ISSN 1364-0380 (on line) 1465-3060 (printed)

Geometry & Topology
Volume 5 (2001) 609–650
Published: 27 June 2001

Heegaard splittings of exteriors of two bridge knots

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Abstract

In this paper, we show that, for each non-trivial two bridge knot K and for each $g \geq 3$, every genus g Heegaard splitting of the exterior $E(K)$ of K is reducible.

AMS Classification numbers Primary: 57M25

Secondary: 57M05

Keywords: Two bridge knot, Heegaard splitting

Proposed: Cameron Gordon
Seconded: Joan Birman, Robion Kirby

Received: 23 February 2001
Accepted: 5 June 2001

1 Introduction

In this paper, we prove the following theorem.

Theorem 1.1 *Let K be a non-trivial two bridge knot. Then, for each $g \geq 3$, every genus g Heegaard splitting of the exterior $E(K)$ of K is reducible.*

We note that since $E(K)$ is irreducible, the above theorem together with the classification of the Heegaard splittings of the 3-sphere S^3 (F Waldhausen [21]) implies the next corollary.

Corollary 1.2 *Let K be a non-trivial two bridge knot. Then, for each $g \geq 3$, every genus g Heegaard splitting of $E(K)$ is stabilized.*

By H Goda, M Scharlemann, and A Thompson [6] (see also K Morimoto's paper [15]) or [13], it is shown that, for each non-trivial two bridge knot K, every genus two Heegaard splitting of $E(K)$ is isotopic to either one of six typical Heegaard splittings (see Figure 11). We note that Y Hagiwara [7] proved that genus three Heegaard splittings obtained by stabilizing the six Heegaard splittings are mutually isotopic. This result together with Corollary 1.2 implies the following.

Corollary 1.3 *Let K be a non-trivial two bridge knot. Then, for each $g \geq 3$, the genus g Heegaard splittings of $E(K)$ are mutually isotopic, ie, there is exactly one isotopy class of Heegaard splittings of genus g.*

We note that this result is proved for figure eight knot by D Heath [9].

The author would like to express his thanks to Dr Kanji Morimoto for careful readings of a manuscript of this paper.

2 Preliminaries

Throughout this paper, we work in the differentiable category. For a submanifold H of a manifold M, $N(H;M)$ denotes a regular neighborhood of H in M. When M is well understood, we often abbreviate $N(H;M)$ to $N(H)$. Let N be a manifold embedded in a manifold M with $\dim N = \dim M$. Then $\mathrm{Fr}_M N$ denotes the frontier of N in M. For the definitions of standard terms in 3-dimensional topology, we refer to [10] or [11].

2.A Heegaard splittings

A 3–manifold C is a *compression body* if there exists a compact, connected (not necessarily closed) surface F such that C is obtained from $F \times [0,1]$ by attaching 2–handles along mutually disjoint simple closed curves in $F \times \{1\}$ and capping off the resulting 2–sphere boundary components which are disjoint from $F \times \{0\}$ by 3–handles. The subsurface of ∂C corresponding to $F \times \{0\}$ is denoted by $\partial_+ C$. Then $\partial_- C$ denotes the subsurface $\mathrm{cl}(\partial C - (\partial F \times [0,1] \cup \partial_+ C))$ of ∂C. A compression body C is said to be *trivial* if either C is a 3–ball with $\partial_+ C = \partial C$, or C is homeomorphic to $F \times [0,1]$ with $\partial_- C$ corresponding to $F \times \{0\}$. A compression body C is called a *handlebody* if $\partial_- C = \emptyset$. A compressing disk $D (\subset C)$ of $\partial_+ C$ is called a *meridian disk* of the compression body C.

Remark 2.1 The following properties are known for compression bodies.

(1) Compression bodies are irreducible.

(2) By extending the cores of the 2–handles in the definition of the compression body C vertically to $F \times [0,1]$, we obtain a union of mutually disjoint meridian disks D of C such that the manifold obtained from C by cutting along D is homeomorphic to a union of $\partial_- C \times [0,1]$ and some (possibly empty) 3–balls. This gives a dual description of compression bodies. That is, a connected 3–manifold C is a compression body if there exists a compact (not necessarily connected) surface F without 2–sphere components and a union of (possibly empty) 3–balls B such that C is obtained from $F \times [0,1] \cup B$ by attaching 1–handles to $F \times \{0\} \cup \partial B$. We note that $\partial_- C$ is the surface corresponding to $F \times \{1\}$.

(3) Let D be a union of mutually disjoint meridian disks of a compression body C, and C' a component of the manifold obtained from C by cutting along D. Then, by using the above fact 2, we can show that C' inherits a compression body structure from C, ie, C' is a compression body such that $\partial_- C' = \partial_- C \cap C'$ and $\partial_+ C' = (\partial_+ C \cap C') \cup \mathrm{Fr}_C C'$.

(4) Let S be an incompressible surface in C such that $\partial S \subset \partial_+ C$. If S is not a meridian disk, then, by using the above fact 2, we can show that S is ∂–compressible into $\partial_+ C$, ie, there exists a disk Δ such that $\Delta \cap S = \partial \Delta \cap S = a$ is an essential arc in S, and $\Delta \cap \partial C = \mathrm{cl}(\partial \Delta - a)$ with $\Delta \cap \partial C \subset \partial_+ C$.

Let N be a cobordism rel ∂ between two surfaces F_1, F_2 (possibly $F_1 = \emptyset$ or $F_2 = \emptyset$), ie, F_1 and F_2 are mutually disjoint surfaces in ∂N with $\partial F_1 = \partial F_2$ such that $\partial N = F_1 \cup F_2 \cup (\partial F_1 \times [0,1])$.

Definition 2.2 We say that $C_1 \cup_P C_2$ (or $C_1 \cup C_2$) is a *Heegaard splitting* of $(N; F_1; F_2)$ (or simply, N) if it satisfies the following conditions.

(1) C_i $(i = 1, 2)$ is a compression body in N such that $\partial_- C_i = F_i$,

(2) $C_1 \cup C_2 = N$, and

(3) $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$.

The surface P is called a *Heegaard surface* of $(N; F_1; F_2)$ (or, N). In particular, if P is a closed surface, then the genus of P is called the *genus* of the Heegaard splitting.

Definition 2.3

(1) A Heegaard splitting $C_1 \cup_P C_2$ is *reducible* if there exist meridian disks D_1, D_2 of the compression bodies C_1, C_2 respectively such that $\partial D_1 = \partial D_2$

(2) A Heegaard splitting $C_1 \cup_P C_2$ is *weakly reducible* if there exist meridian disks D_1, D_2 of the compression bodies C_1, C_2 respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. If $C_1 \cup_P C_2$ is not weakly reducible, then it is called *strongly irreducible*.

(3) A Heegaard splitting $C_1 \cup_P C_2$ is *stabilized* if there exists another Heegaard splitting $C_1' \cup_{P'} C_2'$ such that the pair $(N; P)$ is isotopic to a connected sum of pairs $(N; P') \# (S^3; T)$, where T is a genus one Heegaard surface of the 3–sphere S^3.

(4) A Heegaard splitting $C_1 \cup_P C_2$ is *trivial* if either C_1 or C_2 is a trivial compression body.

Remark 2.4

(1) We note that $C_1 \cup_P C_2$ is stabilized if and only if there exist meridian disks D_1, D_2 of C_1, C_2 respectively such that ∂D_1 and ∂D_2 intersect transversely in one point.

(2) If $C_1 \cup_P C_2$ is stabilized and not a genus one Heegaard splitting of S^3, then $C_1 \cup_P C_2$ is reducible.

2.B Orbifold version of Heegaard splittings

Throughout this subsection, let N be a compact, orientable 3–manifold, γ a 1–manifold properly embedded in N, and F, F_1, F_2, D, S, connected surfaces embedded in N, which are in general position with respect to γ.

Definition 2.5 We say that D is a γ–*disk*, if (1) D is a disk, and (2) either $D \cap \gamma = \emptyset$, or D intersects γ transversely in one point.

Let $\ell(\subset F)$ be a simple closed curve such that $\ell \cap \gamma = \emptyset$.

Definition 2.6 We say that ℓ is γ–*inessential* if ℓ bounds a γ–disk in F. We say that ℓ is γ–*essential* if it is not γ–inessential.

Definition 2.7 We say that D is a γ–*compressing disk* for F if D is a γ–disk, $D \cap F = \partial D$, and ∂D is a γ–essential simple closed curve in F. The surface F is γ–*compressible* if it admits a γ–compressing disk, and F is γ–*incompressible* if it is not γ–compressible. We note that if D is a γ–compressing disk for F, then we can perform a γ–compression on F along D (Figure 1).

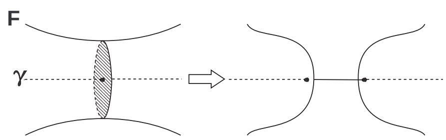

Figure 1

Definition 2.8 Suppose that $\partial F_1 = \partial F_2$, or $\partial F_1 \cap \partial F_2 = \emptyset$. We say that F_1 and F_2 are γ–*parallel*, if there is a submanifold R in N such that $(R, R \cap \gamma)$ is homeomorphic to $(F_1 \times [0,1], P \times [0,1])$ as a pair, where (1) P is a union of points in $\mathrm{Int} F_1$, and (2) $\partial F_1 = \partial F_2$ and F_1 (F_2 respectively) is the subsurface of ∂R corresponding to the closure of the component of $\partial(F_1 \times [0,1]) - (\partial F_1 \times \{1/2\})$ containing $F_1 \times \{0\}$ ($F_1 \times \{1\}$ respectively), or $\partial F_1 \cap \partial F_2 = \emptyset$ and F_1 (F_2 respectively) is the subsurface of ∂R corresponding to $F_1 \times \{0\}$ ($F_1 \times \{1\}$ respectively). The submanifold R is called a γ–*parallelism* between F_1 and F_2.

We say that F is γ–*boundary parallel* if there is a subsurface F' in ∂N such that F and F' are γ–parallel.

Definition 2.9 We say that S is a γ–*sphere* if (1) S is a sphere, and (2) either $S \cap \gamma = \emptyset$, or S intersects γ transversely in two points. We say that a 3–ball B^3 in N is a γ–*ball* if either $B^3 \cap \gamma = \emptyset$, or $B^3 \cap \gamma$ is an unknotted arc properly embedded in B^3. A γ–sphere S is γ–*compressible* if there exists a γ–ball B^3 in N such that $\partial B^3 = S$. A γ–sphere S is γ–*incompressible* if it is not γ–compressible. We say that N is γ–*reducible* if N contains a γ–incompressible 2–sphere. The manifold N is γ–*irreducible* if it is not γ–reducible.

Definition 2.10 We say that F is *γ–essential* if F is γ–incompressible, and not γ–boundary parallel.

Let α be an arc properly embedded in F with $\alpha \cap \gamma = \emptyset$.

Definition 2.11 We say that α is *γ–inessential* if there is a subarc β of ∂F such that $\partial\beta = \partial\alpha$, and $\alpha \cup \beta$ bounds a disk D in F such that $D \cap \gamma = \emptyset$, and α is *γ–essential* if it is not γ–inessential.

Definition 2.12 We say that Δ is a *γ–boundary compressing disk* for F if Δ is a disk disjoint from γ, $\Delta \cap F = \partial\Delta \cap F = \alpha$ is a γ–essential arc in F, and $\Delta \cap \partial N = \partial\Delta \cap \partial N = \mathrm{cl}(\partial\Delta - \alpha)$. The surface F is *γ–boundary compressible* if it admits a γ–boundary compressing disk. The surface F is *γ–boundary incompressible* if it is not γ–boundary compressible. We note that if Δ is a γ–boundary compressing disk for F, then we can perform a *γ–boundary compression* on F along Δ.

Definition 2.13 We say that F_1 and F_2 are *γ–isotopic* if there is an ambient isotopy ϕ_t $(0 \leq t \leq 1)$ of N such that $\phi_0 = id_N$, $\phi_1(F_1) = F_2$, and $\phi_t(\gamma) = \gamma$ for each t.

The next definition gives an orbifold version of compression body (cf (2) of Remark 2.1).

Definition 2.14 Suppose that N is a cobordism rel ∂ between two surfaces G_+, G_-. We say that (N, γ) is an *orbifold compression body* (or *O–compression body*) (with $\partial_+ N = G_+$, and $\partial_- N = G_-$) if the following conditions are satisfied.

(1) G_+ is not empty, and is connected (possibly, G_- is empty).

(2) No component of G_- is a γ–sphere.

(3) $\partial\gamma \subset \mathrm{Int}(G_+ \cup G_-)$.

(4) There exists a union of mutually disjoint γ–compressing disks, say D, for G_+ such that, for each component E of the manifold obtained from N by cutting along D, either E is a γ–ball with $E \cap G_- = \emptyset$, or $(E, \gamma \cap E)$ is homeomorphic to $(G \times [0,1], P \times [0,1])$, where G is a component of G_- with $E \cap G_- = G \times \{0\} = G$ and P is a union of mutually disjoint (possibly empty) points in G (see Figure 2).

