endpoint. Then there exist arcs ${}^{0}_{m};y^{0}_{n}$ between the same respective endpoints as ${}_{m};y_n$ such that $l({}^{0}_{m}) < l({}_{m})$ or $l(y^{0}_{n}) < l(y_n)$; $X({}^{0}_{m};y_n);X(y^{0}_{n};y_n) < X({}_{m};y_n)$; and $X({}^{0}_{m};y_i) = X(y^{0}_{n};y_i) = 0$ for $i\quad n-1$. In particular $C^{0}_{m} = y_1;:::;y_{m-1};\ {}^{0}_{m};C^{00}_{n} = y_1;:::;y_{n-1};y^{0}_{n}$ are both chains.*

Proof This result is essentially Proposition 3.1 in [5], with additional observations upon the number of crossing points. However, upon going through the proof, these observations become clear. □

The following argument gives $P_{m;n;p>1}$: it uses induction on p, the rst induction step being the set of propositions that precede $P_{m;n;p>1}$.

Let $X(\ _m; y_n) = p > 1$ and so by Lemma 3.1 there exist arcs $\ ^0_m; y^0_n$ as stated. Let $p^0 = X(\ ^0_m; y_n) < p;\ p^{00} = X(y^0_m; y_n) < p$. We note that $l(y_m) \quad l(\ ^0_m)$ is either: $P_{m;n;p^0>1}$ if $p^0 > 1$; or a preceding proposition if $p^0 \quad 1$. Likewise, $l(y_n) \quad l(y^0_n)$ is either: $P_{m;n;p^{00}>1}$ if $n = m$ and $p^{00} > 1$; or a preceding proposition if $n > m$ or $p^{00} \quad 1$. Since $l(\ ^0_m) < l(\ _m)$ or $l(y^0_n) < l(y_n)$ it follows, by induction on p, that $l(y_m) \quad l(\ ^0_m) < l(\ _m)$.

So, for the rest of the proof, we may suppose that $X(\ _m; y_n) = 1$.

Lemma 3.2 *Suppose that* $\ _m; y_n$ *have distinct endpoints and that* $k > n + 1$. *Then there exist arcs* $\ ^0_m; y^0_n$ *between* $!_j; !_{n+1}$ *and* $!_n; !_k$ *such that* $l(\ ^0_m) < l(\ _m)$ *or* $l(y^0_n) < l(y_n)$ *and* $X(\ ^0_m; y_i) = X(y^0_n; y_i) = 0$ *for* $i \quad n$. *In particular* $C^0_m = y_1; \dots; y_{m-1};\ ^0_m; C^{00}_n = y_1; \dots; y_{n-1}; y^0_n$ *are both chains.*

Proof This is essentially Lemma 3.3 in [5], again with additional observations upon the number of crossing points. Again, these observations are clear. □

We now give two general arguments using these two lemmas.

Suppose: (1) $\ _m; y_n$ share an endpoint. Again we can apply Lemma 3.1: there exist arcs $\ ^0_m; y^0_n$ as stated. In particular $X(\ ^0_m; y_i) = X(y^0_n; y_i) = 0$ for $i \quad n$. So $l(y_m) \quad l(\ ^0_m);\ l(y_n) \quad l(y^0_n)$ are both preceding propositions. Since $l(\ ^0_m) < l(\ _m)$ or $l(y^0_n) < l(y_n)$, it follows that $l(y_m) \quad l(\ ^0_m) < l(\ _m)$.

Suppose: (2) $\ _m; y_n$ have distinct endpoints and $k > n + 1$. By Lemma 3.2 there exist arcs $\ ^0_m; y^0_n$ as stated. Again $l(y_m) \quad l(\ ^0_m); l(y_n) \quad l(y^0_n)$ are both preceding propositions. As $l(\ ^0_m) < l(\ _m)$ or $l(y^0_n) < l(y_n)$, we have that $l(y_m) \quad l(\ ^0_m) < l(\ _m)$.

For $m = 4 : j = 4; k\ 2\ f5; 6g$ and $n = 4 :\ _4; y_4$ share the endpoint $!_4$ (1).

