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Intersections in hyperbolic manifolds

Igor Belegradek

Department of Mathematics and Statistics
McMaster University
1280 Main St West
Hamilton, ON L8S 4K1
Canada

Email: `belegi@icarus.math.mcmaster.ca`

Abstract

We obtain some restrictions on the topology of infinite volume hyperbolic manifolds. In particular, for any n and any closed negatively curved manifold M of dimension ≥ 3, only finitely many hyperbolic n–manifolds are total spaces of orientable vector bundles over M.

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1 Introduction

A *hyperbolic manifold* is, by definition, a quotient of a negatively curved symmetric space by a discrete isometry group that acts freely. Recall that negatively curved symmetric spaces are hyperbolic spaces over the reals, complex numbers, quaternions, or Cayley numbers.

The homotopy type of a hyperbolic manifold is determined by its fundamental group. Conversely, for finite volume hyperbolic manifolds, fundamental group determines the diffeomorphism type, thanks to the Mostow rigidity theorem. By contrast infinite volume hyperbolic manifolds with isomorphic fundamental groups can be very different topologically. For example, the total spaces of many plane bundles over closed surfaces carry hyperbolic metrics (this and other examples are discussed in the section 2).

In this paper we attempt to count hyperbolic manifolds up to intersection preserving homotopy equivalence. A homotopy equivalence $f\colon N \to L$ of oriented n–manifolds is called *intersection preserving* if, for any k and any pair of (singular) homology classes $\alpha \in H_k(N)$ and $\beta \in H_{n-k}(N)$, their intersection number in N is equal to the intersection number of $f_*\alpha$ and $f_*\beta$ in L. For example, any map that is homotopic to an orientation-preserving homeomorphism is an intersection preserving homotopy equivalence. Conversely, oriented rank two vector bundles over a closed oriented surface are isomorphic iff their total spaces are intersection preserving homotopy equivalent.

Theorem 1.1 *Let π be the fundamental group of a finite aspherical cell complex and let X be a negatively curved symmetric space. Let ρ_k be a sequence of discrete injective representations of π into the group of orientation-preserving isometries of X. Suppose that ρ_k is precompact in the pointwise convergence topology.*

Then the sequence of manifolds $X/\rho_k(\pi)$ falls into finitely many intersection preserving homotopy equivalence classes.

The space of conjugacy classes of faithful discrete representations of a group π into the isometry group of a negatively curved symmetric space is compact provided π is finitely presented, not virtually nilpotent and does not split over a virtually nilpotent group (see 10.1). Since all the intersections in a hyperbolic manifold with virtually nilpotent fundamental group are zero (see 12.1), we get the following.

Corollary 1.2 *Let π be the fundamental group of a finite aspherical cell complex that does not split as an HNN–extension or a nontrivial amalgamated product over a virtually nilpotent group.*

Then, for any n, the class of orientable hyperbolic n–manifolds with fundamental group isomorphic to π breaks into finitely many intersection preserving homotopy equivalence classes.

A particular case of 1.2 was proved by Kapovich in [22]. Namely, he gave a proof for *real* hyperbolic $2m$–manifolds homotopy equivalent to a closed orientable negatively curved manifold of dimension $m \geq 3$.

If π *does* split over a virtually nilpotent group, the space of representations is usually noncompact. For instance, this happens if π is a surface group. Yet, for real hyperbolic 4–manifolds homotopy equivalent to closed surfaces, Kapovich [22] proved a result similar to 1.2.

More generally, Kapovich [22] proved that there is a universal function $C(-,-)$ such that for any incompressible singular surfaces Σ_{g_1}, Σ_{g_2} in an oriented real hyperbolic 4–manifold we have $|\langle[\Sigma_{g_1}],[\Sigma_{g_2}]\rangle| \leq C(g_1,g_2)$. Reznikov [34] showed that for any singular surfaces Σ_{g_1}, Σ_{g_2} in a closed oriented negatively curved 4–manifold M with the sectional curvature pinched between $-k^2$ and $-K^2$ there is a bound $|\langle[\Sigma_{g_1}],[\Sigma_{g_2}]\rangle| \leq C(g_1,g_2,k,K,\chi(M))$ for some universal function C.

Another way to classify hyperbolic manifolds is up to tangential homotopy equivalence. Recall that a homotopy equivalence of smooth manifolds $f\colon N \to L$ is called tangential if the vector bundles f^*TL and TN are stably isomorphic. For example, any map that is homotopic to a diffeomorphism is a tangential homotopy equivalence. Conversely, a tangential homotopy equivalence of open n–manifolds is homotopic to a diffeomorphism provided $n > 4$ and each of the manifolds is the total space of a vector bundle over a manifold of dimension $< n/2$ [25, pp 226–228].

An elementary argument (based on finiteness of the number of connected components of representation varieties and on finiteness of the number of symmetric spaces of a given dimension) yields the following.

Theorem 1.3 *Let π be a finitely presented group. Then, for any n, the class of complete locally symmetric nonpositively curved Riemannian n–manifolds with the fundamental group isomorphic to π falls into finitely many tangential homotopy types.*

Knowing both the intersection preserving and tangential homotopy types sometimes suffices to recover the manifold up to finitely many possibilities.

Theorem 1.4 *Let M be a closed orientable negatively curved manifold of dimension ≥ 3 and $n > \dim(M)$ be an integer. Let $f_k\colon M \to N_k$ be a sequence of smooth embeddings of M into orientable hyperbolic n–manifolds such that f_k induces monomorphisms of fundamental groups. Then the set of the normal bundles $\nu(f_k)$ of the embeddings breaks into finitely many isomorphism classes.*

In particular, only finitely many orientable hyperbolic n–manifolds are total spaces of vector bundles over M.

In some cases it is easy to decide when there exist *infinitely* many rank m vector bundles over a given base. For example, by a simple K–theoretic argument the set of isomorphism classes of rank m vector bundles over a finite cell complex K is infinite provided $m \geq \dim(K)$ and $\oplus_k H^{4k}(K;\mathbb{Q}) \neq 0$. Furthermore, oriented rank two vector bundles over K are in one-to-one correspondence with $H^2(K;\mathbb{Z})$ via the Euler class. Note that many arithmetic closed real hyperbolic manifolds have nonzero Betti numbers in all dimensions [30]. Any closed complex hyperbolic manifold has nonzero even Betti numbers because the powers of the Kähler form are noncohomologous to zero. Similarly, for each k, closed quaternion hyperbolic manifolds have nonzero $4k$th Betti numbers.

Examples of vector bundles with hyperbolic total spaces are given in the section 2. Note that according to a result of Anderson [1] the total space of any vector bundle over a closed negatively curved manifold admits a complete metric with the sectional curvature pinched between two negative constants.

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Outline of the paper. The section 2 is a collection of examples of hyperbolic manifolds. Some invariants of maps and representations are defined in sections 3 and 4. Sections 5, 6, and 7 are devoted to a proof of the theorem 1.3 and other related results. Sections 8, 9, and 10 contain background on algebraic and geometric convergence needed for the main theorem which is proved in section 11. Theorems 1.1 and 1.2 are proved in section 12. Finally, theorem 1.4 is proved in section 13.

2 Examples

To help the reader appreciate the results stated above we collect some relevant examples of hyperbolic manifolds.

Example 2.1 (**Plane bundles over closed surfaces**) Total spaces of rank two vector bundles over closed surfaces often admit hyperbolic metrics. For instance, an orientable $\mathbb{R}^2$–bundle over a closed oriented surface of genus g admits a real hyperbolic structure provided the Euler number e of the bundle satisfies $|e| < g$ [27, 28] (cf [18], [21], [23, 24]). Complex hyperbolic structures exist on orientable $\mathbb{R}^2$–bundles over closed oriented surfaces when $|e + 2g - 2| < g$ [16].

Note that the Euler number is equal to the self-intersection number of the zero section, hence the total spaces of bundles with different Euler classes are not intersection homotopy equivalent.

For nonorientable bundles over nonorientable surfaces the condition $|e| \le [\frac{g}{8}]$ on the twisted Euler number implies the existence of a real hyperbolic structure [3].

Example 2.2 (**Plane bundles over closed hyperbolic 3–manifolds**) Total spaces of plane bundles over closed hyperbolic 3–manifolds sometimes carry hyperbolic metrics [4].

In fact, it can be deduced from [4] that for every k there exists a closed oriented real hyperbolic 3–manifold $M = M(k)$ and oriented real hyperbolic 5–manifolds $N_1, \ldots, N_k$ that are total spaces of plane bundles over M and such that no two of them are intersection preserving homotopy equivalent.