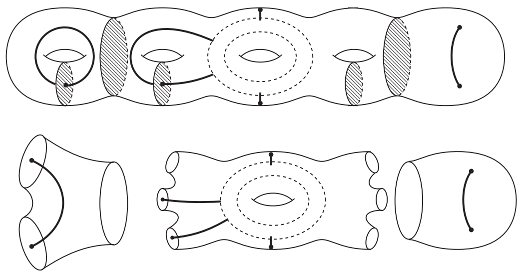

Figure 2

Note that the condition 1 of Definition 2.14 implies that N is connected. We say that an O–compression body (N,γ) is *trivial* if either N is a γ–ball with $\partial_+ N = \partial N$, or (N,γ) is homeomorphic to $(G_- \times [0,1], P^0 \times [0,1])$ with G_- ($\subset \partial N$) corresponding to $G_- \times \{1\}$, and P^0 a union of mutually disjoint points in G_-. An O–compression body (N,γ) is called an *O–handlebody* if $\partial_- N = \emptyset$. A γ–compressing disk of $\partial_+ N$ is called a *(γ–)meridian disk* of the O–compression body (N,γ).

By $\mathbb{Z}_2$–equivariant loop theorem [12, Lemma 3], and $\mathbb{Z}_2$–equivariant cut and paste argument as in [10, Proof of 10.3], we can prove the following (the proof is omitted).

Proposition 2.15 *Let N be a compact, orientable 3–manifold, and γ a 1–manifold properly embedded in N. Suppose that N admits a 2–fold branched cover $p\colon \tilde{N} \to N$ with branch set γ. Let F be a (possibly closed) surface properly embedded in N, which is in general position with respect to γ. Then F is γ–incompressible (γ–boundary incompressible respectively) if and only if $p^{-1}(F)$ is incompressible (boundary incompressible respectively) in $\tilde{N}$.*

By (2) of Remark 2.1, Definition 2.14, $\mathbb{Z}_2$–equivariant cut and paste argument as in [10, Proof of 10.3], and $\mathbb{Z}_2$–Smith conjecture [21], we immediately have the following.

Proposition 2.16 *Let N, γ be as in Proposition 2.15. Then (N,γ) is an O–compression body with $\partial_- N = G_-$, if and only if $\tilde{N}$ is a compression body with $\partial_- \tilde{N} = p^{-1}(G_-)$.*

Since the compression bodies are irreducible (see (1) of Remark 2.1), Proposition 2.16 together with $\mathbb{Z}_2$–Smith conjecture [21] implies the following.

Corollary 2.17 *Let (N, γ) be an O–compression body. Suppose that N admits a 2–fold branched cover with branch set γ. Then N is γ–irreducible.*

By (4) of Remark 2.1, and $\mathbb{Z}_2$–equivariant cut and paste argument as in [10, Proof of 10.3], we have the following.

Corollary 2.18 *Let (N, γ) be an O–compression body such that N admits a 2–fold branched cover with branch set γ. Let F be a connected γ–incompressible surface properly embedded in N, which is not a γ–meridian disk. Suppose that $\partial F \subset \partial_+ N$. Then there exists a γ–boundary compressing disk Δ for F such that $\Delta \cap \partial N \subset \partial_+ N$.*

Let M be a compact, orientable 3–manifold, and γ a 1–manifold properly embedded in M. Let C be a 3–dimensional manifold embedded in M. We say that C is a *γ–compression body* if $(C, \gamma \cap C)$ is an O–compression body. Suppose that M is a cobordism rel ∂ between two surfaces G_1, G_2 (possibly $G_1 = \emptyset$ or $G_2 = \emptyset$) such that $\partial\gamma \subset \mathrm{Int}(G_1 \cup G_2)$.

Definition 2.19 We say that $C_1 \cup_P C_2$ is a *Heegaard splitting* of $(M, \gamma; G_1, G_2)$ (or simply (M, γ)) if it satisfies the following conditions.

(1) C_i $(i = 1, 2)$ is a γ–compression body such that $\partial_- C_i = G_i$,

(2) $C_1 \cup C_2 = M$, and

(3) $C_1 \cap C_2 = \partial_+ C_1 = \partial_+ C_2 = P$.

The surface P is called a *Heegaard surface* of $(M, \gamma; G_1, G_2)$ (or (M, γ)).

Definition 2.20

(1) A Heegaard splitting $C_1 \cup_P C_2$ of (M, γ) is *γ–reducible* if there exist γ–meridian disks D_1, D_2 of the γ–compression bodies C_1, C_2 respectively such that $\partial D_1 = \partial D_2$.

(2) A Heegaard splitting $C_1 \cup_P C_2$ of (M, γ) is *weakly γ–reducible* if there exist γ–meridian disks D_1, D_2 of the γ–compression bodies C_1, C_2 respectively such that $\partial D_1 \cap \partial D_2 = \emptyset$. If $C_1 \cup_P C_2$ is not weakly γ–reducible, then it is called *strongly γ–irreducible*.

(3) A Heegaard splitting $C_1 \cup_P C_2$ of (M, γ) is *trivial* if either C_1 or C_2 is a trivial γ–compression body.

2.C Genus g, n–bridge positions

We first recall the definition of a genus g, n–bridge position of H.Doll [4]. Let $\Gamma = \gamma_1 \cup \cdots \cup \gamma_n$ be a union of mutually disjoint arcs γ_i properly embedded in a 3–manifold N.

Definition 2.21 We say that Γ is *trivial* if there exist mutually disjoint disks $D_1, \ldots, D_n$ in N such that (1) $D_i \cap \Gamma = \partial D_i \cap \gamma_i = \gamma_i$, and (2) $D_i \cap \partial N = \mathrm{cl}(\partial D_i - \gamma_i)$.

Let K be a link in a closed 3–manifold M. Let $X \cup_Q Y$ be a genus g Heegaard splitting of M. Then, with following [4], we say that K is in a *genus g, n–bridge position* (with respect to the Heegaard splitting $X \cup_Q Y$) if $K \cap X$ ($K \cap Y$ respectively) is a union of n arcs which is trivial in X (Y respectively).

A proof of the next lemma is elementary, and we omit it.

Lemma 2.22 *Let Γ be a union of mutually disjoint arcs properly embedded in a handlebody H. Then Γ is trivial if and only if (H, Γ) is a O–handlebody.*

This lemma allows us to generalize the definition of genus g, n–bridge positions as in the following form. Let K, M, and $X \cup_Q Y$ be as above.

Definition 2.23 We say that K is in a *genus g, n–bridge position* (with respect to the Heegaard splitting $X \cup_Q Y$) if $X \cup_Q Y$ gives a Heegaard splitting of (M, K) such that $\mathrm{genus}(Q) = g$, and $K \cap Q$ consists of $2n$ points.

Remark 2.24 This definition allows genus g, 0–bridge position of K.

In this paper, we abbreviate genus 0, n–bridge position to *n–bridge position*.

Definition 2.25 A knot K in the 3–sphere S^3 is called a *n–bridge knot*, if it admits a n–bridge position.

3 Weakly γ–reducible Heegaard splittings

In [8], W Haken proved that the Heegaard splittings of a reducible 3–manifold are reducible. As a sequel of this, Casson–Gordon [2] proved that each non-trivial Heegaard splitting of a ∂–reducible 3–manifold is weakly reducible. In this section, we prove orbifold versions of these results. In fact, we prove the following.

Proposition 3.1 *Let N be a compact orientable 3–manifold, and γ a 1–manifold properly embedded in N such that N admits a 2–fold branched cover with branch set γ. Suppose that N is a cobordism rel ∂ between two surfaces F_1, F_2 (possibly $F_1 = \emptyset$ or $F_2 = \emptyset$) such that $\partial\gamma \subset \mathrm{Int}(F_1 \cup F_2)$, and no component of $F_1 \cup F_2$ is a γ–disk. If N is γ–reducible, then every Heegaard splitting of $(N;\gamma;F_1;F_2)$ is weakly γ–reducible.*

Proposition 3.2 *Let N, γ, F_1, F_2 be as in Proposition 3.1. If $F_1 \cup F_2$ is γ–compressible in N, then every non-trivial Heegaard splitting of $(N;\gamma;F_1;F_2)$ is weakly γ–reducible.*

Remark 3.3 In the conclusion of Proposition 3.1, we can have just “weakly γ–reducible”, not “γ–reducible”. For example, let K be a connected sum of two trefoil knots, and $C_1 \cup C_2$ the Heegaard splitting of $(S^3;K)$ as in Figure 3. We note that $(S^3;K)$ is K–reducible (in fact, a 2–sphere giving prime decomposition of K is K–incompressible). Since the Heegaard splitting gives a minimal genus Heegaard splitting of $E(K)$, we can show that $C_1 \cup C_2$ is not γ–reducible. But $C_1 \cup C_2$ admits a pair of weakly K–reducing disks D_1, D_2 as in Figure 3.

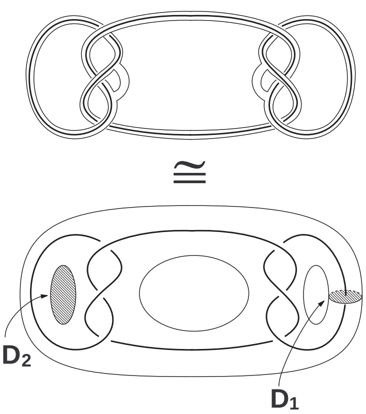

Figure 3

Then, by using Proposition 3.2, we prove an orbifold version of a lemma of Rubinstein–Scharlemann [17, Lemma 4.5].

Proposition 3.4 *Let M be a closed orientable 3–manifold, and K a link in M such that M admits a 2–fold branched cover with branch set K. Let $A \cup_P B$, $X \cup_Q Y$ be Heegaard splittings of $(M;K)$. Suppose that $A \subset \mathrm{Int}X$, and there* *exists a K-meridian disk D of X such that $D \cap A = \emptyset$. Then we have one of the following.*

(1) *M is homeomorphic to the 3-sphere, and either $K = \emptyset$ or K is a trivial knot.*

(2) *$X \cup_Q Y$ is weakly K-reducible.*

3.A Heegaard splittings of $(\hat{N}; \hat{F}_1, \hat{F}_2)$

For the proofs of Propositions 3.1, and 3.2, we show that we can derive Heegaard splittings of $\mathrm{cl}(N - N(\gamma))$ from Heegaard splittings of (N, γ).

Lemma 3.5 *Let $(C, \ )$ be a O-compression body such that C admits a 2-fold branched cover $q\colon \tilde{C} \to C$ with branch set . Let $\hat{C} = cl(C - N(\ ))$, $S = cl(\partial_+ C - N(\ ))$. Then $\hat{C}$ is a compression body with $\partial_+ \hat{C} = S$.*

Proof Let D be the union of mutually disjoint -compressing disks for $\partial_+ C$ as in Definition 2.14. Let D_0 (D_1 respectively) be the union of the components of D which are disjoint from (which intersect respectively). Let E be a component of the manifold obtained from C by cutting along D_0, and $\hat{E} = \mathrm{cl}(E - N(\ ))$. Let $D_{1,E}$ be the union of the components of D_1 that are contained in E. Let E' be the manifold obtained from E by cutting along $D_{1,E}$, and $\hat{E}' = \mathrm{cl}(E' - N(\ ))$. Then we have the following cases.

Case 1 $E \cap \ = \emptyset$.

In this case, $D_{1,E} = \emptyset$, and we have $E = \hat{E} = E' = \hat{E}'$. By the definition of -compression body (Definition 2.14), we see that $\hat{E} (= E)$ is a trivial compression body such that $\hat{E} \cap D_0 \ \ \partial_+ \hat{E}$.

Case 2 $E \cap \ \neq \emptyset$, and $E \cap \partial_- C = \emptyset$.

By the definition of -compression body, we see that each component of E' is a -ball intersecting with $E' \cap \partial_- C = \emptyset$. Hence each component of $\hat{E}'$ is a solid torus, say T, such that $T \cap N(\ )$ is an annulus which is a neighborhood of a longitude of T. This implies that each component of $\hat{E}'$ is a trivial compression body such that the union of the ∂_+ boundaries is $\mathrm{cl}(\partial \hat{E}' - N(\ ))$. Since $\hat{E}$ is recovered from $\hat{E}'$ by identifying pairs of annuli corresponding to $\mathrm{cl}(D_{1,E} - N(\ ))$, we see that the triviality can be pulled back to show that $\hat{E}$ is a trivial compression body with $\partial_+\hat{E} = \mathrm{cl}(\partial\hat{E} - N(\ )) = \mathrm{cl}(\partial_+ E - N(\ ))$, where $\hat{E} \cap D_0 \quad \partial_+\hat{E}$. In fact, we see that either E is a –ball or E is a solid torus with $\cap E$ a core circle.

Case 3 $E \cap \quad \neq \emptyset$, and $E \cap \partial_- C \neq \emptyset$.

By the definition of –compression body, for each component E of E', we have either E is a –ball intersecting with $E \cap \partial_- C = \emptyset$, or $(E\ ; E \cap\ )$ is a trivial –compression body such that the ∂_- boundary is a component of $\partial_- C$. In either case, $\hat{E} = \mathrm{cl}(E - N(\ ))$ is a trivial compression body such that $\partial_+\hat{E} = \mathrm{cl}(\partial_+ E - N(\ ))$. Hence $\hat{E}'$ is a union of trivial compression bodies such that the union of the ∂_+ boundaries is $\mathrm{cl}(\partial_+ E' - N(\ ))$. Since $\hat{E}$ is recovered from $\hat{E}'$ by identifying pairs of annuli corresponding to $\mathrm{cl}(D_{1;E} - N(\ ))$, we see that the triviality can be pulled back to show that $\hat{E}$ is a trivial compression body with $\partial_+\hat{E} = \mathrm{cl}(\partial_+ E - N(\ ))$, where $\hat{E} \cap D_0 \quad \partial_+\hat{E}$.