For $m = 3 : j = 3; k\ 2\ f4; 5; 6g$. For $n = 4$ if $k\ 2\ f4; 5g$ then $\ _3; y_4$ share the endpoint $!_k$ (1); if $k = 6$ then $\ _3; y_4$ have distinct endpoints and $k > n + 1$ (2). For $n = 3 :\ _3; y_3$ share the endpoint $!_3$ (1).

For $m = 2 : j\ 2\ f1; 2g; k\ 2\ f3; \dots; 6g$. For $n = 4$ if $k = 3$ then, by inspection, $\ _2$ is one of $\ ^4_{2;3};\ ^{4;5;6}_{2;3};\ ^{5;4}_{2;3};\ ^{6;4}_{2;3};\ ^{6;5;4}_{2;3}$, and we can apply argument (i) or (ii), or is one of $\ ^4_{1;3};\ ^{4;5;6}_{1;3};\ ^{5;4}_{1;3}$, and we apply Theorem 2.2 in conjunction with argument (ii) { see Figure 5. If $k\ 2\ f4; 5g$ (1); if $k = 6$ (2). For $n = 3$ if $k\ 2\ f3; 4g$ (1); if $k\ 2\ f5; 6g$ (2). For $n = 2$ if $k = 3$ (1); if $k\ 2\ f4; 5; 6g$ (2).

Finally $m = 1$. Suppose $n = 4$. If $fj; kg\ 6=\ f1; 2g$ or $fj; kg\ 6=\ f5; 6g$ then $l(y_1) \quad l(y_i); l(y_i) \quad l(\ _1)$ are both preceding propositions for some

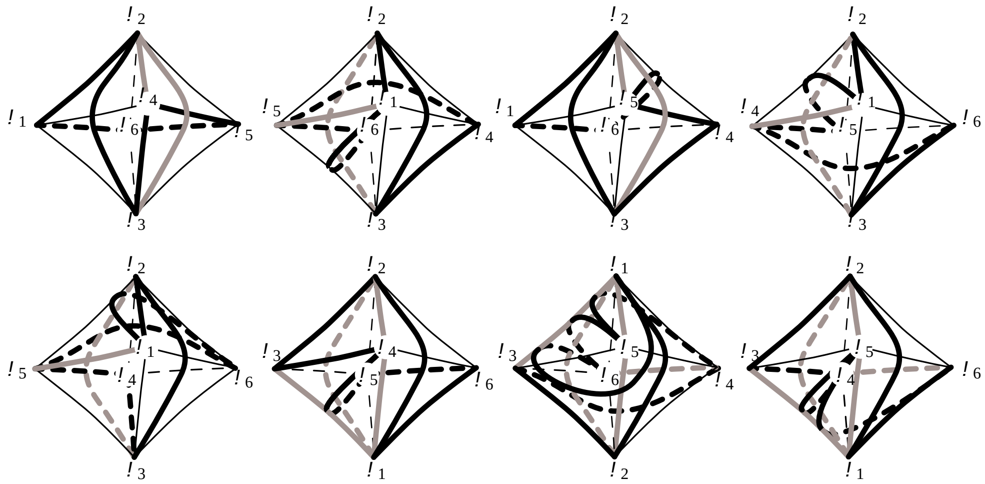

Figure 5: For $\ _2 = \ ^{4}_{2;3}, \ ^{4;5;6}_{2;3}, \ ^{5;4}_{2;3}, \ ^{6;4}_{2;3}$ and $\ ^{6;5;4}_{2;3}$ applications of (i) or (ii); and for $\ _2 = \ ^{4}_{1;3}, \ ^{4;5;6}_{1;3}$ and $\ ^{5;4}_{1;3}$ applications of Theorem 2.2, (ii)

$i \in \{2,3,4\}$. If $\{j,k\} = \{1,2\}$ we can apply (i) or (ii). There is no such $\ _1$ for $\{j,k\} = \{5,6\}$.