Example 2.3 (**Fundamental group at infinity**) There are real hyperbolic 4–manifolds that are intersection homotopy equivalent but not homeomorphic to plane bundles over closed surfaces [18]. The invariant that distinguishes these manifolds from vector bundles is the fundamental group at infinity.

Even more surprising examples were given in [29] and [19]. Namely, there are orientable real hyperbolic 4–manifolds that are homotopy equivalent but not homeomorphic to handlebodies; these manifolds have nontrivial fundamental group at infinity. Note that if N and L are orientable 4–manifolds that are homotopy equivalent to a handlebody, then each homotopy equivalence $N \to L$ is both tangential and intersection preserving.

Example 2.4 (Nonorientable line bundles) Here is a simple way to produce homotopy equivalent hyperbolic manifolds that are not tangentially homotopy equivalent.

First, note that via the inclusion $O(n,1) \hookrightarrow O(n+1,1)$, any discrete subgroup of $O(n,1)$ can be thought of as a discrete subgroup of $O(n+1,1)$ that stabilizes a subspace $\mathbf{H}^n_{\mathbb{R}} \subset \mathbf{H}^{n+1}_{\mathbb{R}}$. The orthogonal projection $\mathbf{H}^{n+1}_{\mathbb{R}} \to \mathbf{H}^n_{\mathbb{R}}$ is an $O(n,1)$–equivariant line bundle over $\mathbf{H}^n_{\mathbb{R}}$. In particular, given a real hyperbolic manifold M, the manifold $M \times \mathbb{R}$ carries a real hyperbolic metric.

If $H^1(M;\mathbb{Z}_2) \neq 0$, this construction can be twisted to produce nonorientable line bundles over $M = \mathbf{H}^n_{\mathbb{R}}/\pi_1(M)$ with real hyperbolic metrics. Indeed, a nonzero element $w \in H^1(M;\mathbb{Z}_2) = \mathrm{Hom}(\pi_1(M);\mathbb{Z}_2)$ defines an epimorphism $w\colon \pi_1(M) \to \mathbb{Z}_2$. Make $\mathbb{Z}_2$ act on $\mathbf{H}^{n+1}_{\mathbb{R}}$ as the reflection in $\mathbf{H}^n_{\mathbb{R}}$. Then let $\pi_1(M)$ act on line bundle $\mathbf{H}^n_{\mathbb{R}} \times \mathbb{R} = \mathbf{H}^{n+1}_{\mathbb{R}}$ by $\gamma(x,t) = (\gamma(x), w(\gamma)(t))$. The quotient $\mathbf{H}^{n+1}_{\mathbb{R}}/\pi_1(M)$ is the total space of a line bundle over M with the first Stiefel–Whitney class w.

In particular, line bundles with different first Stiefel–Whitney classes have total space that are homotopy equivalent but *not* tangentially homotopy equivalent. (Tangential homotopy equivalences preserve Stiefel–Whitney classes yet $w_1(\mathbf{H}^{n+1}_{\mathbb{R}}/\pi_1(M)) = w_1(M) + w$.) Thus, we get many tangentially homotopy inequivalent manifolds in a given homotopy type.

3 Invariants of continuous maps

Definition 3.1 *Let B be a topological space and S_B be a set. Let ι be a map that, given a smooth manifold N, and a continuous map from B into N, produces an element of S_B. We call ι an **invariant of maps of B** if the two following conditions hold:*

(1) *Homotopic maps $f_1\colon B \to N$ and $f_2\colon B \to N$ have the same invariant.*

(2) *Let $h\colon N \to L$ be a diffeomorphism of N onto an open subset of L. Then, for any continuous map $f\colon B \to N$, the maps $f\colon B \to N$ and $h \circ f\colon B \to L$ have the same invariant.*

There is a version of this definition for maps into oriented manifolds. Namely, we require that the target manifold is oriented and the diffeomorphism h preserves orientation. In that case we say that ι is an *invariant of maps of B into oriented manifolds*.

Example 3.2 (Tangent bundle) Assume B is paracompact and S_B is the set of isomorphism classes of real vector bundles over B. Given a continuous map $f\colon B \to N$, set $\theta(f\colon B \to N) = f^{\#}TN$, the isomorphism class of the pullback of the tangent bundle to N under f. Clearly, θ is an invariant.

Example 3.3 (Intersection number in oriented n–manifolds) Assume B is compact. Fix two cohomology classes $\alpha \in H_m(B)$ and $\beta \in H_{n-m}(B)$. (In this paper we always use singular (co)homology with integer coefficients unless stated otherwise.)

Let $f\colon B \to N$ be a continuous map of a compact topological space B into an oriented n–manifold N where $\dim(N) = n$. Set $I_{n;\alpha;\beta}(f)$ to be the intersection number $I(f_*\alpha; f_*\beta)$ of $f_*\alpha$ and $f_*\beta$ in N. We next show that $I_{n;\alpha;\beta}$ is an integer-valued invariant of maps into oriented manifolds.

Recall that the intersection number $I(f_*\alpha; f_*\beta)$ can be defined as follows. Start with an arbitrary compact subset K of N that contains $f(B)$. Let $A \in H^{n-m}(N; N\setminus K)$ and $B \in H^m(N; N\setminus K)$ be the Poincare duals of $f_*\alpha \in H_m(K)$ and $f_*\beta \in H_{n-m}(K)$, respectively. Then, set $I(f_*\alpha; f_*\beta) = \langle A \cup B; [N; N\setminus K]\rangle$ where $[N; N\setminus K]$ is the fundamental class of N near K [13, VII.13.5]. Note that

$$I(f_*\alpha; f_*\beta) = \langle A \cup B; [N; N\setminus K]\rangle = \langle A; B \cap [N; N\setminus K]\rangle = \langle A; f_*\beta\rangle.$$

The following commutative diagram shows that $I(f_*\alpha; f_*\beta)$ is independent of K.

$$\begin{array}{ccccccc}
H_m(B) & \xrightarrow{f_*} & H_m(N; N\setminus f(B)) & \xrightarrow{D^{-1}} & H^{n-m}(f(B)) & \xrightarrow{f^*} & H^{n-m}(B) \\
\Big\downarrow{\scriptstyle f_*} & & \Big\uparrow{\scriptstyle i_*} & & \Big\uparrow{\scriptstyle i^*} & & \Big\| \\
H_m(K) & \xrightarrow{i_*} & H_m(N; N\setminus K) & \xrightarrow{D^{-1}} & H^{n-m}(K) & \xrightarrow{f^*} & H^{n-m}(B)
\end{array}$$

Note that $I_{n;\alpha;\beta}$ is an invariant. Indeed, property (1) holds trivially; property (2) is verified in [13, VIII.13.21(c)].

We say that an invariant of maps is *liftable* if in part (2) of the definition the word "diffeomorphism" can be replaced by a "covering map". For example, tangent bundle is a liftable invariant. Intersection number is not liftable. The following proposition shows to what extent it can be repaired.

Proposition 3.4 *Let $p\colon \tilde N \to N$ be a covering map of manifolds and let B be a finite connected CW–complex. Suppose that $f\colon B \to \tilde N$ is a map such that $p \circ f\colon B \to N$ is an embedding (ie a homeomorphism onto its image).*

Then $\theta(f) = \theta(p \circ f)$ for any invariant of maps θ.

Proof Since $p\circ f\colon B\to L$ is an embedding, so is f. Then, the map $p|_{f(B)}\colon f(B)\to p(f(B))$ is a homeomorphism. Using compactness of $f(B)$, one can find an open neighborhood U of $f(B)$ such that $p|_U\colon U\to p(U)$ is a diffeomorphism. Since invariants $\iota(f)$ and $\iota(p\circ f)$ can be computed in U and $p(U)$, respectively, we conclude $\iota(f)=\iota(p\circ f)$. □

4 Invariants of representations

Assume X is a smooth contractible manifold and let $\mathrm{Diffeo}(X)$ be the group of all self-diffeomorphisms of X equipped with compact-open topology. Let π be the fundamental group of a finite-dimensional CW–complex K with universal cover $\tilde K$.

We refer to a group homomorphism $\rho\colon\pi\to\mathrm{Diffeo}(X)$ as a *representation*. To any representation $\rho\colon\pi\to\mathrm{Diffeo}(X)$, we associate a continuous π–equivariant map $\tilde K\to X$ as follows. Consider the X–bundle $\tilde K\times_\pi X$ over K where $\tilde K\times_\pi X$ is the quotient of $\tilde K\times X$ by the following action of π

$$\gamma(\tilde k,x)=(\gamma(\tilde k),\rho(\gamma)(x)),\quad \gamma\in\pi.$$

Since X is contractible, the bundle has a section that is unique up to homotopy through sections. Any section can be lifted to a π–equivariant continuous map $\tilde K\to\tilde K\times X$. Projecting to X, we get a π–equivariant continuous map $\tilde K\to X$.