By the conclusions of Cases 1, 2 and 3, we see that $\hat{C}$ is recovered from a union of trivial compression bodies by identifying the pairs of disks in ∂_+ boundaries, which are corresponding to D_0, and this implies that $\hat{C}$ is a compression body (see (2) of Remark 2.1). Moreover, since the ∂_+ boundary of each trivial compression body $\hat{E}$ is $\mathrm{cl}(\partial_+ E - N(\ ))$, we see that $\partial_+\hat{C} = \mathrm{cl}(\partial_+ C - N(\ ))$. □

Let $C_1 \cup_P C_2$ be a Heegaard splitting of $(N; \gamma; F_1; F_2)$. Then let $\hat{N} = \mathrm{cl}(N - N(\gamma))$, $\hat{P} = \mathrm{cl}(P - N(\gamma))$, $\hat{C}_i = \mathrm{cl}(C_i - N(\gamma))$, and $\hat{F}_i = \mathrm{cl}(\partial\hat{C}_i - N(\hat{P}; \partial\hat{C}_i))$ $(i = 1; 2)$. By Lemma 3.5, we see that $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$ is a Heegaard splitting of $(\hat{N}; \hat{F}_1; \hat{F}_2)$. By the definitions of strongly irreducible Heegaard splittings, and strongly γ–irreducible Heegaard splittings, we immediately have the following.

Lemma 3.6 *If $C_1 \cup_P C_2$ is strongly γ–irreducible, then $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$ is strongly irreducible.*

3.B Proof of Proposition 3.1

Let N, γ be as in Proposition 3.1, and $C_1 \cup_P C_2$ a Heegaard splitting of $(N; \gamma)$. Let $\hat{N} = \mathrm{cl}(N - N(\gamma))$, and $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$ a Heegaard splitting of $(\hat{N}; \hat{F}_1; \hat{F}_2)$ obtained from $C_1 \cup_P C_2$ as in Section 3.A. Since $(N; \gamma)$ is γ–reducible, there exists a γ–incompressible γ–sphere S in N. Then we have the following two cases.

Case 1 $S \cap \gamma = \emptyset$.

In this case, we may regard that S is a 2–sphere in $\hat{N}$. It is clear that S is an incompressible 2–sphere in $\hat{N}$. Hence, by [2, Lemma 1.1], we see that there exists an incompressible 2–sphere S' in $\hat{N}$ such that S' intersects $\hat{P}$ in a circle. Since $\hat{N} \subset N$, we may regard S' is a 2–sphere in N. It is clear that $S' \cap P$ is a γ–essential simple closed curve in P, hence, $S' \cap C_i$ $(i = 1, 2)$ is a γ–meridian disk of C_i. This shows that $C_1 \cup_P C_2$ is γ–reducible.

Case 2 $S \cap \gamma \neq \emptyset$ (ie, $S \cap \gamma$ consists of two points).

We may suppose that $(S \cap \gamma) \cap P = \emptyset$. Let $\hat{S} = \mathrm{cl}(S - N(\gamma))$. Then $\hat{S}$ is an annulus properly embedded in $\hat{N}$ such that $\partial\hat{S} \subset \mathrm{Fr}_N N(\gamma)$, and $\partial\hat{S} \cap \hat{P} = \emptyset$.

Claim 1 $\hat{S}$ is incompressible in $\hat{N}$.

Proof If there is a compressing disk D for $\hat{S}$, then by compressing S along D, we obtain two 2–spheres, each of which intersects γ in one point. This contradicts the existence of a 2–fold branched cover of N with branch set γ. □

Claim 2 $\hat{S}$ is not ∂–parallel in $\hat{N}$.

Proof Suppose that $\hat{S}$ is parallel to an annulus A in $\partial\hat{N}$. Let $s = \mathrm{cl}(\partial N - (F_1 \cup F_2))$. Note that s is a (possibly empty) union of annulus. Let $F_i' = \mathrm{cl}(F_i - N(\gamma))$. Then $\partial\hat{N} = s \cup F_1' \cup F_2' \cup \mathrm{Fr}_N N(\gamma)$. Since S is γ–incompressible, we see that $(F_1' \cup F_2') \cap A \neq \emptyset$. Since no component of $F_1 \cup F_2$ is a γ–disk, each component of $(F_1' \cup F_2') \cap A$ is an annulus. Let A_* be a component of $\mathrm{Fr}_N N(\gamma)$ such that A_* contains a component of $\partial\hat{S}$. Let F_* be the component of $(F_1' \cup F_2') \cap A$ such that $F_* \cap A_* \neq \emptyset$. Note that $F_* \cap A_*$ is a component of ∂A_* and is also a component of ∂F_*. Let A' be the component of $\mathrm{cl}(\partial\hat{N} - (F_1' \cup F_2'))$ such that $A' \cap F_*$ is the component of ∂F_* other than $F_* \cap A_*$. Then A' is an annulus which is either a component of $\mathrm{Fr}_N N(\gamma)$, or a component of s. If A' is a component of $\mathrm{Fr}_N N(\gamma)$, then the component of $F_1 \cup F_2$ corresponding to F_* is a γ–sphere, hence, a component of $\partial_- C_1$ or $\partial_- C_2$ is a γ–sphere, a contradiction. If A' is a component of s, then the component of $F_1 \cup F_2$ corresponding to F_* is a γ–disk, contradicting the assumption of Proposition 3.1. □

By Claims 1 and 2, $\hat{S}$ is γ–essential in $\hat{N}$. Suppose, for a contradiction, that $C_1 \cup_P C_2$ is strongly γ–irreducible. By Lemma 3.6, $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$ is strongly irreducible. Then, by [19, Lemma 6] or [16, Lemma 2.3], $\hat{S}$ is ambient isotopic rel ∂ to a surface $\hat{S}'$ such that $\hat{S}' \cap \hat{P}$ consists of essential simple closed curves in $\hat{S}'$. We regard $\hat{S} = \hat{S}'$. This means that each component of $S \cap P$ is a simple closed curve which separates the points $S \cap \gamma$. We suppose that $|S \cap P|$ is minimal among the γ–incompressible γ–spheres with this property. Let $n = |S \cap P|$. Suppose that $n = 1$, ie, $S \cap P$ consists of a simple closed curve, say ℓ_1. Then ℓ_1 separates S into two γ–disks, which are γ–meridian disks in C_1 and C_2 respectively. This shows that $C_1 \cup_P C_2$ is γ–reducible, a contradiction. Suppose that $n \geq 2$. Let D_1 be the closure of a component of $S - P$ such that $D_1 \cap \gamma \neq \emptyset$. Note that D_1 is a γ–disk. Without loss of generality, we may suppose that $D_1 \subset C_1$. By the minimality of $|S \cap P|$, we see that D_1 is a γ–meridian disk of C_1. Let A_2 be the closure of the component of $S - P$ such that $A_2 \cap D_1 \neq \emptyset$.

Claim 3 A_2 is γ–incompressible in C_2.

Proof Suppose that there is a γ–compressing disk D for A_2 in C_2. If $D \cap \gamma = \emptyset$, then we have a contradiction as in the proof of Claim 1. Suppose that $D \cap \gamma \neq \emptyset$. Let D_2 be the disk obtained from A_2 by γ–compressing A_2 along D such that $\partial D_2 = \partial D_1$. Since ∂D_1 is γ–essential in P, this shows that D_2 is a γ–meridian disk of C_2. Hence $C_1 \cup C_2$ is γ–reducible, a contradiction. □

Note that $\partial A_2 \subset \partial_+ C_2$. There is a γ–boundary compressing disk Δ for A_2 in C_2 such that $\Delta \cap \partial C_2 \subset \partial_+ C_2$ (Corollary 2.18). By the minimality of $|S \cap P|$, we see that A_2 is not γ–parallel to a surface in $\partial_+ C_2$. Hence, by γ–boundary compressing A_2 along Δ, and applying a tiny isotopy, we obtain a γ–meridian disk D_2 in C_2 such that $D_1 \cap D_2 = \emptyset$. Hence $C_1 \cup_P C_2$ is weakly γ–reducible, a contradiction.

This completes the proof of Proposition 3.1.

3.C Proof of Proposition 3.2

Let N, γ be as in Proposition 3.2 and $C_1 \cup_P C_2$ a Heegaard splitting of $(N; \gamma)$. Let $\hat{N} = \mathrm{cl}(N - N(\gamma))$, and $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$ the Heegaard splitting of $(\hat{N}; \hat{F}_1; \hat{F}_2)$ obtained from $C_1 \cup_P C_2$ as in Section 3.A. Let D be a γ–compressing disk for $F_1 \cup F_2$.

Case 1 $D \cap \gamma = \emptyset$.

In this case, we may regard that D is a disk in $\hat{N}$. It is clear that D is a compressing disk of $\hat{F}_1 \cup \hat{F}_2$. Hence, by [2, Lemma 1.1], we see that $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$ is weakly reducible. This implies that $C_1 \cup_P C_2$ is weakly γ–reducible.

Case 2 $D \cap \gamma \neq \emptyset$ (ie, $D \cap \gamma$ consists of a point).

Let $\hat{D} = \mathrm{cl}(D - N(\gamma))$.

Claim $\hat{D}$ is an essential annulus in $\hat{N}$.

Proof By using the argument as in Claim 1 of Case 2 of Section 3.B, we can show that $\hat{D}$ is incompressible in $\hat{N}$. Suppose that $\hat{D}$ is parallel to an annulus A in $\partial\hat{N}$. Let s, F_i' $(i = 1, 2)$ be as in Claim 2 of Case 2 of Section 3.B. Let A be the component of $\mathrm{Fr}_N N(\gamma)$ such that $\partial D \subset A$, and F the component of $F_1' \cup F_2'$ such that $F \supset \partial D$. By using the argument of the proof of Claim 2 of Case 2 of Section 3.B, we see that A is disjoint from $s \cup (\mathrm{Fr}_N N(\gamma) - A)$, hence $\mathrm{cl}(A - A) \subset F$. Hence $F \cap A$ is an annulus, and this shows that ∂D bounds a γ-disk in $F_1 \cup F_2$, a contradiction. □

Suppose, for a contradiction, that $C_1 \cup_P C_2$ is strongly γ-irreducible. By Lemma 3.6, $\hat{C}_1 \cup_{\hat{P}} \hat{C}_2$ is strongly irreducible. Then, by [19, Lemma 6] or [16, Lemma 2.3], $\hat{D}$ is ambient isotopic rel ∂ to a surface $\hat{D}'$ such that $\hat{D}' \cap \hat{P}$ consists of essential simple closed curves in $\hat{D}'$. We regard $\hat{D} = \hat{D}'$. This means that each component of $D \cap P$ is a simple closed curve bounding a disk in D, which contains the point $D \cap \gamma$. We suppose that $|D \cap P|$ is minimal among the γ-compressing disks for $F_1 \cup F_2$ with this property. Let $n = |D \cap P|$.

Suppose that $n = 1$, ie, $D \cap P$ consists of a simple closed curve, say ℓ_1. Then the closures of the components of $D - \ell_1$ consists of a disk, say D_1, and an annulus, say A_2. Without loss of generality, we may suppose that $D_1 \subset C_1$, and $A_2 \subset C_2$. Note that a component of ∂A_2 is contained in $\partial_- C_2$, and the other in $\partial_+ C_2$. Since C_2 is not trivial, there exists a γ-meridian disk D_2 in C_2. It is elementary to show, by applying cut and paste arguments on D_2 and A_2, that there is a γ-meridian disk D_2' in C_2 such that $D_2' \cap A_2 = \emptyset$. Hence $D_1 \cap D_2' = \emptyset$, and this shows that $C_1 \cup_P C_2$ is weakly γ-reducible, a contradiction.

Suppose that $n \geq 2$. Let D_1 be the closure of the component of $D - P$ such that $D_1 \cap \gamma \neq \emptyset$. Note that D_1 is a γ-disk. Without loss of generality, we may suppose that $D_1 \subset C_1$. By the minimality of $|D \cap P|$, we see that D_1 is a γ-meridian disk of C_1. Let A_2 be the closure of the component of $D - P$ such that $A_2 \cap D_1 \neq \emptyset$. Then, by using the arguments as in the proof of Claim 3 of Case 2 of Section 3.B, we can show that A_2 is γ-incompressible in C_2. Note that $\partial A_2 \subset \partial_+ C_2$. There is a γ-boundary compressing disk for A_2 in C_2 such that $\cap\, \partial C_2 \subset \partial_+ C_2$ (Corollary 2.18). By the minimality of $|S \cap P|$, we see that A_2 is not γ-parallel to a surface in $\partial_+ C_2$. Hence, by γ-boundary compressing A_2 along Δ, and applying a tiny isotopy, we obtain a γ-meridian disk D_2 of C_2 such that $D_1 \cap D_2 = \emptyset$. Hence $C_1 \cup_P C_2$ is weakly γ-reducible, a contradiction.

This completes the proof of Proposition 3.2.

3.D Proof of Proposition 3.4

Let D be a union of mutually disjoint, non K-parallel, K-meridian disks for X such that $D \cap A = \emptyset$. We suppose that D is maximal among the unions of K-meridian disks with the above properties. Let $Z' = N(\partial X; X) \cup N(D; X)$. Then we have the following two cases.

Case 1 A component of $\partial Z' - \partial X$ bounds a K-ball, say B_K, such that $B_K \supset A$.

In this case, since $\partial B_K \subset B$, and B is K-irreducible, ∂B_K bounds a K-ball B'_K in B (Corollary 2.17). Hence $M = B_K \cup B'_K$ is a 3-sphere. In particular, if $K \neq \emptyset$, then $K \cap B_K$ ($K \cap B'_K$ respectively) is a trivial arc properly embedded in B_K (B'_K respectively). Hence K is a trivial knot. This shows that we have conclusion 1.

Case 2 No component of $\partial Z' - \partial X$ bounds a K-ball which contains A.