Now suppose $n = 3$. If $\{j,k\} \neq \{1,2\}$ or $\{j,k\} \not\subset \{4,5,6\}$ then $l(y_1) \ \ l(y_i); l(y_i) \ \ l(\ _1)$ are both preceding propositions for some $i \in \{2,3\}$. Again, if $\{j,k\} = \{1,2\}$ we can apply (i) or (ii). For $\{j,k\} \ \ \{4,5,6\}$ either $j = 4$ (1) or $j = 5$ (2).

Now suppose $n = 2$. If $\{j,k\} \neq \{1,2\}$ or $\{j,k\} \not\subset \{3,\dots,6\}$ (ie $j \in \{1,2\}, k \in \{3,\dots,6\}$) then $l(y_1) \ \ l(y_2); l(y_2) \ \ l(\ _1)$ are both preceding propositions. For $\{j,k\} = \{1,2\}$ (1). For $\{j,k\} \ \ \{3,\dots,6\}$ either $j = 3$ (1); or $j \in \{4,5,6\}$ (2).

Finally $n = 1$. Either j or $k \in \{1,2\}$ (1); or $\{j,k\} \ \ \{3,\dots,6\}$ (2). □

Proof of Theorem 2.4 As $l(\ _{3;0}) \ \ l(\ _{0;5}); l(\ _{2;3}) \ \ l(\ _{2;5}); l(\ _{0;1}) \ \ l(\ _{0;4})$, by Corollary 2.3, we have that $l(\ _{1;2}) \ \ l(\ _{2;4})$. Likewise, since $l(\ _{3;0}) \ \ l(\ _{0;4}); l(\ _{2;3}) \ \ l(\ _{2;4}); l(\ _{0;1}) \ \ l(\ _{0;5})$ we have that $l(\ _{1;2}) \ \ l(\ _{2;5})$. That is $l(\ _{1;2}) \ \ l(\ _{2;l})$.

The arc set K divides O into eight triangles. We label these as follows: let t_k (respectively T_k) denote the triangle with one edge $\ _{k;k+1}$ and one vertex c_4 (respectively c_5). We shall use $\angle c_l t_k$ to denote the angle at the c_l–vertex of t_k, et cetera. Cut O open along $\ _{3;0} \cup \ _{0;1} \cup \ _{1;4} \cup \ _{1;2} \cup \ _{1;5}$ to obtain a domain Ω.

We show that $l(\ _{2;3})\ \ l(\ _{2;l}); l(\ _{3;0})\ \ l(\ _{1;2})\ \ f l(\ _{0;l}); l(\ _{1;l}) g; l(\ _{0;1})$ $l(\ _{0;l})$ implies that $\min_l\ l(\ _{3;l})\ \ l(\ _{0;1})$ with equality *if and only if O* is the octahedral orbifold. First we show that: $\angle c_2 t_2\ \ \angle c_4 t_0$ *or* $\angle c_2 T_2\ \ \angle c_5 T_0$.

Now $l(\ _{1;2})\ \ l(\ _{1;l})$, $l(\ _{3;0})\ \ l(\ _{0;l}$ so $\angle c_2 t_1\ \ \angle c_4 t_1$, $\angle c_2 T_1\ \ \angle c_5 T_1$, $\angle c_3 t_3\ \ \angle c_4 t_3$, $\angle c_3 T_3\ \ \angle c_5 T_3$, which imply

$$\angle c_2 t_1 + \angle c_2 T_1 + \angle c_3 t_3 + \angle c_3 T_3\ \ \angle c_4 t_1 + \angle c_5 T_1 + \angle c_4 t_3 + \angle c_5 T_3$$
$$,\ (\ - \angle c_2 t_1 - \angle c_2 T_1) + (\ - \angle c_3 t_3 - \angle c_3 T_3)$$
$$(\ - \angle c_4 t_1 - \angle c_4 t_3) + (\ - \angle c_5 T_1 - \angle c_5 T_3)$$
$$,\ (\angle c_2 t_2 + \angle c_2 T_2) + (\angle c_3 t_2 + \angle c_3 T_2)\ \ (\angle c_4 t_2 + \angle c_4 t_0) + (\angle c_5 T_2 + \angle c_5 T_0)$$