Note that any two π–equivariant continuous maps $\tilde g,\tilde f\colon\tilde K\to X$, are π–equivariantly homotopic. (Indeed, $\tilde f$ and $\tilde g$ descend to sections $K\to\tilde K\times_\pi X$ that must be homotopic. This homotopy lifts to a π–equivariant homotopy of $\tilde f$ and $\tilde g$.)

Assume now that $\rho(\pi)$ acts freely and properly discontinuously on X. Then the map $\tilde f$ descends to a continuous map $f\colon K\to X/\rho(\pi_1(K))$. We say that ρ is *induced* by f.

Let ι be an invariant of continuous maps of K. Given a representation ρ such that $\rho(\pi)$ acts freely and properly discontinuously on X, set $\iota(\rho)$ to be $\iota(f)$ where ρ is induced by f. We say ι is an *invariant of representations of* $\pi_1(K)$.

Similarly, any invariant ι of continuous maps of K into oriented manifolds defines an invariant of representations into the group of orientation-preserving diffeomorphisms of X.

Note that representations conjugate by a diffeomorphism ϕ of X have the same invariants. (Indeed, if $\tilde f\colon\tilde K\to X$ is a π–equivariant map, the map $\phi\circ\tilde f$ is

$^{-1}${equivariant.) The same is true for invariants of orientation-preserving representations when is orientation-preserving.

Example 4.1 (Tangent bundle) Let be the invariant of maps de ned in 3.2. Then, for any representation such that ( ) acts freely and properly discontinuously on X, let ( ) be the pullback of the tangent bundle to $X/$( ) via a map $f\colon K \to X/$( ) that induces .

In fact, can be de ned for any representation as follows. Look at the \vertical" bundle $\tilde{K}$ TX over $\tilde{K}$ X where TX is the tangent bundle to X. Set ( ) to be the pullback of the vertical bundle via a section $K \to \tilde{K}$ X. Thus, to every representation : $_1(K) \to$ Di eo(X) we associated a vector bundle ( ) of rank dim(X) over K.

Example 4.2 (Intersection number for orientation preserving actions) Assume the cell complex K is nite and choose an orientation on X (which makes sense because, like any contractible manifold, X is orientable).

Fix two cohomology classes $\in H_m(K)$ and $\in H_{n-m}(K)$. Let $I_{n; \, ;}$ is an invariant of maps de ned in 3.3 where $n = \dim(X)$.

Let be a representation of into the group of orientation-preserving di eomorphisms of X such that ( ) acts freely and properly discontinuously on X. Then let $I_{n; \, ;}$( ) be the intersection number $I(f \; ; f \;)$ of f and f in $X/$( ) where $f\colon K \to X/$( ) is a map that induces .

5 Spaces of representations and tangential homotopy equivalence

Let X be a smooth contractible manifold and let Di eo(X) be the group of all self-di eomorphisms of X equipped with compact{open topology. We equip the space of representations Hom(; Di eo(X)) with the pointwise convergence topology, ie a sequence of representations $_k$ converges to provided $_k(\gamma)$ converges to (γ) for each $\gamma \in$.

In the next two sections we explore the consequences of the following observation.

Proposition 5.1 *Let* $_0$ *and* $_1$ *be injective representations of into* Di eo(X) *such that the groups* $_0$( ) *and* $_1$( ) *act freely and properly discontinuously on* X. *Suppose that* $_0$ *and* $_1$ *can be joined by a continuous path* *of representations* $_t\colon \ \to$ Diffeo(X) *(where continuous means that, for every* $y \in$ *, the map* $_t(y)\colon [0,1] \to$ Diffeo(X) *is continuous).*

Then the homotopy equivalence of manifolds $X/_0()$ *and* $X/_1()$ *induced by* $_1 \ (_0)^{-1}$ *is tangential.*

Proof Since $_t\colon \ \to$ Diffeo(X) is a continuous path of representations, the covering homotopy theorem implies that the bundles $(_0)$ and $(_1)$ are isomorphic (the invariant is defined in 4.1).

Let $f_0\colon K \to X/_0()$ and $f_1\colon K \to X/_1()$ be homotopy equivalences that induce $_0$ and $_1$, respectively. For $i = 1, 2$ the bundle $(_i)$ is isomorphic to the pullback of the tangent bundle to $X/_i()$ via f_i. Thus $f_1 \ (f_0)^{-1}$ is a tangential homotopy equivalence. □

Remark 5.2 In fact, the covering homotopy theorem implies that is constant on any path-connected component of the space Hom$(_1(K), \mathrm{Diffeo}(X))$.

Corollary 5.3 *Under the assumptions of* 5.1, *suppose that* dim(X) 5 *and that each of the manifolds* $X/_0()$ *and* $X/_1()$ *is homeomorphic to the total space of a vector bundle over a manifold of dimension* $<$ dim$(X)/2$.

Then $_0$ *and* $_1$ *are smoothly conjugate on* X.

Proof According to [25, pp 226–228], the homotopy equivalence $f_1 \ (f_0)^{-1}$ is homotopic to a diffeomorphism. Hence, $_0$ and $_1$ are smoothly conjugate on X. □

Remark 5.4 More precisely, the result proved in [25, pp 226–228] is as follows. Let $f\colon N_0 \to N_1$ be a tangential homotopy equivalence of smooth n–manifolds with n 5. If each of the manifolds is homeomorphic to the interior of a regular neighbourhood of a simplicial complex of dimension $< n/2$, then f is homotopic to a diffeomorphism.

6 Discrete representations in stable range

Suppose X is a symmetric space of nonpositive sectional curvature. Note that for any discrete torsion-free subgroup Γ Isom(X) that stabilizes a totally geodesic submanifold Y, the exponential map identifies the quotient X/Γ and the total space of normal bundle of Y/Γ in X/Γ. Applying 5.1, we deduce the following.

Theorem 6.1 *Let X be a nonpositively curved symmetric space of dimension 5 and let be a group. Let $_1$ and $_2$ be injective discrete representations of into the isometry group of X that lie in the same path-connected component of the space* Hom(; Isom(X)). *Suppose that each of the representation $_1$ and $_2$ stabilizes a totally geodesic subspace of dimension $<$* dim(X)=2.

Then $_1$ and $_2$ are smoothly conjugate on X.

Remark 6.2 Of course, the above argument works in other geometries as well. Here is a sample result for complete a ne manifolds.

Let $_0$ and $_1$ be injective representations of a group into A ($\mathbb{R}^n$) that lie in the same path-connected component of the space of representations Hom(; A ($\mathbb{R}^n$)). Assume that the groups $_0$( ) and $_1$( ) act freely and properly discontinuously on $\mathbb{R}^n$ and, furthermore, suppose that $_0$( ) and $_1$( ) are contained in A ($\mathbb{R}^k$) A ($\mathbb{R}^n$).

Since the coordinate projection $\mathbb{R}^n$! $\mathbb{R}^k$ is A ($\mathbb{R}^k$){equivariant, the manifolds $\mathbb{R}^n$= $_0$( ) and $\mathbb{R}^n$= $_1$( ) are the total spaces of vector bundles over the manifolds $\mathbb{R}^k$= $_0$( ) and $\mathbb{R}^k$= $_1$( ), respectively. Then 5.3 implies that $_0$ and $_1$ are conjugate by a di eomorphism of $\mathbb{R}^n$ provided n 5 and $k < n$=2.

7 Locally symmetric nonpositively curved manifolds up to tangential homotopy equivalence

Let G be a subgroup of Di eo(X) such that the space of representations Hom(; G) has nitely many path-connected components. Then 5.1 implies that the class of manifolds of the form X= ( ) where is injective and ( ) acts freely and properly discontinuously on X falls into nitely many tangential equivalence classes.

For example, Hom(; G) has nitely many path-connected components if is nitely presented and G is either real algebraic, or complex algebraic, or semisimple with nite center [15]. In particular, the following is true.