Since X is K-irreducible, each of the K-sphere components of $\partial Z' - \partial X$ (if exists) bounds K-balls in X. By the construction of Z', it is easy to see that the K-balls are mutually disjoint. Let $Z = Z' \cup$ (the K-balls). By (3) of Remark 2.1 and Proposition 2.16, we see that Z is a K-compression body with $\partial_+ Z = \partial X$, and by the maximality of D, we see that $\partial_- Z$ consists of one component, say F, such that F bounds a K-handlebody which contains A. Let $N = Y \cup Z$. Note that $Y \cup_Q Z$ is a Heegaard splitting of $(N; K \cap N)$. Since $\partial N = F$ is a closed surface contained in B, it is K-compressible in B (Proposition 2.15). By the maximality of D, we see that the compressing disk lies in N. Hence, by Proposition 3.2, we see that $Y \cup_Q Z$ is weakly K-reducible. This obviously implies that $X \cup_Q Y$ is weakly K-reducible, and we have conclusion 2.

This completes the proof of Proposition 3.4.

4 The Casson–Gordon theorem

A Casson and C McA Gordon proved that if a Heegaard splitting of a closed 3–manifold M is weakly reducible, then either the splitting is reducible, or M contains an incompressible surface [2, Theorem 3.1]. In this section, we generalize this result for compact M. The author thinks that this generalization is well known (eg, [20]). However, the formulation given here will be useful for the proof of Theorem 1.1 (Section 7.C).

Let M be a compact, orientable 3–manifold, and $C_1 \cup_P C_2$ a Heegaard splitting of M such that P is a closed surface, ie, $\partial_- C_1 \cup \partial_- C_2 = \partial M$. Let $\Delta = \Delta_1 \cup \Delta_2$ be a weakly reducing collection of disks for P, ie, Δ_i $(i = 1, 2)$ is a union of mutually disjoint, non-empty meridian disks of C_i such that $\Delta_1 \cap \Delta_2 = \emptyset$. Then $P(\Delta)$ denotes the surface obtained from P by compressing P along Δ. Let $\hat{P}(\Delta) = P(\Delta) -$ (the components of $P(\Delta)$ which are contained in C_1 or C_2).

Lemma 4.1 *If there is a 2–sphere component in* $\hat{P}(\Delta)$, *then* $C_1 \cup_P C_2$ *is reducible.*

Proof Suppose that there is a 2–sphere component S of $\hat{P}(\Delta)$. We note that $S \cap C_i$ $(i = 1, 2)$ is a union of non-empty meridian disks of C_i. Let $\hat{S} = \mathrm{cl}(S - (C_1 \cup C_2))$. Note that $\hat{S}$ is a planar surface in P. Let $A_1 \cup A_2$ be a union of mutually disjoint arcs properly embedded in $\hat{S}$ such that $\partial A_i \subset \partial(S \cap C_i)$, and that $\mathrm{cl}(\hat{S} - N(A_1 \cup A_2, \hat{S}))$ is an annulus, say A'. Let S' be a 2–sphere obtained from S by pushing A_1 into C_1, and A_2 into C_2 such that $S' \cap P = A'$. It is clear that $S' \cap C_i$ $(i = 1, 2)$ consists of a disk, say D_i, obtained from $S \cap C_i$ by banding along A_i.

Claim D_i is a meridian disk of the compression body C_i $(i = 1, 2)$.

Proof Suppose, for a contradiction, that either D_1 or D_2, say D_1, is not a meridian disk, ie, there is a disk D in P such that $\partial D = \partial D_1$. Note that we have either $N(A_1, \hat{S}) \subset D$, or $N(A_1, \hat{S}) \subset \mathrm{cl}(P - D)$. If $N(A_1, \hat{S}) \subset D$, then $\partial(S \cap C_1)$ is recovered from ∂D by banding along arcs properly embedded in D. This shows that $\partial(S \cap C_1) \subset D$, and this implies that each component of $S \cap C_1$ is not a meridian disk, a contradiction. On the other hand, if $N(A_1, \hat{S}) \subset \mathrm{cl}(P - D)$, then $\mathrm{cl}(\hat{S} - N(A_1, \hat{S})) \subset D$. This shows that $\partial(S \cap C_2) \subset D$, and this implies that each component of $S \cap C_2$ is not a meridian disk, a contradiction. □

Since ∂D_1 and ∂D_2 are parallel in P, we see by Claim that $C_1 \cup_P C_2$ is reducible. ◻

Now we define a complexity $c(F)$ of a closed surface F as follows.

$$c(F) = \sum (\chi(F_i) - 1),$$

where the sum is taken for all components of F. Then we suppose that $c(\hat{P}(\Delta))$ is maximal among all weakly reducing collections of disks for P. By Lemma 4.1, we see that if the complexity of a component of $\hat{P}(\Delta)$ is positive, then $C_1 \cup_P C_2$ is reducible. Suppose that the complexities of the components of $\hat{P}(\Delta)$ are strictly negative, ie, each component of $\hat{P}(\Delta)$ is not a 2–sphere. Then, by the argument of the proof of [2, Theorem 3.1], we see that $\hat{P}(\Delta)$ is incompressible in M. Hence we have the next proposition.

Proposition 4.2 *Let M be a compact, orientable 3–manifold, and $C_1 \cup_P C_2$ a Heegaard splitting of M with $\partial_- C_1 \cup \partial_- C_2 = \partial M$. Suppose that $C_1 \cup_P C_2$ is weakly reducible. Then either*

(1) *$C_1 \cup_P C_2$ is reducible, or*

(2) *there exists a weakly reducing collection of disks Δ for P such that each component of $\hat{P}(\Delta)$ is an incompressible surface in M, which is not a 2–sphere.*

Note that, in [2], M is assumed to be closed. However, it is easy to see that the arguments there work for Heegaard splittings $C_1 \cup C_2$ such that $\partial_- C_1 \cup \partial_- C_2 = \partial M$.

The following is a slight extension of [1, Lemme 1.4]. Let M, $C_1 \cup_P C_2$, Δ be as above. Suppose that we have conclusion 2 of Proposition 4.2. Let $M_1, \ldots, M_n$ be the closures of the components of $M - \hat{P}(\Delta)$. Let $M_{j,i} = M_j \cap C_i$ ($j = 1, \ldots, n;\ i = 1, 2$).

Lemma 4.3 *For each j, we have either one of the following.*

(1) *$M_{j,2} \cap P \subset \mathrm{Int}(M_{j,1} \cap P)$, and $M_{j,1}$ is connected.*

(2) *$M_{j,1} \cap P \subset \mathrm{Int}(M_{j,2} \cap P)$, and $M_{j,2}$ is connected.*

Proof Recall that Δ_i is the union of the components of Δ that are contained in C_i ($i = 1, 2$). We see, from the definition of $\hat{P}(\Delta)$, that each M_j is obtained as in the following manner.

(*) Take a component N of $\mathrm{cl}(C_i - N(\Gamma_i; C_i))$ ($i = 1$ or 2, say 1) such that there exists a component D_2 of Γ_2 such that $\partial D_2 \subset N$. Let $N' = N \cup$ (the components of $N(\Gamma_2; C_2)$ intersecting N). Then $M_j = N' \cup$ (the union of components N_2 of $\mathrm{cl}(C_2 - N(\Gamma_2; C_2))$ such that $(N_2 \cap P) \subset (N \cap P)$).

It is clear that this construction process gives conclusion 1. If N is a component of $\mathrm{cl}(C_2 - N(\Gamma_2; C_2))$, then we have conclusion 2. □

We note that each component of $\mathrm{Fr}_{C_i}(M_{j,i})$ is a meridian disk of C_i, which is parallel to a component of Γ. Recall that $\hat{P}(\Gamma)$ is obtained from $P(\Gamma)$ by discarding the components each of which is contained in C_1 or C_2. These imply that each component E of $M_{j,i}$ inherits a compression body structure from C_i (see (3) of Remark 2.1), ie, $\partial_+ E = (E \cap \partial_+ C_i) \cup \mathrm{Fr}_{C_i}(E)$. Then we can obtain a splitting, denoted by $C_{j,1} \cup_{P_j} C_{j,2}$, of M_j as follows ([1, Lemme 1.4]).

> Suppose that M_j satisfies conclusion 1 (2 respectively) of Lemma 4.3. Recall that $M_{j,1}$ ($M_{j,2}$ respectively) inherits a compression body structure from C_1 (C_2 respectively). Then let
> $C_{j,1} = \mathrm{cl}(M_{j,1} - N(\partial_+ M_{j,1}; M_{j,1}))$ ($C_{j,2} = \mathrm{cl}(M_{j,2} - N(\partial_+ M_{j,2}; M_{j,2}))$ respectively), and
> $C_{j,2} = N(\partial_+ M_{j,1}; M_{j,1}) \cup M_{j,2}$ ($C_{j,1} = N(\partial_+ M_{j,2}; M_{j,2}) \cup M_{j,1}$ respectively).

Lemma 4.4 *Suppose that each component of $\hat{P}(\Gamma)$ is not a 2–sphere. Then each $C_{j,i}$ is a compression body such that, for each j, we have $\partial_+ C_{j,2} = \partial_+ C_{j,1} = C_{j,1} \cap C_{j,2}$, ie, $C_{j,1} \cup_{P_j} C_{j,2}$ is a Heegaard splitting of M_j.*

Proof Since the argument is symmetric, we may suppose that M_j satisfies conclusion 1 of Lemma 4.3. Since $M_{j,1}$ is a compression body, it is clear that $C_{j,1}$ is a compression body. Let $D_1 = \mathrm{Fr}_{C_2} M_{j,2}$. There is a union of mutually disjoint meridian disks, say D_2, of $M_{j,2}$ such that $D_2 \cap D_1 = \emptyset$, and each component of the manifold obtained from $M_{j,2}$ by cutting along D_2 is homeomorphic to either a 3–ball or $G \times [0,1]$, where G is a component of $\partial_- M_{j,2}$ with $G \times \{0\}$ corresponding to G. Hence $C_{j,2}$ ($= N(\partial_+ M_{j,1}; M_{j,1}) \cup M_{j,2}$) is homeomorphic to a manifold obtained from $N(\partial_+ M_{j,1}; M_{j,1})$ ($= \partial_+ M_{j,1} \times [0,1]$) by attaching 2–handles along the simple closed curves corresponding to $\partial(D_1 \cup D_2)$ in $\partial_+ M_{j,1} \times \{1\}$, and capping off some of the resulting 2–sphere boundary components. By the definition of compression body (Section 2.A), this implies that $C_{j,2}$ is a compression body, unless there exists a 2–sphere component S of $\partial C_{j,2}$, which is disjoint from $N(\partial_+ M_{j,1}; M_{j,1}) \cap C_{j,1}$ ($= \partial_+ M_{j,1} \times \{0\}$). However such S must be a component of $\hat{P}(\Gamma)$, a contradiction. □

Let M, $C_1 \cup_P C_2$, , M_j, $M_{j,i}$, and $C_{j,1} \cup_{P_j} C_{j,2}$ be as above.

Lemma 4.5 *Suppose that each component of $\hat{P}(\ )$ is not a 2–sphere. If ∂M is incompressible in M, then each compression body $C_{j,i}$ is not trivial.*

Proof Suppose that some compression body is trivial. By changing subscripts if necessary, we may suppose that $C_{1,1}$ is trivial. Then we claim that $M_{1,2} \cap P$ Int$(M_{1,1} \cap P)$, ie, we have conclusion 1 of Lemma 4.3. In fact, if we have conclusion 2 of Lemma 4.3, then $C_{1,1} = N(\partial_+ M_{1,2}, M_{1,2}) \cup M_{1,1}$. However this expression obviously implies $C_{1,1}$ is not trivial, a contradiction. Hence $C_{1,1} = \mathrm{cl}(M_{1,1} - N(\partial_+ M_{1,1}, M_{1,1}))$, and this implies that $M_{1,1}$ is a trivial compression body such that $\partial_- M_{1,1}$ is a component of ∂M. Let D be any component of $\mathrm{Fr}_{C_2} M_{1,2}$. Then by extending D vertically to $M_{1,1}$, we obtain a disk $\tilde{D}$ properly embedded in M. Since each component of $\hat{P}(\ )$ is not a 2–sphere, $\partial\tilde{D}$ is not contractible in ∂M. Hence $\tilde{D}$ is a compressing disk of ∂M, a contradiction. □

Lemma 4.6 *If some $C_{j,1} \cup_{P_j} C_{j,2}$ is reducible, then $C_1 \cup_P C_2$ is reducible.*

Proof We prove this by using an argument of C.Frohman [5, Lemma 1.1]. If M is reducible, then by [2, Lemma 1.1], we see that any Heegaard splitting of M is reducible. Hence we may suppose that M is irreducible. If a component of $\hat{P}(\ )$ is a 2–sphere, then $C_1 \cup_P C_2$ is reducible (Lemma 4.1). Hence we may suppose that each component of $\hat{P}(\ )$ is not a 2–sphere (hence, $C_{j,1} \cup_{P_j} C_{j,2}$ is a Heegaard splitting of M_j). Since the argument is symmetric, we may suppose that the pair $M_{j,1}$, $M_{j,2}$ satisfies conclusion 1 of Lemma 4.3. By [2, Lemma 1.1], there exists an incompressible 2–sphere S in M_j such that S intersects P_j in a circle. Let $D_1 = S \cap C_{j,1}$. Note that D_1 is a meridian disk of $C_{j,1}$. Since M is irreducible, S bounds a 3–ball B^3 in M_j. Let $C'_{j,1}$ be the closure of the component of $C_{j,1} - D_1$ such that $C'_{j,1}$ B^3. Since $\partial M \cap B^3 = \emptyset$, we see that $C'_{j,1}$ is a handlebody, ie, $\partial_- C'_{j,1} = \emptyset$. Let X be a spine of $C'_{j,1}$, and $M_X = \mathrm{cl}(M - N(X, C_1))$. It is clear that S is an incompressible 2–sphere in M_X, and P is a Heegaard surface of M_X. Hence, by [2, Lemma 1.1], there exists an incompressible 2–sphere S_X in M_X such that S_X intersects P in a circle. It is obvious that the 2–sphere S_X gives a reducibility of $C_1 \cup_P C_2$. □

5 Reducing genus g, n–bridge positions

Let K be a knot in a closed, orientable 3–manifold M. Let $V_1 \cup V_2$ be a Heegaard splitting of M, which gives a genus g, n–bridge position of K with

$n \geq 1$. Let a be a component of $K \cap V_i$ ($i = 1$ or 2, say 2). Let $V_1' = V_1 \cup N(a; V_2)$, and $V_2' = \mathrm{cl}(V_2 - N(a; V_2))$. By the definition of genus g, n–bridge positions, it is easy to see that $V_1' \cup V_2'$ gives a genus $(g+1)$, $(n-1)$–bridge position of K. We say that the Heegaard splitting $V_1' \cup V_2'$ is obtained from $V_1 \cup V_2$ by a *tubing* (along a). See Figure 4.