and $l(\ _{2;3})\ \ l(\ _{2;l})$ so $\angle c_3 t_2\ \ \angle c_4 t_2; \angle c_3 T_2\ \ \angle c_5 T_2\ )\ \angle c_2 t_2 + \angle c_2 T_2$ $\angle c_4 t_0 + \angle c_5 T_0\ )\ \angle c_2 t_2\ \ \angle c_4 t_0$ or $\angle c_2 T_2\ \ \angle c_5 T_0$:

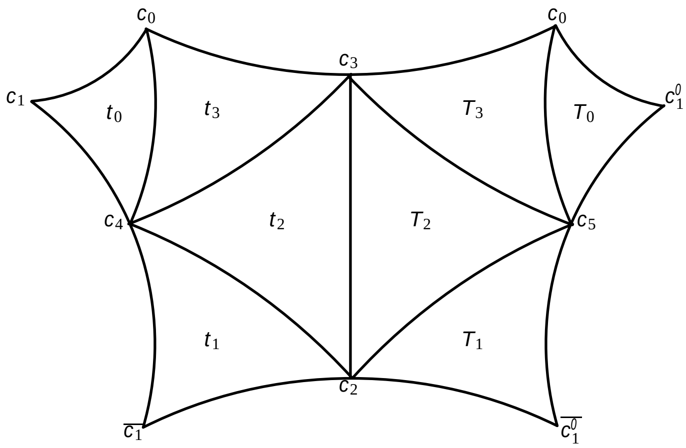

Figure 6: The triangles $t_k; T_k$ in the domain Ω

Up to relabelling, we may suppose that $\angle c_2 t_2\ \ \angle c_4 t_0$. We now show that $l(\ _{3;4})\ \ l(\ _{0;1})$. There are two arguments. Firstly we show that if $\angle c_3 t_2\ \ \ -$ then $l(\ _{0;4}) < l(\ _{3;0})$ { contradicting a hypothesis. So $\angle c_3 t_2 <\ \ -$ and we then show that $l(\ _{3;4})\ \ l(\ _{0;1})$. The angle is given as follows. Let I_2 be an isoceles triangle with vertices $v_2; v_3; v_4$ and edges $"_{2;3}; "_{2;4}; "_{3;4}$ such that $l("_{2;3}) = l("_{2;4}) = l(\ _{2;4})$ and $\angle v_2 I_2 = \angle c_2 t_2$. Then $\ = \angle v_3 I_2 = \angle v_4 I_2$.

Let $C_2; C_4$ denote circles of radius $l(\ _{2;4})$ about $c_2; c_4$ respectively. As in Figure 7 c_3 must lie inside C_2 since $l(\ _{2;3})\ \ l(\ _{2;4})$. Likewise c_0 must lie outside C_4 since $l(\ _{0;4})\ \ l(\ _{1;2})\ \ l(\ _{2;4})$. Similarly c_1 must lie outside C_4 since $l(\ _{1;4})\ \ l(\ _{1;2})\ \ l(\ _{2;4})$. Moreover since the angle sum at any cone point is $\ $: $\angle c_3 t_2 + \angle c_3 t_3 <\ $. In Figure 6 we have also constructed the point x as the intersection of the radius through $_{2;3}$ and C_4. Let t_x denote the triangle spanning $x; c_3; c_4$.

Now $\angle c_3 t_2$ $-$ is equivalent to $\angle c_3 t_x$. It follows that $\angle c_4 t_x$ $\angle c_3 t_x$. By inspection $\angle c_4 t_3 > \angle c_4 t_x$ and $\angle c_3 t_x > \angle c_3 t_3$. So $\angle c_4 t_3 > \angle c_4 t_x$ $\angle c_3 t_x >$ $\angle c_3 t_3$ or equivalently $l(\ _{0;4}) < l(\ _{0;3})$.