Theorem 7.1 *Let be a nitely presented torsion-free group and let X be a nonpositively curved symmetric space. Then the class of manifolds of the form X= ( ), where 2* Hom(; Isom(X)) *is a faithful discrete representation, falls into nitely many tangential homotopy types.*

Proof Represent X as a Riemannian product $Y \times \mathbb{R}^k$ where Y is a nonpositively curved symmetric space without Euclidean factors. By de Rham's theorem this decomposition is unique, so $\mathrm{Isom}(X) = \mathrm{Isom}(Y) \times \mathrm{Isom}(\mathbb{R}^k)$. The group $\mathrm{Isom}(Y)$ is semisimple with trivial center, hence the analytic variety $\mathrm{Hom}(\ ;\mathrm{Isom}(Y))$ has finitely many path-connected components [15, p 567]. The same is true for $\mathrm{Hom}(\ ;\mathrm{Isom}(\mathbb{R}^k))$ because $\mathrm{Isom}(\mathbb{R}^k)$ is real algebraic [15, p 567]. Hence the analytic variety

$$\mathrm{Hom}(\ ;\mathrm{Isom}(X)) = \mathrm{Hom}(\ ;\mathrm{Isom}(Y)) \times \mathrm{Hom}(\ ;\mathrm{Isom}(\mathbb{R}^k))$$

has finitely many connected components. □

Corollary 7.2 *Let   be a finitely presented torsion-free group. Then, for any n, the class of locally symmetric complete nonpositively curved n–manifolds with the fundamental group isomorphic to   falls into into finitely many tangential homotopy types.*

Proof For any n there exist only finitely many nonpositively curved symmetric spaces of dimension n. Hence 7.1 applies. □

8 Geometric convergence

In this section we discuss some basic facts on geometric convergence. The notion of geometric convergence was introduced by Chaubaty (see [7]). More details relevant to our exposition can be found in [11], [26] and [5]. In this section we let G be a Lie group equipped with some left invariant Riemmanian metric.

Definition 8.1 *Let $C(G)$ be the set of all closed subgroups of G. Define a topology on $C(G)$ as follows. We say that a sequence $\{\Gamma_n\} \in C(G)$ converges to $\Gamma_{\mathrm{geo}} \in C(G)$ geometrically if the following two conditions hold:*

(1) *If $\gamma_{n_k} \in \Gamma_{n_k}$ converges to $\gamma \in G$, then $\gamma \in \Gamma_{\mathrm{geo}}$.*

(2) *If $\gamma \in \Gamma_{\mathrm{geo}}$, then there is a sequence $\gamma_n \in \Gamma_n$ with $\gamma_n \to \gamma$ in G.*

Fact 8.2 ([11]) The space $C(G)$ is compact and metrizable.

Fact 8.3 ([26]) $\Gamma_n \to \Gamma_{\mathrm{geo}}$ iff for every compact subset $K \subset G$ the sequence $\Gamma_n \cap K \to \Gamma_{\mathrm{geo}} \cap K$ in the Hausdorff topology (ie for any $\varepsilon > 0$, there is N such that, if $n > N$, then $\Gamma_n \cap K$ lies in the ε–neihgborhood of $\Gamma_{\mathrm{geo}} \cap K$ and $\Gamma_{\mathrm{geo}} \cap K$ lies in the ε–neihgborhood of $\Gamma_n \cap K$).

Fact 8.4 ([26]) Let $\Gamma_{\mathrm{geo}} \subset G$ is a discrete subgroup. Then there is $\varepsilon > 0$ such that, for any $\Gamma_n \to \Gamma_{\mathrm{geo}}$ in $C(G)$ and any compact $K \subset G$, there is N such that, if $n > N$ and $\gamma \in \Gamma_{\mathrm{geo}} \cap K$, then there is a unique $\gamma_n \in \Gamma_n$ that is ε–close to γ.

In particular, Γ_n is discrete for $n > N$, since $e \in \Gamma_n$ is the only element of Γ_n, that is in the ε–neighbourhood of the identity.

Remark 8.5 Let Γ_n be a sequence of torsion-free groups converging to a discrete group Γ_{geo} in $C(G)$. Then Γ_{geo} is torsion-free. Indeed, choose $\gamma \in \Gamma_{\mathrm{geo}}$ with $\gamma^k = e$. Find a sequence $\gamma_n \to \gamma$. Then $\gamma_n^k \to e$. By 8.4 we have $\gamma_n^k = e$ for large n. Since Γ_n are torsion-free, $k = 1$ as desired.

Theorem 8.6 ([26]) *Let X be a simply connected homogeneous Riemannian manifold and G be a transitive group of isometries. Let Γ_n be a sequence of torsion-free subgroups of G converging geometrically to a discrete subgroup Γ_{geo}. Let $U \subset X$ be a relatively compact open set.*

Then, if n is large enough, there exists a relatively compact open set V with $U \subset V$ and smooth embeddings $\tilde{\varphi}_n \colon U \to V$ such that $\tilde{\varphi}_n$ descend to embeddings $\varphi_n \colon U/\Gamma_{\mathrm{geo}} \to V/\Gamma_n$.

$$\begin{tikzcd} U \arrow[r, "\tilde{\varphi}_n"] \arrow[d, "p"'] & V \arrow[d, "p_n"] \\ U/\Gamma_{\mathrm{geo}} \arrow[r, "\varphi_n"'] & V/\Gamma_n \end{tikzcd}$$

The embeddings $\tilde{\varphi}_n$ converge C^r–uniformly to the inclusion.

Sketch of the Proof Let $K = \{g \in G \colon g(\overline{V}) \cap \overline{V} \neq \emptyset\}$, so K is compact. By 8.4, if we take n sufficiently large, then for any $\gamma \in \Gamma_{\mathrm{geo}} \cap K$ there exists a unique $\gamma_n \in \Gamma_n$ that is close to γ. It defines an injective map r_n of $\Gamma_{\mathrm{geo}} \cap K$ into Γ_n.

Our goal is to construct embeddings $\varphi_n \colon U \to V$ that are close to the inclusion and r_n–equivariant (in the sense that $\varphi_n(\gamma(x)) = r_n(\gamma)\varphi_n(x)$ for $\gamma \in \Gamma_{\mathrm{geo}}$). The construction is non-trivial and we refer to [26] for a complete proof. □

Lemma 8.7 *Let G be a Lie group and Γ_n be a sequence of discrete groups that converges geometrically to $\Gamma = \Gamma_{\mathrm{geo}}$. If the identity component Γ_0 of Γ_{geo} is compact, the sequence $\{\Gamma_n\}$ has unbounded torsion (ie for any positive integer N, there is $\gamma \in \Gamma_{n(N)}$ of finite order which is greater than N).*

Proof Choose $\varepsilon > 0$ so small that the closed 4ε–neighborhood U of Γ_0 is disjoint from $\Gamma \setminus \Gamma_0$ (this is possible since Γ_0 is compact). Given $M > 1$, for large n, Γ_n and Γ are ε/M–Hausdorff close on the compact set U. Take an arbitrary element $g \in \Gamma_0$ in ε–neighborhood of the identity e and choose $g_n \in \Gamma_n$ so that it is ε/M–close to g.

First, show that g_n has finite order. Suppose not. Since U is compact and $\langle g_n \rangle$ is discrete and infinite, there is a smallest $k > 1$ with $(g_n)^k \notin U$. So, $(g_n)^{k-1} \in U$. The metric is left invariant, hence,

$$d((g_n)^{k-1}, (g_n)^k) = d(g_n, e) < d(g, e) + d(g_n, g) < \varepsilon + \varepsilon/M < 2\varepsilon.$$

Since Γ_n and Γ are ε/M–Hausdorff close on U and since $\Gamma \cap U = \Gamma_0$, we get $d((g_n)^{k-1}, \Gamma_0) < \varepsilon/M < \varepsilon$. This implies $d((g_n)^k, \Gamma_0) < 3\varepsilon$ and, thus, $(g_n)^k \in U$, a contradiction. Thus g_n has finite order.