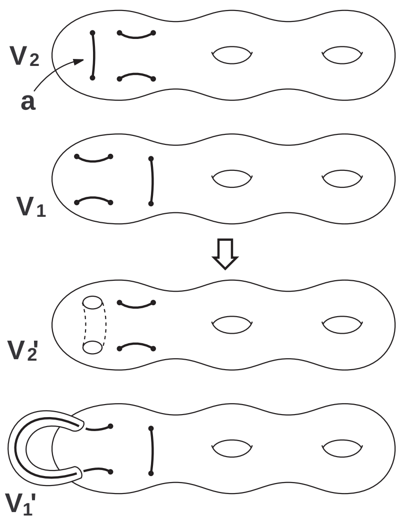

Figure 4

We say that a knot K in M is a *core knot* if there is a genus one Heegaard splitting $V \cup W$ of M such that K is a core curve of the solid torus V, ie, K admits a genus one, 0–bridge position. Note that if M is a 3–sphere, then K is a core knot if and only if K is a trivial knot. We say that K is *small* if the exterior $E(K)$ of K does not contain a closed essential surface. We say that a surface F properly embedded in $E(K)$ is *meridional* if ∂F is a union of non-empty meridian loops. We note that [3, Theorem 2.0.3] implies that if M is a 3–sphere and K is small, then $E(K)$ does not contain a meridional essential surface.

Proposition 5.1 *Let K be a knot in a closed orientable 3–manifold M with the following properties.*

(1) *M is K–irreducible.*

(2) *There exists a 2–fold branched covering space of M with branch set K.*

(3) *K is not a core knot.*

(4) *K is small and there does not exists a meridional essential surface in $E(K)$.*

Let $C_1 \cup_P C_2$ be a Heegaard splitting of M, which gives a genus g, n-bridge position of K. Suppose that $C_1 \cup_P C_2$ is weakly K-reducible. Then we have either one of the following.

(1) *There exists a weakly K-reducing pair of disks E_1, E_2 in C_1, C_2 respectively such that $E_1 \cap K = \emptyset$, and $E_2 \cap K = \emptyset$.*

(2) *There exists a Heegaard splitting $H_1 \cup H_2$ of M, which gives a genus $(g-1)$ $(n+1)$-bridge position of K such that $C_1 \cup C_2$ is obtained from $H_1 \cup H_2$ by a tubing.*

Remark 5.2 Note that, in Proposition 5.1, if $g = 0$, then we always have conclusion 1.

Proof Let D_1, D_2 be a pair of K-essential disks in C_1, C_2 respectively, which gives a weak K-reducibility of $C_1 \cup C_2$. If $D_1 \cap K = \emptyset$ and $D_2 \cap K = \emptyset$, then we have conclusion 1. Hence in the rest of the proof, we may suppose that $D_1 \cap K \neq \emptyset$. We have the following two cases.

Case 1 $D_2 \cap K = \emptyset$.

In this case, we first show the following.

Claim 1 If D_1 is separating in C_1, then we have conclusion 1.

Proof Let C_1', C_1'' be the closures of the components of $C_1 - D_1$ such that $\partial D_2 \subset \partial C_1''$. Then C_1' is a K-handlebody which is not a K-ball. Hence there exists a K-essential disk D_1' in C_1' such that $D_1' \cap K = \emptyset$, and $D_1' \cap D_1 = \emptyset$ (hence, D_1' is properly embedded in C_1). See Figure 5. It is clear that D_1' is a K-meridian disk of C_1. Hence, by regarding, $E_1 = D_1'$, $E_2 = D_2$, we have conclusion 1. □

By Claim 1, we may suppose that D_1 is non-separating in C_1. Let P' be the surface obtained from P by K-compressing along D_1, and $\hat{P}' = P' \cap E(K)$. We note that P' separates M into two components, say C_1' and C_2', where C_1' is obtained from C_1 by cutting along D_1. Let $\hat{C}_i' = C_i' \cap E(K)$ $(i = 1, 2)$. Hence, by Section 3.B, we see that $\hat{C}_1'$ is a compression body with $\partial_+ \hat{C}_1' = \hat{P}'$. Let D be a union of maximal mutually disjoint, non parallel compressing disks for $\hat{P}'$ such that $D \subset \hat{C}_2'$. Note that since D_2 is a compressing disk for $\hat{P}'$ such that $D_2 \subset \hat{C}_2'$, there actually exists such D. Let $\hat{C}_2'' = N(\hat{P}'; \hat{C}_2') \cup N(D; \hat{C}_2')$. Note that $\hat{C}_2'$ is homeomorphic to $C_2 \cap E(K) = \mathrm{cl}(C_2 - N(K))$, hence, $\hat{C}_2'$ is

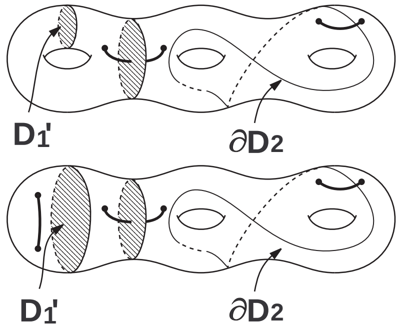

Figure 5

irreducible (Section 3.A). Hence the 2–sphere components S (possibly $S = \emptyset$) of $\partial \hat{C}_2'' - \hat{P}'$ bounds mutually disjoint 3–balls in $\hat{C}_2'$. Let $C_2 = \hat{C}_2'' \cup$ (the 3–balls). Then C_2 is a compression body such that $\partial_+ C_2 = \hat{P}'$. Let $P^* = \partial_- C_2$.

Claim 2 If P^* is compressible in $E(K)$, then we have conclusion 1.

Proof Suppose that there exists a compressing disk E of P^* in $E(K)$. Let $M^* = \hat{C}_1' \cup C_2$. By the maximality of D, we see that E is contained in M^*. Note that $\hat{C}_1' \cup_{\hat{P}'} C_2$ is a Heegaard splitting of M^*, and $\partial E \subset \partial_- C_2$. Hence, by [2, Lemma 1.1], we see that $\hat{C}_1' \cup_{\hat{P}'} C_2$ is weakly reducible, and this implies conclusion 1. □

By Claim 2, we may suppose that P^* is incompressible in $E(K)$. Note that each component of ∂P^* is a meridian loop of K. Since K is small and there does not exist a meridional essential surface in $E(K)$, we see that each component of P^* is a boundary parallel annulus properly embedded in $E(K)$. Recall that S is the union of the 2–sphere components of $\partial \hat{C}_2'' - \hat{P}'$. Note that we can assign labels C_1 and C_2 to the components of $E(K) - (P^* \cup S)$ alternately so that the C_2 region are contained in $\hat{C}_2'$, and that $\hat{P}'$ is recovered from $P^* \cup S$ by adding tubes along mutually disjoint arcs in C_1–regions. Recall that $\hat{P}'$ is connected. Since each component of $P^* \cup S$ is separating in $E(K)$, this shows that exactly one component of $E(K) - (P^* \cup S)$ is a C_1–region. Let $\bar{P}$ be a surface in M obtained from P^* by capping off the boundary components by mutually disjoint K–disks in $N(K)$ (hence, via isotopy, P' is recovered from $\bar{P} \cup S$ by adding the tubes used for recovering $\hat{P}'$ from $P^* \cup S$). Then each component of $\bar{P} \cup S$ is a K–sphere. Since M is K–irreducible, the components of $\bar{P} \cup S$ bounds K–balls, say $B_1, \ldots, B_m$, in M.

Claim 3 The K–balls $B_1, \ldots, B_m$ are mutually disjoint.

Proof Suppose not. By exchanging the subscript if necessary, we may suppose that $B_2 \subset B_1$. Since there exists exactly one C_1–region, this implies that the K–balls $B_2, \ldots, B_m$ are included in B_1 in a non-nested configuration. Hence P' is contained is the K–ball B_1. See Figure 6. Note that P is recovered from P' by adding a tube along the component of $K - P'$, which intersects D_1. Hence we see that P is contained in a regular neighborhood of K, say N_K.

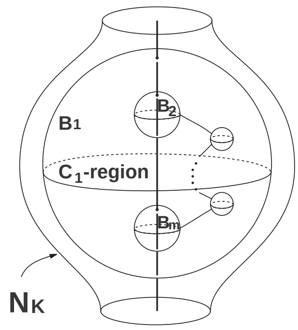

Figure 6

Note that $\mathrm{cl}(M - N_K)$ is contained in C_2. Since C_2 is a K–compression body, there exists a K–compressing disk D_N for ∂N_K in C_2. Suppose that $D_N \subset N_K$. Since N_K is a regular neighborhood of K, we see that $\mathrm{cl}(N_K - N(D_N; N_K))$ is a K–ball. Since C_2 is K–irreducible, $\mathrm{cl}(M - N_K) \cup N(D_N; N_K)$ is also a K–ball. These show that M is the 3–sphere, and K is a trivial knot, contradicting the condition 3 of the assumption of Proposition 5.1. Suppose that $D_N \subset \mathrm{cl}(M - N_K)$. Since M is K–irreducible, we see that $\mathrm{cl}(M - N_K)$ is irreducible. This shows that we obtain a 3–ball by cutting $\mathrm{cl}(M - N_K)$ along D_N. This shows that $\mathrm{cl}(M - N_K)$ is a solid torus. Hence $N_K \cup \mathrm{cl}(M - N_K)$ is a genus one Heegaard splitting of M. Hence K is a core knot, contradicting the condition 3 of the assumption of Proposition 5.1.

This completes the proof of Claim 3. □

Recall that P' is the surface obtained from P by K–compressing along D_1, and C_1', C_2' the closures of the components of $M - P'$, where C_1' is obtained from C_1 by cutting along D_1. By Proposition 2.16 and (3) of Remark 2.1, C_1' is a K–handlebody. By Claim 3, we see that the C_1–region is $\mathrm{cl}(M - (\cup_{i=1}^m B_i)) \cap E(K)$. Hence P' is recovered from $\partial B_1 \cup \cdots \cup \partial B_m$ by adding tubes along arcs properly embedded in $\mathrm{cl}(M - (\cup_{i=1}^m B_i))$. Hence, we see that C_2' is obtained from the

K–balls $B_1, \ldots, B_m$ by adding 1–handles disjoint from K. Hence C_2' is also a K–handlebody. These show that P' is a Heegaard surface for (M, K). It is clear that $C_1' \cup C_2'$ gives a genus $(g-1)$, $(n+1)$ bridge position of K, and $C_1 \cup C_2$ is obtained from $C_1' \cup C_2'$ by a tubing along a component of $K \cap (\cup B_i)$. Hence, by regarding $H_1 = C_1'$, $H_2 = C_2'$, we have conclusion 2 of Proposition 5.1.

Case 2 $D_2 \cap K \neq \emptyset$.

In this case, we first show the following.

Claim 1 If ∂D_i ($i = 1$ or 2) is separating in P, then we have conclusion of Proposition 5.1.

Proof Since the argument is symmetric, we may suppose that ∂D_2 is separating in P. This implies that D_2 is separating in C_2. Let C_2', C_2'' be the closures of the components of $C_2 - D_2$ such that $\partial D_1 \subset C_2''$. Then C_2' is a K–handlebody which is not a K–ball. Hence there exists a K–essential disk D_2' in C_2' such that $D_2' \cap K = \emptyset$, and $D_2' \cap D_2 = \emptyset$ (hence, D_2' is properly embedded in C_2). See Figure 5. It is clear that D_2' is a K–meridian disk of C_2. Hence by applying the arguments of Case 1 to the pair D_1, D_2', we have conclusion of Proposition 5.1. □

Let P' be the surface obtained from P by K–compressing along $D_1 \cup D_2$, and $\hat{P}' = P' \cap E(K)$. Let C_1', C_2' be the closures of the components of $M - P'$ such that C_1' is obtained from C_1 by cutting along D_1 and attaching $N(D_2; C_2)$, and C_2' is obtained from C_2 by cutting along D_2 and attaching $N(D_1; C_1)$. Then let $\hat{C}_i' = C_i' \cap E(K)$ $(i = 1, 2)$.

Claim 2 If $\hat{P}'$ is compressible in $E(K)$, then we have conclusion of Proposition 5.1.