So $\angle c_3 t_2 <$ $-$ and we will compare $t_2; t_0$. Firstly, $\angle c_3 t_2 <$ $-$ implies that $l(\ _{3;4})$ $l(''_{3;4})$. (Recall that $''_{3;4}$ is an edge of I_2.) Let I_0 be an isoceles triangle with vertices $v_0; v_1; v_4$ and edges $''_{0;1}; ''_{1;4}; ''_{0;4}$ such that $l(''_{1;4}) = l(''_{0;4}) =$ $l(\ _{2;4})$ and $\angle v_4 I_0 = \angle c_4 t_0$. Since $l(\ _{0;4}); l(\ _{1;4})$ $l(\ _{1;2})$ $l(\ _{2;4})$ we then observe that $l(\ _{0;1})$ $l(''_{0;1})$. As $\angle c_2 t_2$ $\angle c_4 t_0$ we have that $l(''_{3;4})$ $l(''_{0;1})$. Therefore $l(\ _{0;1})$ $l(''_{0;1})$ $l(''_{3;4})$ $l(\ _{3;4})$.

We have equality *if and only if* $\angle c_2 t_2 = \angle c_4 t_0$ and $l(\ _{2;3}) = l(\ _{2;4}) = l(\ _{0;4}) =$ $l(\ _{1;4})$. From above $\angle c_2 t_2 = \angle c_4 t_0$ *if and only if* $l(\ _{1;2}) = l(\ _{1;l}); l(\ _{3;0}) =$ $l(\ _{0;l})$ and $l(\ _{2;3}) = l(\ _{2;l})$. So we have that $l(\ _{0;1}) = l(\ _{3;4})$ and $l(\ _{1;2}) =$ $l(\ _{2;3}) = l(\ _{3;0}) = l(\ _{0;l}) = l(\ _{1;l}) = l(\ _{2;l})$.

That is: $t_1; T_1$ are isometric equilateral triangles and $t_0; T_0; t_2; t_3$ (respectively $T_2; T_3$) are isometric isoceles triangles. By considering angle sums at $c_4; c_5$: $\angle c_4 t_2 = \angle c_4 t_3 = \angle c_5 T_2 = \angle c_5 T_3$. So: $t_1; T_1$ are isometric equilateral triangles and $t_0; T_0; t_2; t_3; T_2; T_3$ are isometric isoceles triangles. By the angle sum at c_3 : $\angle c_3 t_2 = \angle c_3 t_3 = \angle c_3 T_2 = \angle c_3 T_3 =$ $=4$ and so $\angle c_0 t_0 = \angle c_1 t_0 = \angle c_0 T_0 =$ $\angle c_1 T_0 =$ $=4$. Again, by considering angle sums at $c_0; c_1$ all the angles are $=4$, all of the edges are of equal length. So O is the octahedral orbifold. □

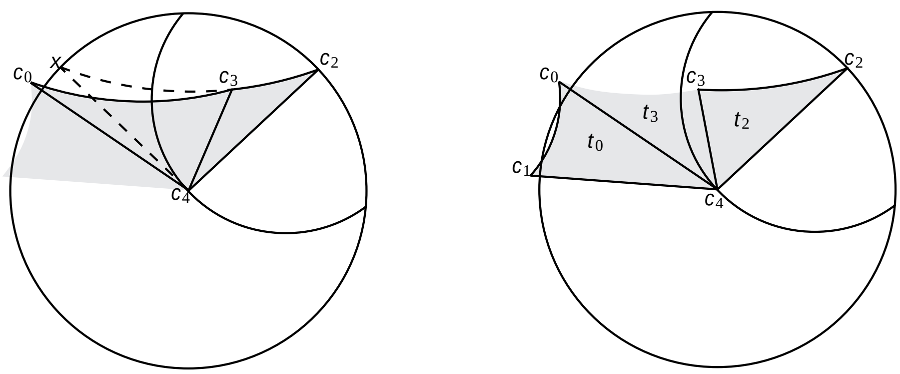

Figure 7: Arguments for $\angle c_3 t_2$ $-$ and for $\angle c_3 t_2 <$ $-$

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