We have just showed that any neighborhood of g contains an element of finite order $g_n \in \Gamma_n$ (for n large enough). So we get a sequence g_n converging to g. Assume the torsion of Γ_n is uniformly bounded by N. Then, for any finite order element $h \in \Gamma_n$, we have $h^{N!} = e$. In particular $e = (g_n)^{N!}$ converges to $g^{N!}$. We, thus, proved that any element g in Γ_0 that is ε–close to the identity satisfies $g^{N!} = e$. This is absurd. (Indeed, it would mean that the map $\Gamma_0 \to \Gamma_0$ that takes g to $g^{N!}$ maps the ε–neighborhood of the identity to the identity. But the map is clearly a diffeomorphism on some neighborhood of the identity). Thus, Γ_n has unbounded torsion as desired. □

Lemma 8.8 *Let G be a Lie group and let Γ_n be a sequence of discrete groups that converges geometrically to $\Gamma = \Gamma_{\mathrm{geo}}$. Then the identity component Γ_0 of Γ_{geo} is nilpotent.*

Proof Denote the identity component by G_0. The group Γ_{geo} is closed, so it is a Lie group. So the identity component Γ_0 of Γ_{geo} is a Lie group; we are to show that $\Gamma_0 \subset G_0$ is nilpotent. Let U be a neighborhood of the identity that lies in a Zassenhaus neighborhood in G_0 (see [33, 8.16]). Being a Lie group Γ_0 is generated by any neighborhood of the identity, in particular by $V = U \cap \Gamma_0$. To show Γ_0 is nilpotent, it suffices to check that, for some k, $V^{(k)} = \{e\}$, that is, any iterated commutator of weight k with entries in V is trivial [33, 8.17]. Fix an iterated commutator $[v_1, \dots [v_{k-2}[v_{k-1}, v_k]] \dots]$ with $v_k \in V$ and choose a sequences $g^n_k \to v_k$ where $g^n_k \in \Gamma_n$. For n large enough, the elements $g^n_k, \dots, g^n_k$ lie in a Zassenhaus neighborhood. Then by Zassenhaus–Kazhdan–Margulis lemma [33, 8.16] the group $\langle g^n_k, \dots, g^n_k \rangle$ lies in a connected nilpotent Lie subgroup of G_0. The class of any connected nilpotent

Lie subgroup of G_0 is bounded by $\dim(G_0)$. Thus, for $k > \dim(G_0)$, the k–iterated commutator $[g_1^n \ldots [g_{k-2}^n[g_{k-1}^n, g_k^n]] \ldots]$ is trivial, for large n. This implies $[v_1 \ldots [v_{k-2}[v_{k-1}, v_k]] \ldots]$ is trivial and, therefore, Γ_0 is nilpotent. □

9 Algebraic convergence

The set of representations Hom($;G$) of a group into a topological group G can be given the so-called *algebraic* topology (also called pointwise convergence topology). Namely, a sequence of representations $_k$ converges to provided $_k(\gamma)$ converges to ( ) for each $\gamma \in$. In this section we discuss how algebraic and geometric convergences interact.

It follows from de nitions that geometric limit always contains the algebraic one. More precisely, if $_k$ converges to algebraically and $_k$( ) converges to Γ_{geo} geometrically, then $_k$( ) Γ_{geo}. In particular, if Γ_{geo} is discrete, so is ( ).

Theorem 9.1 *Let be a nitely generated group and let X be a negatively curved symmetric space. Let $_k \in$ Hom($;G$) be a sequence of representations converging algebraically to a representation where, for every k, the group $_k$( ) is discrete and the sequence $\{ _k( )\}$ has uniformly bounded torsion. Suppose that ( ) is an in nite group without nontrivial normal nilpotent subgroups.*

Then the closure of $\{ _k( )\}$ in the geometric topology consists of discrete groups. In particular, the group ( ) is discrete.

Proof Passing to subsequence, we can assume that $\{ _k( )\}$ converges geometrically to Γ_{geo}. Being a closed subgroup Γ_{geo} is a Lie group. Suppose, arguing by contradiction, that the identity component Γ_0 of Γ_{geo} is nontrivial. By lemma 8.8, Γ_0 is a nilpotent Lie group.

Since ( ) does not have a nontrivial normal nilpotent subgroup, ( ) $\cap\ \Gamma_0 = \{e\}$, so ( ) is discrete. Using the Selberg lemma we nd a torsion-free subgroup ( ) of nite index. The group is in nite since ( ) is. Notice that the group may have at most two xed points at in nity (any isometry that xes at least three points is elliptic [2, p 84]). The next goal is to show that is virtually nilpotent.

According to [9, 3.3.1], the nilpotence of Γ_0 implies one the following mutually exclusive conclusions:

Case 1 Γ_0 has a fixed point in X (hence Γ_0 is compact which is is impossible by 8.7).

Case 2 Γ_0 fixes a point $p \in \partial X$ and acts freely on $X \cup \partial X \setminus \{p\}$.

Since normalizes Γ_0 (in fact, Γ_0 is a normal subgroup of Γ_{geo}) it has to fix p too. If p is the only fixed point of , then every nontrivial element of is parabolic [14, 8.9P]. Any parabolic preserves horospheres centered at p [2, p 84], therefore, according to [8] is virtually nilpotent.

If fixes two points at infinity, then it acts freely and properly discontinuously on the geodesic joining the points. Hence $= \mathbb{Z}$, the fundamental group of a circle.

Case 3 Γ_0 has no fixed points in X and preserves setwise some bi-infinite geodesic. In this case the fixed point set of Γ_0 is the endpoints p and q of the geodesic. Indeed, Γ_0 fixes each of the endpoints, since Γ_0 is connected. Assume Γ_0 fixes some other point of ∂X. Then any element of Γ_0 is elliptic (see [2, p 85]) and, as such, it fixes the bi-infinite geodesic pointwise. Thus Γ_0 has a fixed point in X which contradicts the assumptions of Case 3.

Since normalizes Γ_0, it leaves the set $\{p;q\}$ invariant. Moreover, preserves $\{p;q\}$ pointwise because contains no elliptics (any isometry that flips p and q has a fixed point on the geodesic joining p and q). Therefore, $= \mathbb{Z}$ as before. Hence, is virtually nilpotent.

Thus, ( ) has a nilpotent subgroup of finite index and, therefore, has a normal nilpotent subgroup of finite index. This contradicts the assumption that ( ) is an infinite group without nontrivial normal nilpotent subgroups. □

Remark 9.2 Let Γ be a *torsion-free discrete* subgroup of $\mathrm{Isom}(X)$ that has a nontrivial normal nilpotent subgroup. Then Γ is, in fact, virtually nilpotent. Indeed, repeating the arguments above, we see that Γ fixes a point at infinity and, hence, is virtually nilpotent. Conversely, any virtually nilpotent group clearly has a normal nilpotent subgroup of finite index.

Lemma 9.3 *Let G be a Lie group and let be a finitely generated group without nontrivial normal nilpotent subgroups. Let $_k \in \mathrm{Hom}(;G)$ be a sequence of discrete faithful representations that converges algebraically to a representation . Then is faithful.*

Proof Denote the identity component of G by G_0. It suffices to prove that the group $K = \mathrm{Ker}( )$ is nilpotent. Let V be a set of generators for K (maybe infinite). To show K is nilpotent, it suffices to check that, for some m, $V^{(m)} = \{e\}$, that is, any iterated commutator of weight m with entries in V is trivial [33, 8.17]. Fix an iterated commutator $[v_1 \ldots [v_{m-2}[v_{m-1}, v_m]] \ldots]$ with $v_m \in V$.

For k large enough, the elements $\rho_k(v_1), \ldots, \rho_k(v_m)$ lie in a Zassenhaus neighborhood of G_0. Then by Zassenhaus–Kazhdan–Margulis lemma [33, 8.16] the group $\langle \rho_k(v_1), \ldots, \rho_k(v_m) \rangle$ lies in a connected nilpotent Lie subgroup of G_0. The class of any nilpotent subgroup of G_0 is bounded by $\dim(G_0)$. Thus, for $m > \dim(G_0)$, the m–iterated commutator

$$[\rho_k(v_1) \ldots [\rho_k(v_{m-2})[\rho_k(v_{m-1}), \rho_k(v_m)]] \ldots]$$

is trivial, for large k. Since ρ_k is faithful $[v_1 \ldots [v_{m-2}[v_{m-1}, v_m]] \ldots]$ is trivial and we are done. □

Corollary 9.4 *Let X be a negatively curved symmetric space and let Γ be a finitely generated, torsion-free, discrete subgroup of* $\mathrm{Isom}(X)$ *that is not virtually nilpotent. Then the set of faithful discrete representations of Γ into* $\mathrm{Isom}(X)$ *is a closed subset of* $\mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$.