Proof Suppose that there is a compressing disk D for $\hat{P}'$ in $E(K)$. Since the argument is symmetric, we may suppose that $D \subset \hat{C}_2'$. We may regard that D is a compressing disk for P'. Since P is recovered from P' by adding two tubes along a component of $K \cap C_1'$ and a component of $K \cap C_2'$, we may suppose that $D \cap P = \partial D$. Hence D is a K–meridian disk of C_2 such that $D \cap K = \emptyset$. Hence, by applying the arguments of Case 1 to the pair D_1, D, we have the conclusion of Proposition 5.1. □

By Claims 1 and 2, we see that, for the proof of Proposition 5.1, it is enough to show that either (1) ∂D_i ($i = 1$ or 2) is separating in P, or (2) $\hat{P}'$ is compressible in $E(K)$. Suppose that ∂D_i ($i = 1, 2$) is non-separating in P, and that $\hat{P}'$ is incompressible in $E(K)$. Then, by the argument preceding Claim 3 of Case 1, we see that each component of P' is a K–sphere, and P is recovered from P' by adding tubes along two arcs a_1, a_2 such that a_i is a component of $K \cap C_i'$ ($i = 1, 2$), and that $a_1 \cap a_2 = \emptyset$. Note that P is connected. Since ∂D_1, ∂D_2 are non-separating in P, we see that P' consists of one K–sphere, or two K–spheres, and this shows that $K \cap C_i'$ consists of one arc, or two arcs. But since K is a knot, we have $a_1 \cap a_2 \neq \emptyset$ in either case, a contradiction. Hence we have the conclusion of Proposition 5.1 in Case 2.

This completes the proof of Proposition 5.1. □

6 Heegaard splittings of (S^3, two bridge knot)

In this section, we prove the following.

Proposition 6.1 *Let K be a non-trivial two bridge knot, and $X \cup_Q Y$ a Heegaard splitting of S^3, which gives a genus g, n–bridge position of K. Suppose that $(g, n) \neq (0, 2)$. Then $X \cup_Q Y$ is weakly K–reducible.*

Proposition 6.2 *Let K be a non-trivial two bridge knot. Then, for each $g \geq 3$, every genus g Heegaard splitting of the exterior $E(K)$ of K is weakly reducible.*

6.A Comparing $X \cup_Q Y$ with a two bridge position

Let $A \cup_P B$ be a genus 0 Heegaard splitting of S^3, which gives a 2–bridge position of K. Then, by [14, Corollary 6.22] (if $n \geq 1$) or by [13, Corollary 3.2] (if $n = 0$), we have the following.

Proposition 6.3 *Let $X \cup_Q Y$ be a Heegaard splitting of S^3, which gives a genus g, n–bridge position of K. If $X \cup_Q Y$ is strongly K–irreducible, then Q is K–isotopic to a position such that $P \cap Q$ consists of non-empty collection of transverse simple closed curves which are K–essential in both P and Q.*

In this subsection, we prove the following proposition.

Proposition 6.4 *Let $X \cup_Q Y$ be a Heegaard splitting of S^3, which gives a genus g, n–bridge position of K with $(g,n) \neq (0,2)$. Suppose that $P \cap Q$ consists of non-empty collection of transverse simple closed curves which are K–essential in both P and Q. Then $X \cup_Q Y$ is weakly K–reducible.*

We note that Proposition 6.1 is a consequence of Propositions 6.3 and 6.4.

Proof of Proposition 6.1 from Propositions 6.3 and 6.4 Let $X \cup_Q Y$ be a Heegaard splitting of S^3, which gives a genus g, n–bridge position of K with $(g,n) \neq (0,2)$. Suppose, for a contradiction, that $X \cup_Q Y$ is strongly K–irreducible. Then, by Propositions 6.3, we may suppose that $P \cap Q$ consists of non-empty collection of transverse simple closed curves which are K– essential in both P and Q. By Propositions 6.4, we see that $X \cup_Q Y$ is weakly K–reducible, a contradiction. □

Proof of Proposition 6.4 First of all, we would like to remark that the proof given below is just an orbifold version of the proof of [17, Corollary 6.4]. We suppose that $|P \cap Q|$ is minimal among all surfaces P such that P gives a two bridge position of K, and that $P \cap Q$ consists of non-empty collection of simple closed curves which are K–essential in both P and Q. Note that the closure of each component of $P - Q$ is either an annulus which is disjoint from K, or a disk intersecting K in two points. We divide the proof into several cases.

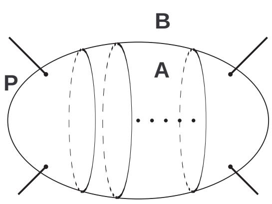

Figure 7

Case 1 Each component of $P \cap X$ is not K–boundary parallel in X, and each component of $P \cap Y$ is not K–boundary parallel in Y.

Case 1 is divided into the following subcases.

Case 1.1 $P \cap X$ contains a component which is K–compressible in X, and $P \cap Y$ contains a component which is K–compressible in Y

In this case, by K–compressing the components in X and Y, we obtain K–meridian disks D_X, D_Y in X, Y respectively. By applying a slight K–isotopy if necessary, we may suppose that $D_X \cap D_Y = \emptyset$, and this shows that $X \cup_Q Y$ is weakly K–reducible.

Case 1.2 Either $P \cap X$ or $P \cap Y$, say $P \cap X$, contains a component which is K–compressible in X, and each component of $P \cap Y$ is K–incompressible in Y.

Let D_X be a K–meridian disk obtained by K–compressing the component of $P \cap X$. Note that $P \cap Y$ is K–boundary compressible in Y (Corollary 2.18). Let D_Y be a disk in Y obtained by K–boundary compressing $P \cap Y$. By the minimality of $|P \cap Q|$, we see that D_Y is a K–meridian disk in Y. Since ∂D_X is a component of $P \cap Q$, we may suppose that $D_X \cap D_Y = \emptyset$ by applying a slight K–isotopy if necessary. This shows that $X \cup_Q Y$ is weakly K–reducible.

Case 1.3 Each component of $P \cap X$ is K–incompressible in X, and each component of $P \cap Y$ is K–incompressible in Y.

Let D_X (D_Y respectively) be a K–meridian disk obtained by K–boundary compressing $P \cap X$ ($P \cap Y$ respectively). By applying slight isotopies, we may suppose that $\partial D_X \cap P = \emptyset$, $\partial D_Y \cap P = \emptyset$ (hence, $\partial D_X \subset A$ or B, $\partial D_Y \subset A$ or B). If one of ∂D_X or ∂D_Y is contained in A, and the other in B, then $D_X \cap D_Y = \emptyset$, and this shows that $X \cup_Q Y$ is weakly K–reducible. Suppose that $\partial D_X \cup \partial D_Y$ is contained in A or B, say A. Let D_B be a K–meridian disk in B (ie, D_B is a disk properly embedded in B such that $D_B \cap K = \emptyset$, and D_B separates the components of $K \cap B$). Note that since each component of $P \cap X$, $P \cap Y$ is K–incompressible, $D_B \cap Q \neq \emptyset$. We take D_B so that $|D_B \cap Q|$ is minimal among all K–essential disks D' in B such that each component of $D' \cap (P \cap X)$ ($D' \cap (P \cap Y)$ respectively) is a K–essential arc properly embedded in $P \cap X$ ($P \cap Y$ respectively). Suppose that $D_B \cap Q$ contains a simple closed curve component. Let D ($\subset D_B$) be an innermost disk. Since the argument is symmetric, we may suppose that $D \subset X$. By the minimality of $|D_B \cap Q|$, we see that D is a K–meridian disk in X. Since $D \subset B$, $\partial D \cap \partial D_Y = \emptyset$. Hence the pair D, D_Y gives a weak K–reducibility of $X \cup_Q Y$. Suppose that each component of $D_B \cap Q$ is an arc. Let Δ ($\subset D_B$) be an outermost disk. Since the argument is symmetric, we may suppose that $\Delta \subset X$. Recall that $\Delta \cap (P \cap X)$ is a K–essential arc in $P \cap X$. By the minimality of $|D_B \cap Q|$, we see that at least one component, say D^*, of the surface obtained from $P \cap X$ by K–boundary compressing along Δ is a K–meridian disk in X. Since $\Delta \subset B$,

$\partial D \cap \partial D_Y = \emptyset$. Hence the pair D, D_Y gives a weak K–reducibility of $X \cup_Q Y$.

Case 2 A component of $P \cap X$ or $P \cap Y$, say $P \cap Y$, is K–boundary parallel in Y.

By the minimality of $|P \cap Q|$, we have either $|P \cap Q| = 1$ (and $P \cap Y$ ($P \cap X$ respectively) is a disk intersecting K in two points) or, $|P \cap Q| = 2$ (and $P \cap Y$ is an annulus disjoint from K).

Case 2a $|P \cap Q| = 1$.

Let $P_X = P \cap X$ and $P_Y = P \cap Y$. Let E be the closure of the component of $Q - P$ such that E and P_Y are K–parallel in Y. Since the argument is symmetric, we may suppose that $E \subset A$. We have the following subcases.

Case 2a.1 P_X is K–boundary parallel in X

Since $(g, n) \neq (0, 2)$, P_X is parallel to E in A, and cannot be parallel to $\mathrm{cl}(Q - E)$.

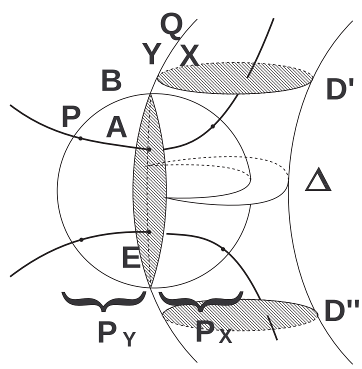

Figure 8

Let D_B be a K–meridian disk in B. Since K is not a trivial knot, ∂D_B and ∂E are not isotopic in $P - K$. Hence $D_B \cap Q \neq \emptyset$. We suppose that $|D_B \cap Q|$ is minimal among all K–meridian disks D' in B such that each component of $D' \cap P_X$ ($D' \cap P_Y$ respectively) is a K–essential arc in P_X (P_Y respectively). Suppose that $D_B \cap Q$ contains a simple closed curve. Let D ($\subset D_B$) be an innermost disk. Since the argument is symmetric, we may suppose that $D \subset X$. By the minimality of $|D_B \cap Q|$, we see that D is a

K{meridian disk in X. Then by pushing D_Y into X along the parallelism through E, we can K{isotope P to P^0 such that P^0 Int$X$, and $P^0 \setminus D = ;$. Hence, by Proposition 3.4, we see that $X [_Q Y$ is weakly K{reducible. Suppose that each component of $D_B \setminus Q$ is an arc. Let (D_B) be an outermost disk. Since the argument is symmetric, we may suppose that X. See Figure 8.

Claim At least one component of the disks obtained from P_X by K{boundary compressing along is a K{meridian disk.

Proof Let D^0, D^{00} be the disks obtained from P_X by K{boundary compressing along . Suppose that D^0 is K{boundary parallel, ie, there exists a K{disk D_Q in Q such that $@D_Q = @D^0$. Note that since $D^0 [D^{00}$ is obtained from P_X by K{boundary compressing along , there is an annulus A_Q is Q such that $@A_Q = @D^0 [@D^{00}$, and that $A_Q \setminus K = E \setminus K$: two points. Note also that $D_Q \setminus K$ consists of one point. Hence A_Q is not contained in D_Q, and this implies that $A_Q \setminus D_Q = @D^0$. Then $A_Q [D_Q$ is a disk intersecting K in three points, whose boundary is $@D^{00}$. Since $(g; n) \neq (0; 2)$, cl$(Q - (A_Q [D_Q))$ is not a K{disk. Hence D^{00} is a K{meridian disk in X. □

Let D^{00} be a K{meridian disk in X obtained as in Claim. By applying a slight isotopy, we may suppose that $P \setminus D^{00} = ;$. Then by pushing P_Y into X along the parallelism through E, we can K{isotope P to P^0 such that P^0 Int$X$, and $P^0 \setminus D^{00} = ;$. Hence, by Proposition 3.4, we see that $X [_Q Y$ is weakly K{reducible.

Case 2a.2 P_X is not K{boundary parallel in X, and P_X is K{incompressible in X, ie, P_X is K{essential in X.

Since P_X is K{incompressible, there is a K{boundary compressing disk for P_X in X.

Claim B.

Proof Suppose that A. Note that $K \setminus E$ consists of two points in Int$E$, and $\setminus E$ is an arc properly embedded in E, which separates the points. Then, by K{boundary compressing P_X along , we obtain two K{disks. Since X is K{irreducible, these K{disks are K{boundary parallel in X. This shows that P_X is K{boundary parallel in X, contradicting the condition of Case 2a.2. □

Then, by using the argument of the proof of Claim of Case 2a.1, we see that at least one component, say D'', of the K-disks obtained from P_X by K-boundary compressing along is a K-meridian disk in X. By applying a slight isotopy, we may suppose that $D'' \cap P = \emptyset$. By Claim, we see that $\partial D''$ B. Then by pushing P_Y into X along the parallelism through E, we can K-isotope P to P' such that P' Int$X$, and $P' \cap D'' = \emptyset$. Hence, by Proposition 3.4, we see that $X \cup_Q Y$ is weakly K-reducible.

Case 2a.3 P_X is not K-boundary parallel in X, and P_X is K-compressible in X.

Let D be the K-compressing disk for P_X. Since there does not exist a 2-sphere (S^3) intersecting K in three points, $D \cap K = \emptyset$. Let D be the disk component of a surface obtained from P_X by K-compressing along D. Since $(g, n) \neq (0, 2)$, we see that D is a K-meridian disk of X. By applying a slight isotopy, we may suppose that $D \cap P = \emptyset$. Suppose that D B. Then by pushing D_Y into X along the parallelism through E, we can K-isotope P to P' such that P' Int$X$, and $P' \cap D = \emptyset$. Hence, by Proposition 3.4, we see that $X \cup_Q Y$ is weakly K-reducible. Hence, in the rest of this subcase, we suppose that D A (Figure 9). Let D_B be a K-meridian disk in B. Since K is not a trivial two component link, ∂D and ∂D_B are not isotopic in $P - K$. Hence $D_B \cap Q \neq \emptyset$. We suppose that $|D_B \cap Q|$ is minimal among all K-meridian disks D' in B such that each component of $D' \cap P_X$ ($D' \cap P_Y$ respectively) is a K-essential arc in P_X (P_Y respectively).

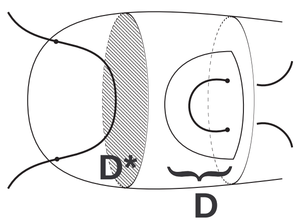

Figure 9

Suppose that $D_B \cap Q$ contains a simple closed curve. Let D (D_B) be an innermost disk. By the minimality of $|D_B \cap Q|$, we see that ∂D is K-essential in Q. Note that ∂D B. If D Y, then the pair D , D gives a weak K-reducibility of $X \cup_Q Y$. If D X, then by pushing P_Y into X along the parallelism through E, we can K-isotope P to P' such that P' Int$X$, and $P' \cap D = \emptyset$. Hence, by Proposition 3.4, we see that $X \cup_Q Y$ is weakly K-reducible.

Suppose that each component of $D_B \cap Q$ is an arc. Let Δ ($\subset D_B$) be an outermost disk. If $\Delta \subset X$, then by using the argument as in the proof of Case 2a.1, we see that $X \cup_Q Y$ is weakly K-reducible. Suppose that $\Delta \subset Y$. Then, by using the argument as in the proof of Claim of Case 2a.1, we can show that at least one component, say D'', of the K-disks obtained from P_Y by K-boundary compressing along Δ is a K-meridian disk in Y. By applying slight K-isotopy, we may suppose that $D'' \subset B$. Hence the pair D^*, D'' gives a weak K-reducibility of $X \cup_Q Y$.

Case 2b $|P \cap Q| = 2$.

Let D_1, D_2 be the components of $P \cap X$, and $A_1 = P \cap Y$. Recall that A_1 is a K-boundary parallel annulus in Y such that $A_1 \cap K = \emptyset$, and that D_1, D_2 are not K-boundary parallel. We also note that $\partial D_1 \cup \partial D_2$ bounds an annulus A' in Q such that A_1 and A' are K-parallel in Y. Without loss of generality, we may suppose that A' is contained in the 3-ball A.

Case 2b.1 $D_1 \cup D_2$ is K-incompressible in X.

Since $D_1 \cup D_2$ is K-incompressible, there is a K-boundary compressing disk Δ for $D_1 \cup D_2$. Without loss of generality, we may suppose that $\Delta \cap D_1 \neq \emptyset$, $\Delta \cap D_2 = \emptyset$. Since D_1 is not K-boundary parallel, at least one component, say D^*, of the K-disks obtained from D_1 by K-boundary compressing along Δ is a K-meridian disk in X. By applying a slight K-isotopy, we may suppose that $D^* \cap P = \emptyset$.

Claim $D^* \subset B$.

Proof Suppose, for a contradiction, that $D^* \subset A$. Then ∂D^* is contained in the annulus A' bounded by $\partial D_1 \cup \partial D_2$. We note that D^* intersects K in one point. Hence ∂D^* is not contractible in Q. This shows that ∂D^* is a core curve of A'. Let A'' be the annulus in A' bounded by $\partial D^* \cup \partial D_1$. Then the 2-sphere $D_1 \cup A'' \cup D^*$ intersects K in three points, a contradiction. □

By Claim we see that, by pushing A_1 into X along the parallelism through A', we can K-isotope P to P' such that $P' \subset \mathrm{Int} X$. By the above claim, we may suppose that $P' \cap D^* = \emptyset$. Hence, by Proposition 3.4, we see that $X \cup_Q Y$ is weakly K-reducible.

Case 2b.2 $D_1 \cup D_2$ is K-compressible.

Let D be the K–compressing disk for $D_1 \cup D_2$. Without loss of generality, we may suppose that $D \cap D_1 \neq \emptyset$, $D \cap D_2 = \emptyset$. Let D^* be a K–meridian disk of X obtained from D_1 by K–compressing along D. By applying slight isotopy, we may suppose that $D^* \cap P = \emptyset$. Suppose that $D^* \subset B$. By pushing A_1 into X along the parallelism through A', we can K–isotope P to P' such that $P' \subset \text{Int}X$, and $P' \cap D^* = \emptyset$. Hence, by Proposition 3.4, we see that $X \cup_Q Y$ is weakly K–reducible.

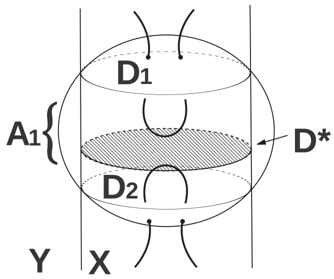

Figure 10

Suppose that $D^* \subset A$ (Figure 10). Let D_B be a K–meridian disk in B. Since K is not a trivial two component link, ∂D^* and ∂D_B are not isotopic in $P - K$. Hence $D_B \cap Q \neq \emptyset$. We suppose that $|D_B \cap Q|$ is minimal among all K–essential disks D' in B such that each component of $D' \cap D_1$ ($D' \cap D_2$, $D' \cap A_1$ respectively) is a K–essential arc in D_1 (D_2, A_1 respectively). Suppose that $D_B \cap Q$ contains a simple closed curve component. Let D' be an innermost disk. By the minimality of $|D_B \cap Q|$, we see that $\partial D'$ is K–essential in Q. Note that $\partial D' \subset B$. If $D' \subset Y$, then the pair D', D^* gives a weak K–reducibility of $X \cup_Q Y$. If $D' \subset X$, then by pushing A_1 into X along the parallelism through A', we can K–isotope P to P' such that $P' \subset \text{Int}X$, and $P' \cap D' = \emptyset$. Hence, by Proposition 3.4, we see that $X \cup_Q Y$ is weakly K–reducible. Suppose that each component of $D_B \cap Q$ is an arc. Let Δ ($\subset D_B$) be an outermost disk. If $\Delta \subset X$, then by using the argument as in the proof of Case 2b.1, we see that $X \cup_Q Y$ is weakly K–reducible. Suppose that $\Delta \subset Y$. Let D^* be the disk obtained from A_1 by K–boundary compressing along Δ.

Claim D^* is a K–meridian disk of Y.

Proof Suppose that D^* is not a K–meridian disk of Y, ie, D^* is K–parallel to a disk, say D'', in $\partial Y (= Q)$. Since $\Delta \subset B$, we see that $D'' \subset \text{cl}(Q - A')$. Note that $\text{cl}(Q - A')$ is recovered from D'' by adding a band along an arc intersecting $\Delta \cap Q$ in one point. This shows that $\text{cl}(Q - A')$ is an annulus not intersecting K. Hence Q is a torus, and X is a solid torus such that $Q \cap K = \emptyset$. However, since D is a meridian disk of X, this implies that X is K–reducible, contradicting Corollary 2.17. □

By Claim, we see that, by applying a slight isotopy, we may suppose that $D \cap P = \emptyset$, and $D \subset B$. Hence the pair D, D gives a weak K–reducibility of $X \cup_Q Y$.

This completes the proof of Proposition 6.4 □

6.B Proof of Proposition 6.2

Let K be a non-trivial two bridge knot, and $C \cup_P V_2$ a genus g Heegaard splitting of $E(K)$ with $g \geq 3$. Note that K satisfies the conditions of the assumption of Proposition 5.1. Let V_1 be the handlebody in S^3 such that $\partial V_1 = P$, and $C \subset V_1$. Then $V_1 \cup V_2$ is a Heegaard splitting of S^3 which gives a genus g, 0–bridge position of K. By Propositions 6.1 and 5.1, we have either one of the following.

(1.1) There exists a weakly K–reducing pair of disks D_1, D_2 for $V_1 \cup V_2$ such that $D_1 \cap K = \emptyset$, and $D_2 \cap K = \emptyset$.

(1.2) There exists a Heegaard splitting $V_{1,1} \cup_{P_1} V_{1,2}$ of (S^3, K) which gives a genus $(g-1)$, 1–bridge position of K such that $V_1 \cup V_2$ is obtained from $V_{1,1} \cup_{P_1} V_{1,2}$ by a tubing.

If (1.1) holds, then we immediately have the conclusion of Propositions 6.2. If (1.2) holds, then we further apply Propositions 6.1 and 5.1, and we have either one of the following.

(2.1) There exists a weakly K–reducing pair of disks D_1, D_2 for $V_{1,1} \cup_{P_1} V_{1,2}$ such that $D_1 \cap K = \emptyset$, and $D_2 \cap K = \emptyset$.

(2.2) There exists a Heegaard splitting $V_{2,1} \cup_{P_2} V_{2,2}$ of (S^3, K) which gives a genus $(g-2)$, 2–bridge position of K such that $V_{1,1} \cup_{P_1} V_{1,2}$ is obtained from $V_{2,1} \cup_{P_2} V_{2,2}$ by a tubing.

We claim that if (2.1) holds, then we have the conclusion of Propositions 6.2. In fact, since $D_1 \cap K = \emptyset$, and $D_2 \cap K = \emptyset$, and tubing operations are performed in a small neighborhood of K, the pair D_1, D_2 survives in $V_1 \cup V_2$ to give a weak reducibility. If (2.2) holds, then we further apply Propositions 6.1 and 5.1, and we have either one of the following.

(3.1) There exists a weakly K–reducing pair of disks D_1, D_2 for $V_{2,1} \cup_{P_2} V_{2,2}$ such that $D_1 \cap K = \emptyset$, and $D_2 \cap K = \emptyset$.

(3.2) There exists a Heegaard splitting $V_{3,1} \cup_{P_3} V_{3,2}$ of (S^3, K) which gives a genus $(g-3)$, 3–bridge position of K such that $V_{2,1} \cup_{P_2} V_{2,2}$ is obtained from $V_{3,1} \cup_{P_3} V_{3,2}$ by a tubing.

Then we apply the same argument as above, and so on. Then either we have the conclusion of Propositions 6.2, or the procedures are repeated $(g-1)$ times to give the following.

(g.1) There exists a weakly K–reducing pair of disks D_1, D_2 for $V_{g-1,1} \cup_{P_{g-1}} V_{g-1,2}$ such that $D_1 \cap K = \emptyset$, and $D_2 \cap K = \emptyset$.

(g.2) There exists a Heegaard splitting $V_{g,1} \cup_{P_g} V_{g,2}$ of (S^3, K) which gives a genus 0, g–bridge position of K such that $V_{g-1,1} \cup_{P_{g-1}} V_{g-1,2}$ is obtained from $V_{g,1} \cup_{P_g} V_{g,2}$ by a tubing.

If (g.1) holds, then by using the arguments as above, we see that we have the conclusion of Propositions 6.2. Suppose that (g.2) holds. Then we see that there exists a weakly reducing pair of disks D_1, D_2 for $V_{g,1} \cup_{P_g} V_{g,2}$ such that $D_1 \cap K = \emptyset$, and $D_2 \cap K = \emptyset$ (see Remark 5.2), and this together with the arguments as for the case (g.1), we see that we have the conclusion of Propositions 6.2.

This completes the proof of Propositions 6.2.

7 Proof of Theorem 1.1

Let K be a knot in a closed 3–manifold M.

Definition 7.1 A *tunnel* for K is an embedded arc γ in S^3 such that $\gamma \cap K = \partial\gamma$. We say that a tunnel γ for K is *unknotting* if $S^3 - \mathrm{Int}\, N(K \cup \gamma; S^3)$ is a genus two handlebody.

For a tunnel for K, let ^ = \ $E(K)$. Then ^ is an arc properly embedded in $E(K)$, and we may regard that $N(K$ [ ) is obtained from $N(K)$ by attaching N(^; $E(K)$), where N(^; $E(K)$) \ $N(K)$ consists of two disks, ie, N(^; $E(K)$) is a 1{handle attached to $N(K)$.

De nition 7.2 Let $_1$, $_2$ be tunnels for K. We say that $_1$ is *isotopic* to $_2$ if there is an ambient isotopy h_t (0 t 1) of $E(K)$ such that $h_0 = \mathrm{id}_{E(K)}$, and h_1(^$_1$) = ^$_2$.

Remark 7.3 Let be an unknotting tunnel for K, and let $V = N(K$ [; $M)$, and $W = \mathrm{cl}(M - V)$. Note that V [W is a Heegaard splitting of $(M; K)$, which gives a genus two, 0{bridge position of K. Let $_1$, $_2$ be unknotting tunnels for K, and V_1 [$_{P_1}$ W_1, V_2 [$_{P_2}$ W_2 Heegaard splittings obtained from $_1$, $_2$ respectively as above. Then it is known that $_1$ is isotopic to $_2$ if and only if P_1 is K{isotopic to P_2.

Now, in the rest of this paper, let K be a non-trivial 2{bridge knot, and A [$_P$ B a genus 0 Heegaard splitting of S^3, which gives a two bridge position of K (Figure 11).