Proof Let a sequence ρ_k of faithful discrete representations converge to $\rho \in \mathrm{Hom}(\Gamma, \mathrm{Isom}(X))$. According to 9.2 the group Γ has no normal nilpotent subgroup. Then 9.3 implies ρ is faithful. By compactness $\rho_k(\Gamma)$ has a subsequence that converges geometrically to Γ_{geo} which is a discrete group by 9.1. Therefore, $\rho(\Gamma) \subset \Gamma_{\mathrm{geo}}$ is also discrete as wanted. □

Corollary 9.5 *Let X be a negatively curved symmetric space and let Γ be a finitely generated, torsion-free, discrete subgroup of* $\mathrm{Isom}(X)$ *that is not virtually nilpotent. Suppose ρ_k is a sequence of injective, discrete representations of Γ into* $\mathrm{Isom}(X)$ *that converges algebraically. Then the closure of $\{\rho_k(\Gamma)\}$ in the geometric topology consists of discrete groups.*

Proof Combine 9.1, 9.2, and 9.4. □

10 A compactness theorem

In this section we state a compactness theorem for the space of conjugacy classes of faithful discrete representations of a group Γ into the isometry group of a negatively curved symmetric space. The proof follows from the work of Bestvina and Feighn [6] based on ideas of Rips (see the review of [6] by Paulin in

MR96h : 20056). Earlier versions of the theorem have been proved by Thurston, Morgan and Shalen [32].

Let X be a negatively curved symmetric space and $\mathrm{Isom}(X)$ be the isometry group of X. We equip $\mathrm{Hom}(\pi; \mathrm{Isom}(X))$ with the algebraic topology.

Theorem 10.1 *Let X be a negatively curved symmetric space and let π be a discrete, finitely presented subgroup of the isometry group of X. Suppose that π is not virtually nilpotent and does not split as an HNN–extension or a nontrivial amalgamated product over a virtually nilpotent group.*

Then the space of conjugacy classes of faithful discrete representations of π into $\mathrm{Isom}(X)$ *is compact.*

Here is an example of a group that does not split over a virtually nilpotent group.

Proposition 10.2 *Let M be a closed aspherical n–manifold such that any nilpotent subgroup of $\pi_1(M)$ has cohomological dimension $\le n-2$. Then $\pi_1(M)$ does not split as a nontrivial amalgamated product or HNN–extension over a virtually nilpotent group.*

Proof Assume, by contradiction, that $\pi = \pi_1(M)$ is of the form $\Gamma_1 *_N \Gamma_2$ or $\Gamma_0 *_N$ where N is a virtually nilpotent group and Γ_k is a proper subgroup of π, for $k = 0; 1; 2$.

First, suppose that both Γ_1 and Γ_2 have infinite index in π. Note that by the definition of HNN–extension, Γ_0 has infinite index in π. Then it follows from the Mayer–Vietoris sequence [10, VII.9.1, VIII.2.2.4(c)] that

$$\mathrm{cd}(\pi) \le \max_k(\mathrm{cd}(\Gamma_k); \mathrm{cd}(N) + 1).$$

The cohomological dimension of $\pi = \pi_1(M)$ is n because M is a closed aspherical n–manifold. Since $|\pi : \Gamma_k| = \infty$, Γ_k is the fundamental group of a noncompact manifold of dimension n, hence $\mathrm{cd}(\Gamma_k) < \mathrm{cd}(\pi)$. Finally, $\mathrm{cd}(N) \le n-2$ and we get a contradiction.

Second, assume that $\pi = \Gamma_1 *_N \Gamma_2$ and, say, Γ_1 has finite index in π. Look at the map $i_* : H_n(\Gamma_1) \to H_n(\pi)$ induced by the inclusion $i \colon \Gamma_1 \to \pi$ in the homology with $w_1(M)$–twisted integer coefficients. It is proved in [10, III.9.5(b)] that the image of i_* is generated by $|\pi : \Gamma_1| [M]$ where $[M]$ is the fundamental class of M.

Since Γ_1 has finite index in , the group N is of finite index in Γ_2 because $N = \Gamma_1 \cap \Gamma_2$. In particular, $\mathrm{cd}(\Gamma_2) = \mathrm{cd}(N) \le n-2$ [10, VIII.3.1].

Look at the nth term of the Mayer–Vietoris sequence with $w_1(M)$–twisted integer coefficients [10, VII.9.1]:

$$\longrightarrow H_n N \longrightarrow H_n\Gamma_1 \oplus H_n\Gamma_2 \longrightarrow H_n\Gamma \longrightarrow H_{n-1}N \longrightarrow$$

Using the cohomological dimension assumption on N, we get $0 = H_n(N) = H_{n-1}(N) = H_n(\Gamma_2)$, and hence the inclusion $i\colon \Gamma_1 \to$ induces an isomorphism of nth homology.

Thus $|\ :\Gamma_1| = 1$ which contradicts the assumption the Γ_1 is a proper subgroup of . □

Corollary 10.3 *Let M be a closed aspherical manifold of dimension ≥ 3 with word-hyperbolic fundamental group. Then $\pi_1(M)$ does not split as a nontrivial amalgamated product or HNN–extension over a virtually nilpotent group.*

Proof It is well known that any virtually nilpotent subgroup of a word-hyperbolic group is virtually cyclic. Since any virtually cyclic group has cohomological dimension one, proposition 10.2 applies. □

11 The Main theorem

Throughout this section is an invariant of continuous maps of K into oriented manifolds. We also use the letter for the corresponding invariant of discrete representations. Let be the fundamental group of a finite CW–complex K.

Theorem 11.1 *Let X be a contractible homogeneous Riemannian manifold and G be a transitive group of orientation-preserving isometries. Let $_k \in$* Hom$(;G)$ *be a sequence of representations such that*

- *the groups $_k(\ )$ are torsion-free, and*
- *$_k$ converges algebraically to , and*
- *$_k(\ )$ converges geometrically to a discrete group Γ_{geo}.*

Let $f\colon K \to X/\Gamma_{\mathrm{geo}}$ be a continuous map that induces the homomorphism $\ :\ \to (\ ) \subset \Gamma_{\mathrm{geo}}$. Then $(\ _k) = (f)$ for all large k.

Proof According to 8.4, we can assume that the groups $_k(\ )$ are discrete. Being a limit of torsion-free groups the discrete group Γ_{geo} is itself torsion-free by 8.5. Thus Γ_{geo} acts freely and properly discontinuously on X, so X/Γ_{geo} is a manifold.

We consider the universal covering $q\colon \tilde{K} \to K$. Since K is a finite complex, one can choose a finite connected subcomplex $D \ \tilde{K}$ with $q(D) = K$. (Pick a representing cell in every orbit, the union of the cells is a finite subcomplex that is mapped onto Y by q. Adding finitely many cells, one can assume the complex is connected.)

According to section 4, the representation defines a continuous –equivariant map $\tilde{f}\colon \tilde{K} \to X$ which is unique up to –equivariant homotopy. The map $\tilde{f}$ descends to a continuous map $f\colon K \to X/(\ )$.

The identity map $\mathrm{id}\colon X \to X$ is equivariant with respect to the inclusion $(\ ) \hookrightarrow \Gamma_{\text{geo}}$, therefore, $\tilde{f}$ is equivariant with respect to the homomorphism $\colon \ \to \Gamma_{\text{geo}}$. We denote by f the composition of f and the covering $X/(\ ) \to X/\Gamma_{\text{geo}} = N$ induced by the inclusion $(\ ) \ \Gamma_{\text{geo}}$.

Let $U \ X$ be an open relatively compact neighborhood of $\tilde{f}(D)$. We are in position to apply Theorem 8.6. Thus, if k is large enough, there is a sequence of embeddings $\tilde{\ }_k\colon U \to V \ X$ that converges to the inclusion and is r_k–equivariant (recall that r_k was defined in the proof of 8.6). So the map $\tilde{\ }_k \ \tilde{f}\colon D \to V$ is $r_k \ $–equivariant. By the very definition of r_k we have $r_k \ = \ _k$ whenever the left hand side is defined. We now extend the map $\tilde{\ }_k \ \tilde{f}\colon D \to V$ by equivariance to a $_k$–equivariant map $\tilde{\ }_k\colon \tilde{K} \to X$. The map $\tilde{\ }_k$ descends to $_k\colon K \to V/_k(\ ) \ X/_k(\ )$. Notice that by construction $_k = {'}_k \ f$. Since ${'}_k$ converge to the inclusion, ${'}_k$ is orientation-preserving for large k. Thus $(_k) = (f)$ for large k because is an invariant of maps into oriented manifolds.

Notice that $(_k) = (_k)$ since the map $_k$ is $_k$–equivariant (according to section 4 any $_k$–equivariant map can be used to define $(_k)$). Therefore, for large k, $(_k) = (f)$ as desired. □

Remark 11.2 Let X be a negatively curved symmetric space. Then, according to 9.1, the assumption "Γ_{geo} is discrete" of the theorem 11.1 can be replaced by "$_k(\ )$ are discrete and $(\ )$ is an infinite group without nontrivial normal nilpotent subgroups".

Remark 11.3 In some cases the conclusion of the theorem 11.1 can be improved to "$(_k) = (\ )$ for all large k". Since $(\ ) = (f)$, it suffices to show that $(f) = (f)$.