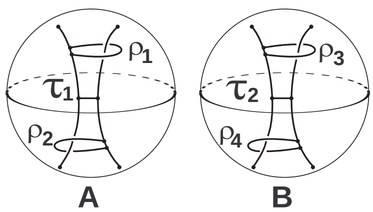

Figure 11

7.A Genus two Heegaard splittings of $E(K)$

Here we show the next lemma on unknotting tunnels of K, which is used in the proof of Theorem 1.1.

Lemma 7.4 *Let be an unknotting tunnel for K, and V [W a Heegaard splitting obtained from as in Remark 7.3. Then there exist meridian disks D_1, D_2 of V, W respectively such that D_1 intersects K transversely in one point, D_1 \ N(^; $E(K)$) = ; , and @D_1 intersects @D_2 transversely in one point.*

Proof We note that   is isotopic to either one of the six unknotting tunnels $_1$, $_2$, $_1$, $_2$, $_3$, or $_4$ in Figure 11 (see [6] or [13]). Suppose that   is isotopic to $_i$, $i = 1$ or 2, say 1. Then we may regard that $V = A$ [$N(K$ \ $B;B)$ (Figure 12). Here $N(\hat{}; E(K)) = N(D_A;A)$, where D_A is a disk properly embedded in A, such that D_A separates the components of K \ A, and $N(D_A;A)$ \ $N(K$ \ $B;B) = ;$ (hence, D_A is properly embedded in V). Then we can take a pair D_1, D_2 satisfying the conclusion of Lemma 7.4 as in Figure 12.

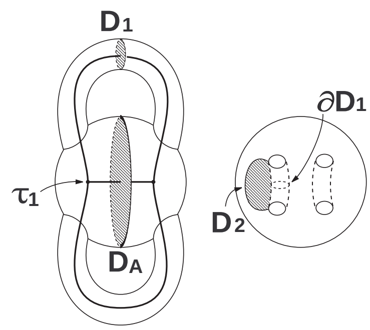

Figure 12

Suppose that   is isotopic to $_i$, $i = 1$, 2, 3, or 4, say 1. Then we may regard that V is obtained from the Heegaard splitting A [$_P$ B of $(S^3;K)$ as follows.

Let a be the component of K \ A, which is disjoint from , and $V^0 = \mathrm{cl}(A - N(a;A))$, $W^0 = B$ [$N(a;A)$. Let $a^0 = a$ [$(K$ \ $B)$. Note that a^0 is an arc properly embedded in W^0. Then $V = V^0$ [$N(a^0;W^0)$. See Figure 13. That is, V [W is obtained from A [$_P$ B by successively tubing along a, and a^0. We can take a pair D_1, D_2 satisfying the conclusion of Lemma 7.4, as in Figure 13. □

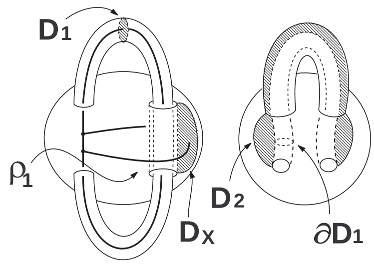

Figure 13

7.B Irreducible Heegaard splittings of (torus) $\times$ $[0,1]$

In [1], M Boileau, and J-P Otal gave a classification of Heegaard splittings of (torus) $\times$ $[0,1]$, and M.Scharlemann, and A.Thompson [18] proved that the same kind of results hold for $F \times [0,1]$, where F is any closed orientable surface. The result of Boileau–Otal will be used for the proof of Theorem 1.1, and in this section we quickly state it.

Let T be a torus. Let Q_1 be the surface $T \times \{1/2\}$ in $T \times [0,1]$. It is clear that Q_1 separates $T \times [0,1]$ into two trivial compression bodies. Hence Q_1 is a Heegaard surface of $T \times [0,1]$. We call this Heegaard splitting *type 1*. Let a be a vertical arc in $T \times [0,1]$. Let $V_1 = N((T \times \{0,1\}) \cup a; T \times [0,1])$, and $V_2 = \mathrm{cl}(T \times [0,1] - V_1)$. It is easy to see that V_1 is a compression body, V_2 is a genus two handlebody, and $V_1 \cap V_2 = \partial_+ V_1 = \partial_+ V_2 (= \partial V_2)$. Hence $V_1 \cup V_2$ is a Heegaard splitting of $T \times I$. We call this Heegaard splitting *type 2*. Then in [1, Theoreme 1.5], or [18, Main theorem 2.11], the following is shown.

Theorem 7.5 *Every irreducible Heegaard splitting of* $T \times [0,1]$ *is isotopic to either a Heegaard splitting of type 1 or type 2.*

7.C Proof of Theorem 1.1

Let $C_1 \cup_P C_2$ be a genus g Heegaard splitting of the exterior of K, $E(K) = \mathrm{cl}(S^3 - N(K))$, with $g \geq 3$ and $\partial_- C_1 = \partial E(K)$. Then, by Proposition 6.2, we see that $C_1 \cup_P C_2$ is weakly reducible. By Proposition 4.2, either $C_1 \cup_P C_2$ is reducible, or there is a weakly reducing collection of disks Δ for P such that each component of $\hat{P}(\Delta)$ is an incompressible surface in $E(K)$, which is not a 2–sphere. Suppose that the second conclusion holds and let M_j $(j = 1, \ldots, n)$, $M_{j,i}$ $(i = 1,2)$, and $C_{1,1} \cup_{P_1} C_{1,2}, \cdots, C_{n,1} \cup_{P_n} C_{n,2}$ be as in Section 4. Note that each component of $\partial_- C_{i,j}$ is either $\partial E(K)$ or a closed incompressible surface in $\mathrm{Int}E(K)$. Since every closed incompressible surface in $\mathrm{Int}E(K)$ is a ∂–parallel torus, we see that the submanifolds $M_1, \ldots, M_n$ lie in $E(K)$ in a linear configuration, ie, by exchanging the subscripts if necessary, we may suppose that

1. (1) $\partial_- C_{1,1} = \partial E(K)$,
2. (2) For each i $(1 \leq i \leq n-1)$, M_i is homeomorphic to (torus) $\times$ $[0,1]$, and $M_i \cap M_{i+1} = F_i$: a ∂–parallel torus in $E(K)$.

Claim 1 If $n > 2$, then $C_1 \cup_P C_2$ is reducible.

Proof Let $M_1^0 = \mathrm{cl}(C_1 - M_{n,1})$, and $M_2^0 = \mathrm{cl}(C_2 - M_{n,2})$. Then from the pair M_1^0, M_2^0 we can obtain, as in Section 4, a Heegaard splitting, say $C_1^0 \cup_{P^0} C_2^0$, of the product region between F_{n-1} and $\partial E(K)$. Since $n > 2$, we see, by [20, Remark 2.7], that genus$(P^0) > 2$. Hence by Theorem 7.5, $C_1^0 \cup_{P^0} C_2^0$ is reducible. Hence, by Lemma 4.6, $C_1 \cup_P C_2$ is reducible. □

By Claim 1, we may suppose, in the rest of the proof, that $n = 2$. Now we prove Theorem 1.1 by the induction on g.

Suppose that $g = 3$. By Lemma 4.6, we may suppose that both $C_{1,1} \cup_{P_1} C_{1,2}$, and $C_{2,1} \cup_{P_1} C_{2,2}$ are irreducible. By Lemma 4.5 and Theorem 7.5, we see that $C_{1,1}$ is a genus 2 compression body with $\partial_- C_{1,1} = \partial E(K) \cup F_1$, and $C_{1,2}$ is a genus 2 handlebody.

Claim 2 $(M_{1,1} \cap P) \quad (M_{1,2} \cap P)$.

Proof Suppose not. Then, by Lemma 4.3, we see that $(M_{1,1} \cap P) \quad (M_{1,2} \cap P)$. Recall that $C_{1,1} = \mathrm{cl}(M_{1,1} - N(\partial_+ M_{1,1}, M_{1,1}))$. This implies that $\partial_- M_{1,1} = \partial_- C_{1,1}$. Note that $C_{1,1} \cup_{P_1} C_{1,2}$ is a Heegaard splitting of type 2 in Section 7.B. These show that $\partial_- M_{1,1} = \partial E(K) \cup F_1$. However, this is impossible since $\partial_- M_{1,1} \quad \partial E(K)$. □

By Claim 2, we see that $M_{1,2}$ is a genus two handlebody. Hence ${}_2$ is either one of Figure 14, ie, either (1) ${}_2$ consists of a non-separating disk in C_2, (2) ${}_2$ consists of a separating disk in C_2, or (3) ${}_2$ consists of two disks, one of which is a separating disk, and the other is a non-separating disk in C_2.

Suppose that ${}_2$ is of type (1) in Figure 14. Since no component of $\hat{P}(\ )$ is a 2-sphere, we see that $\partial\ {}_1 \quad M_{1,2}$. By Claim 2, we see that $(M_{2,1} \cap P) \quad (M_{2,2} \cap P)$. Since $\partial(M_{2,1} \cup M_{2,2}) = \partial M_2 = F_1$: a torus, we see that $M_{2,1}$ is a genus two handlebody, and ${}_1$ consists of a separating disk in C_1 (Figure 15).

Let $N_K = \mathrm{cl}(S^3 - M_2)$. Since F_1 is a ∂-parallel torus in $E(K)$, we see that N_K is a regular neighborhood of K, hence M_2 is an exterior of K. Note that $M_{2,2}$ is a 1-handle attached to N_K such that $\mathrm{cl}(S^3 - (N_K \cup M_{2,2})) = M_{2,1}$, a genus two handlebody. This shows that $M_{2,2}$ is a regular neighborhood of an arc properly embedded in M_2, which comes from an unknotting tunnel of K. Hence, by Lemma 7.4, we see that there is a pair of disks D_1, D_2 in $N_K \cup M_{2,2}$, $M_{2,1}$ respectively such that D_1 intersects K transversely in one point, $D_1 \cap M_{2,2} = \emptyset$, and ∂D_1 intersects ∂D_2 transversely in one point. Here, by deforming D_2 by an ambient isotopy of $M_{2,1}$ if necessary, we may suppose that $D_2 \cap\ {}_1 = \emptyset$

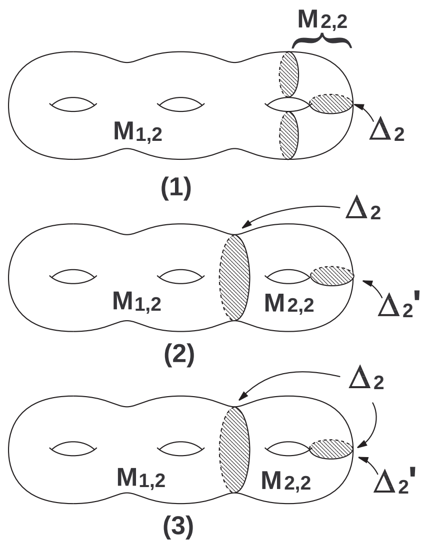

Figure 14

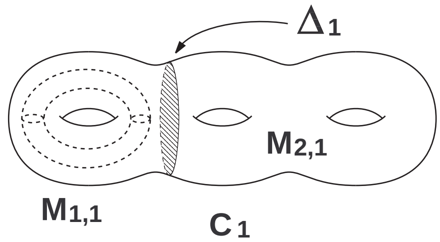

Figure 15

(hence, D_2 is a meridian disk of C_1). Since D_1 and K intersect transversely in one point, we may suppose that $D_1 \cap E(K)$ $(= D_1 \cap M_1)$ is a vertical annulus, say A_1, properly embedded in M_1 $(= T^2 \times [0,1])$. Recall that $C_{1,1} \cup_{P_1} C_{1,2}$ is a type 2 Heegaard splitting of M_1. This implies that there exists a vertical arc a in M_1 such that $M_{1,1} = N(\partial E(K) \cup a; M_1)$. Since a is vertical, we may suppose, by isotopy, that $a \subset A_1$, ie, a is an essential arc properly embedded in A_1. Let ℓ be the component of ∂A_1 contained in $\partial E(K)$. Hence $A_1 \cap C_2 = A_1 \cap M_{1,2}$ $= \mathrm{cl}(A_1 - N(\ell \cup a; M_1))$, and this is a disk, say D_1', properly embedded in C_2. Obviously $\partial D_1'$ and ∂D_2 intersect transversely in one point. Recall that D_2 (D_1' respectively) is a disk properly embedded in C_1 (C_2 respectively). Hence $C_1 \cup_P C_2$ is stabilized and this shows that $C_1 \cup_P C_2$ is reducible if $g = 3$ (see 2 of Remark 2.3).

Suppose that $_2$ is of type (2) or (3) in Figure 14. Then we take $^{0}_{2}$ as in Figure 14, and let $^{0} = {}_1 \cup {}^{0}_{2}$. We note that ^{0} is a weakly reducing collection of disks for P, where ^{0} is of type (1) in Figure 14. Let F_1^0 be the torus obtained from ^{0}, which is corresponding to F_1. It is directly observed from Figure 14 that F_1^0 is isotopic to F_1. Hence we can apply the argument for type 1 weakly reducing collection of disks to ^{0}, and we can show that $C_1 \cup_P C_2$ is reducible.

Suppose that g 4. If genus$(P_1) > 2$, then by Theorem 7.5 and Lemma 4.6, we see that $C_1 \cup_P C_2$ is reducible. Suppose that genus$(P_1) = 2$. Then, by [20, Remark 2.7], we see that genus$(P_2) = g - 1$. Hence, by the assumption of the induction, we see that $C_{2;1} \cup_{P_2} C_{2;2}$ is reducible. Hence, by Lemma 4.6, $C_1 \cup_P C_2$ is reducible.

This completes the proof of Theorem 1.1.

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