This is clearly true when $(\ ) = \Gamma_{\mathrm{geo}}$ in which case it is usually said that $_k$ converges to strongly. Another obvious example is when is a liftable invariant. Finally, according to 3.4, $(f) = (f)$ provided f is homotopic to an embedding (ie a homeomorphism onto its image).

Corollary 11.4 *Let X be a negatively curved symmetric space and let K be a nite cell complex such that $= {}_1(K)$ is a torsion-free, not virtually nilpotent, discrete subgroup of* $\mathrm{Isom}(X)$. *Let $_k$ be a sequence of discrete injective representations of into the group of orientation-preserving isometries of X. Suppose that $_k$ converges in the pointwise convergence topology.*

Then $(\ _k) = (\ _{k+1})$ for all large k.

Proof The result follows from 11.1 and 9.5. □

Remark 11.5 The results of this section certainly hold if is any invariant of maps rather that an invariant of maps into *oriented* manifolds. For such a we do not have to assume that the isometry groups preserve orientation.

12 Hyperbolic manifolds up to intersection preserving homotopy equivalence

A homotopy equivalence $f\colon N \;!\; L$ of oriented n{manifolds is called *intersection preserving* if, for any m and any pair of (singular) homology classes $2\, H_m(N)$ and $2\, H_{n-m}(N)$, their intersection number in N is equal to the intersection number of f and f in L. For example, any map that is homotopic to an orientation-preserving homeomorphism is an intersection preserving homotopy equivalence [13, 13.21].

Proposition 12.1 *Let N be an oriented hyperbolic manifold with virtually nilpotent fundamental group. Then the intersection number of any two homology classes in N is zero.*

Proof It su ces to prove that N is homeomorphic to $\mathbb{R}\ \ Y$ for some space Y. First note that any torsion-free, discrete, virtually nilpotent group Γ acting on a hyperbolic space X must have either one or two xed points at in nity (see the proof of 9.1). If Γ has only one xed point, Γ is parabolic and, hence, it preserves all horospheres at the xed point. Therefore, if H is such a horosphere, $X{=}\Gamma$ is homeomorphic to $\mathbb{R}\ \ H{=}\Gamma$. If Γ has two xed points, Γ preserves a bi-in nite geodesic. Hence $X{=}\Gamma$ is the total space of a vector bundle over a circle and the result easily follows. □

Theorem 12.2 *Let Γ be the fundamental group of a finite aspherical cell complex and let X be a negatively curved symmetric space. Let ρ_k be a sequence of discrete injective representations of Γ into the group of orientation-preserving isometries of X. Suppose that ρ_k is precompact in the pointwise convergence topology.*

Then the sequence of manifolds $X/\rho_k(\Gamma)$ falls into finitely many intersection preserving homotopy equivalence classes.

Proof According to 12.1, we can assume that $\pi_1(K)$ is not virtually nilpotent. Argue by contradiction. Pass to a subsequence so that no two of the manifolds $X/\rho_k(\Gamma)$ are intersection preserving homotopy equivalent and so that ρ_k converges algebraically.

Being the fundamental group of a finite aspherical cell complex, Γ has finitely generated homology. Choose a finite set of generators of $H_*(\Gamma)$. Using 11.4 we pass to a subsequence so that $I_{n;\alpha,\beta}(\rho_k) = I_{n;\alpha,\beta}(\rho_{k+1})$ for any pair of generators $\alpha \in H_m(N)$ and $\beta \in H_{n-m}(N)$. Hence, the homotopy equivalence that induces $\rho_{k+1} \circ (\rho_k)^{-1}$ is intersection preserving. □

Theorem 12.3 *Let K be a finite connected cell complex such that $\pi_1(K)$ does not split as an HNN–extension or a nontrivial amalgamated product over a virtually nilpotent group. Then, given a nonnegative integer n and homology classes $\alpha \in H_m(K)$ and $\beta \in H_{n-m}(K)$, there exists $C > 0$ such that,*

(1) *for any continuous map $f\colon K \to N$ of K into an oriented hyperbolic n–manifold N that induces an isomorphism of fundamental groups, and*

(2) *for any embedding $f\colon K \to N$ of K into an oriented hyperbolic n–manifold N that induces a monomorphism of fundamental groups,*

the intersection number of $f_*\alpha$ and $f_*\beta$ in N satisfies $|\langle f_*\alpha, f_*\beta\rangle| \le C$.

Proof First, we prove (1). Arguing by contradiction, consider a sequence of maps $f_k\colon K \to N_k$ that induce π_1–isomorphisms of K and oriented hyperbolic n–manifolds N_k and such that $I_{n;\alpha,\beta}(f_k)$ is an unbounded sequence of integers. Pass to a subsequence so that no two integers $I_{n;\alpha,\beta}(f_k)$ are the same. Each map f_k induces an injective discrete representation ρ_k of $\pi_1(K)$ into the group of orientation-preserving isometries of a negatively curved symmetric space; in particular $I_{n;\alpha,\beta}(\rho_k) = I_{n;\alpha,\beta}(f_k)$. Since in every dimension there are no more than four negatively curved symmetric spaces, we can pass to a subsequence to ensure that all the symmetric spaces where $\rho_k(\pi_1(K))$ acts are isometric.

According to 12.1, we can assume that $\pi_1(K)$ is not virtually nilpotent. By 10.1 there exists a sequence of orientation-preserving isometries γ_k of X such that the sequence $\gamma_k \circ \rho_k \circ (\gamma_k)^{-1}$ is algebraically precompact. As we explained in the section 4, $I_{n;\pi;\rho}(\rho_k) = I_{n;\pi;\rho}(\gamma_k \circ \rho_k \circ (\gamma_k)^{-1})$. Thus 11.4 provides a contradiction. Finally, according to 3.4, (1) implies (2). □

Corollary 12.4 *Let π be the fundamental group of a finite aspherical cell complex that does not split as an HNN–extension or a nontrivial amalgamated product over a virtually nilpotent group.*

Then, for any n, the class of hyperbolic n–manifolds with fundamental group isomorphic to π breaks into finitely many intersection preserving homotopy equivalence classes.

Proof By 12.1, we can assume that $\pi_1(K)$ is not virtually nilpotent. Thus the result follows from 12.2 and 10.1. □

13 Vector bundles with hyperbolic total spaces

I am most grateful to Jonathan Rosenberg for explaining to me the following nice fact.

Proposition 13.1 *Let K be a finite CW–complex and m be a positive integer. Then the set of isomorphism classes of oriented real (complex, respectively) rank m vector bundles over K with the same rational Pontrjagin classes and the rational Euler class (rational Chern classes, respectively) is finite.*

Proof To simplify notation we only give a proof for complex vector bundles and then indicate necessary modifications for real vector bundles. We need to show that the “Chern classes map” $(c_1, \dots, c_m)\colon [K, BU(m)] \to H^*(K;\mathbb{Q})$ is finite-to-one. First, notice that c_1 classifies line bundles [20, I.3.8, I.4.3.1], so we can assume $m > 1$.

The integral ith Chern class $c_i \in H^{2i}(BU(m);\mathbb{Z}) = [BU(m), K(\mathbb{Z}, 2i)]$ can be represented by a continuous map $f_i\colon BU(m) \to K(\mathbb{Z}, 2i)$ such that $c_i = f_i^*(\iota_{2i})$ where ι_{2i} is the fundamental class of $K(\mathbb{Z}, 2i)$. (Recall that, by the Hurewicz theorem $H_{n-1}(K(\mathbb{Z}, n);\mathbb{Z}) = 0$ and $H_n(K(\mathbb{Z}, n);\mathbb{Z}) = \mathbb{Z}$; the class in $H^n(K(\mathbb{Z}, n);\mathbb{Z}) = \mathrm{Hom}(H_n(K(\mathbb{Z}, n);\mathbb{Z});\mathbb{Z})$ corresponding to the identity homomorphism is called the fundamental class and is denoted ι_n.)

It defines "Chern classes map"

$$c = (f_1, \dots, f_m)\colon BU(m) \to K(\mathbb{Z}, 2) \times \cdots \times K(\mathbb{Z}, 2m).$$

We now check that the map induces an isomorphism on rational cohomology. According to [17, 7.5, 7.6] for even n, $H^n(K(\mathbb{Z}, n); \mathbb{Q}) = \mathbb{Q}[\iota_n]$. By the Künneth formula

$$H^*\left(\prod_{i=1}^m K(\mathbb{Z}, 2i); \mathbb{Q}\right) = \bigotimes_{i=1}^m H^*(K(\mathbb{Z}, 2i); \mathbb{Q}) = \bigotimes_{i=1}^m \mathbb{Q}[\iota_{2i}] = \mathbb{Q}[\iota_2, \dots, \iota_{2m}]$$

and under this ring isomorphism $\iota_2^{s_1} \otimes \cdots \otimes \iota_{2m}^{s_m}$ corresponds to $\iota_2^{s_1} \cdots \iota_{2m}^{s_m}$. It is well known that $H^*(BU(m); \mathbb{Q}) = \mathbb{Q}[c_1, \dots, c_m]$ [31, 14.5]. Thus the homomorphism $c^*\colon H^*(\prod_{i=1}^m K(\mathbb{Z}, 2i); \mathbb{Q})) \to H^*(BU(m); \mathbb{Q})$ defines a homomorphism $\mathbb{Q}[\iota_2, \dots, \iota_{2m}] = \mathbb{Q}[c_1, \dots, c_m]$ that takes ι_{2i} to c_i. Since this is an isomorphism, so is c^* as promised.

Therefore c induces an isomorphism on rational homology. Then, since $BU(m)$ is simply connected, the map c must be a rational homotopy equivalence [17, 7.7]. In other words the homotopy theoretic fiber F_c of the map c has finite homotopy groups.

Consider an oriented rank m vector bundle $f\colon K \to BU(m)$ with characteristic classes $c \circ f\colon K \to \prod_{i=1}^m K(\mathbb{Z}, 2i)$. Our goal is to show that the map $c \circ f$ has at most finitely many nonhomotopic liftings to $BU(m)$. Look at the set of liftings of $c \circ f$ to $BU(m)$ and try to construct homotopies skeleton by skeleton using the obstruction theory. The obstructions lie in the groups of cellular cochains of K with coefficients in the homotopy groups of F_c. (Note that the fibration $F_c \to BU(m) \to \prod_{i=1}^m K(\mathbb{Z}, 2i)$ has simply connected base and fiber (since $m > 1$), so the coefficients are not twisted.) Since the homotopy groups of F_c are finite, there are at most finitely many nonhomotopic liftings. This completes the proof for complex vector bundles.

For oriented real vector bundles of odd rank the same argument works with $c = (p_1, \dots, p_{[m/2]})$, where p_i is the ith Pontrjagin class. Similarly, for oriented real vector bundles of even rank we set $c = (e, p_1, \dots, p_{m/2-1})$, where e is the Euler class. (The case $m = 2$ can be treated separately: since $SO(2) = U(1)$ any oriented rank two vector bundle has a structure of a complex line bundle with e corresponding to c_1. Thus, according to [20, I.3.8, I.4.3.1], oriented rank two vector bundles are in one-to-one correspondence with $H^2(K; \mathbb{Z})$.) □

Remark 13.2 A similar result is probably true for nonorientable bundles. However, the argument given above fails due to the fact $BO(m)$ is not simply connected (ie the map $c = (p_1, \dots, p_{[m/2]})$ of $BO(m)$ to the product of Eilenberg–MacLane spaces is *not* a rational homotopy equivalence even though it induces an isomorphism on rational cohomology). For simplicity, we only deal with oriented bundles.

Remark 13.3 Note that, if either m is odd or $m > \dim(K)$, the rational Euler class is zero, and, hence rational Pontrjagin classes determine an oriented vector bundle up to a finite number of possibilities.

Corollary 13.4 *Let M be a closed smooth manifold and let m be an integer such that either m is odd or $m > \dim(M)$. Let $f_k\colon B \to N_k$ be a sequence of smooth immersions of B into complete locally symmetric nonpositively curved Riemannian manifolds N_k with orientable normal bundle and $\dim(N_k) = m + \dim(M)$.*

Then the set of the normal bundles $\nu(f_k)$ of the immersions falls into finitely many isomorphism classes.

Proof We can assume that all the manifolds N_k are quotients of the same symmetric space X since in every dimension there exist only finitely many symmetric spaces.

According to section 4, the immersions f_k induce representations $\rho_k\colon \pi_1(M) \to \mathrm{Isom}(X)$ such that $\tau(f_k) = \xi(\rho_k)$. The sequence $\xi(\rho_k)$ of vector bundles breaks into finitely many isomorphism classes because the representation variety $\mathrm{Hom}(\pi_1(M); \mathrm{Isom}(X))$ has finitely many connected components (see 7.1). In particular, there are only finitely many possibilities for the total Pontrjagin class of $\xi(\rho_k)$.

The normal bundle of the immersion f_k satisfies $\nu(f_k) \oplus TM = \tau(f_k)$. Applying the total Pontrjagin class, we get $p(\nu(f_k)) \cup p(TM) = p(\tau(f_k))$. The total Pontrjagin class of any bundle is a unit, hence we can solve for $p(\nu(f_k))$. Thus, there are only finitely many possibilities for $p(\nu(f_k))$. Finally, 13.1 and 13.3 imply that there are only finitely many possibilities for $\nu(f_k)$. □

Theorem 13.5 *Let M be a closed negatively curved manifold of dimension ≥ 3 and let $n > \dim(M)$ be an integer. Let $f_k\colon B \to N_k$ be a sequence of smooth embeddings of M into hyperbolic n–manifolds such that for each k*

- *f_k induces a monomorphism of fundamental groups, and*
- *the normal bundle $\nu(f_k)$ of the embedding f_k is orientable.*

Then the set of the normal bundles $\nu(f_k)$ falls into finitely many isomorphism classes. In particular, up to diffeomorphism, only finitely many hyperbolic n–manifolds are total spaces of orientable vector bundles over M.

Proof Passing to covers corresponding to $f_{k*}\pi_1(M)$, we can assume that $f_k\colon M \to N_k$ induce isomorphisms of fundamental groups. Arguing by contradiction, assume that $\xi_k = \xi(f_k)$ are pairwise nonisomorphic.

Arguing as in the proof of 13.4, we deduce that there are only finitely many possibilities for the total Pontrjagin classes of ξ_k. Thus, according to 13.1, we can pass to a subsequence so that the (rational) Euler classes of ξ_k are all different. Denote the integral Euler class by $e(\xi_k)$.

First, assume that M is orientable. Recall that, by definition, the Euler class $e(\xi_k)$ is the image of the Thom class $U(\xi_k) \in H^m(N_k, N_k \setminus f_k(M))$ under the map $f_k^*\colon H^m(N_k, N_k \setminus f_k(M)) \to H^m(M)$. According to [13, VIII.11.18] the Thom class has the property $U(\xi_k) \cap [N_k, N_k \setminus f_k(M)] = f_{k*}[M]$ where $[N_k, N_k \setminus f_k(M)]$ is the fundamental class of the pair $(N_k, N_k \setminus f_k(M))$ and $[M]$ is the fundamental class of M.

Therefore, for any $\alpha \in H_m(M)$, the intersection number of $f_{k*}\alpha$ and $f_{k*}[M]$ in N_k satisfies

$$I(f_{k*}[M], f_{k*}\alpha) = \langle U(\xi_k), f_{k*}\alpha\rangle = \langle f_k^* U(\xi_k), \alpha\rangle = \langle e(\xi_k), \alpha\rangle.$$

Since M is compact, $H_m(M)$ is finitely generated; we fix a finite set of generators. The (rational) Euler classes are all different, hence the homomorphisms $\langle e(\xi_k), -\rangle \in \mathrm{Hom}(H_m(M), \mathbb{Z})$ are all different. Then there exists a generator $\alpha \in H_m(M)$ such that $\{\langle e(\xi_k), \alpha\rangle\}$ is an infinite set of integers. Hence $\{I(f_{k*}[M], f_{k*}\alpha)\}$ is an infinite set of integers. Combining 12.3 and 10.3, we get a contradiction.

Assume now that M is nonorientable. Let $q\colon \tilde{M} \to M$ be the orientable two-fold cover. Any finite cover of aspherical manifolds induces an injection on rational cohomology [10, III.9.5(b)]. Hence $e(q^\# \xi_k) = q^* e(\xi_k)$ implies that the rational Euler classes of the pullback bundles $q^\# \xi_k$ are all different, and there are only finitely many possibilities for the total Pontrjagin classes of $q^\# \xi_k$. Furthermore, the bundle map $q^\# \xi_k \to \xi_k$ induces a smooth two-fold cover of the total spaces, thus the total space of $q^\# \xi_k$ is hyperbolic. Finally, $\tilde{M}$ is a closed orientable negatively curved manifold. Thus, we get a contradiction as in the oriented case. □

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