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What is a virtual link?

Greg Kuperberg

Abstract Several authors have recently studied virtual knots and links because they admit invariants arising from R-matrices. We prove that every virtual link is uniquely represented by a link $L \subset S \times I$ in a thickened, compact, oriented surface S such that the link complement $(S \times I) \setminus L$ has no essential vertical cylinder.

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A *virtual link* L is an equivalence class of decorated, finite, tetravalent graphs Γ. The edges at each vertex must be cyclically ordered, and two opposite edges are labelled as an *overcrossing*, while the other two are labelled as an *undercrossing*. The equivalence relation is the one given by Reidemeister moves. The notion was proposed by Kauffman [5] in light of the fact that R-matrices and quandles, which are commonly used to make link invariants, also yield invariants of virtual links.

We will borrow from the analysis of virtual links by Carter, Kamada, and Saito [1]. They show that virtual links are equivalent to stable equivalence classes of links projections onto compact, oriented surfaces. (Fenn, Rourke, and Sanderson have written a self-contained proof [2].) The surface S need not be connected, but we require that each component contains at least one component of the projection P. The projection P is considered up to Reidemeister moves, and the stabilization operation consists of adding a handle to S. Note that the feet of a stabilizing handle may lie in any two regions of the complement of P (Figure 1).

As Figure 1 also shows, the reverse destabilization operation consists of cutting the surface S along a circle C which is disjoint from P, and capping the resulting boundary. If C separates the connected component of S' of S containing it, then we require that both components of $S' \setminus C$ contain part of P; otherwise destabilization would create a naked surface component.

It is also well-known that a link drawn on a surface S, considered up to Reidemeister moves, is equivalent to a (tame) link $L \subset S \times I$ in the thickened surface

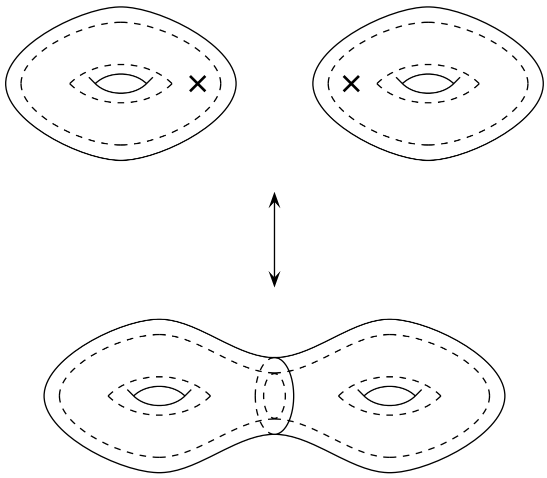

Figure 1: The stabilization move on $S \times I$.

S, considered up to isotopy. The destabilization operation, then, consists of cutting S along a vertical annulus $C \times I$ which is disjoint from L, and capping the two resulting annuli with thickened disks $D^2 \times I$. Since L is only considered up to isotopy, it is equivalent to allow this operation with any topologically vertical annulus A, *i.e.*, any properly embedded annulus isotopic to some $C \times I$. This will be our working definition of virtual links.

Theorem 1 *Every stable equivalence class of links in thickened surfaces has a unique irreducible representative.*

Theorem 1 is proved in the same way as several classical results in 3-manifold topology, from the unique factorization of knots and 3-manifolds [8, 7] to the Jaco-Shalen-Johannson theorem [3, 4], from which Theorem 1 also follows as a corollary. Another similar result is the author's classification of scalar involutory Hopf words [6]. The motivation is also similar, since a virtual link can be viewed as a scalar R-matrix word.

Proof In outline, we induct on the complexity of intersection between two destabilization annuli by compressing one along an innermost disk of the other.

First generalize the definition of destabilization of a link $L \subset S \times I$ to include two other operations:

(1) If a sublink $L' \subset L$ is separated from the rest of L by a sphere or disk $A \subset S \times I$, then we can remove L' from $S \times I$ and place it in a separate thickened sphere $S^2 \times I$.

(2) If an annulus A divides $S \times I$ with L entirely on one side and some genus of S on the other side, we can cut $S \times I$ along A, discard the naked component, and cap the remaining component. Both operations can easily be reproduced by destabilization as defined previously.

Say that a surface is *admissible* if it is a vertical annulus, a sphere, or a proper disk; and that an admissible surface is *essential* if it does not bound a ball in $(S \times I) \setminus L$. Thus, admissible, essential surfaces are those along which we can destabilize $L \subset S \times I$.

Suppose, to the contrary of the conclusion, that some link $L \subset S \times I$ has more than one irreducible descendant. If S has c components with total genus g, and L has n components, assume that $g+n-c$ is minimal among counterexamples. (Note that $n \geq c$.) Then every destabilization of $L \subset S \times I$ has a unique irreducible descendant, since destabilization always reduces $g+n-c$. Say that two such destabilizations, $L \subset S_1 \times I$ and $L \subset S_2 \times I$, are *descent equivalent* if their irreducible descendants are isomorphic. The aim is to show that all destabilizations of $L \subset S \times I$ are descent equivalent.

For example, if A_1 and A_2 are disjoint admissible, essential surfaces, then the resulting destabilizations $L \subset S_1 \times I$ and $L \subset S_2 \times I$ are descent equivalent. This is immediate if A_1 and A_2 are parallel. If they are not, then we can destabilize each $L \subset S_i \times I$ along A_{3-i} to produce a common descendant.

Suppose that A_1 and A_2 are descent-inequivalent surfaces in general position, and that they intersect in the fewest curves among descent-inequivalent pairs in $(S \times I) \setminus L$. If a curve $C \subset A_1 \cap A_2$ is a circle, then it is either *horizontal* in A_i, if A_i is an annulus and C is parallel to ∂A_i, or it bounds a disk in A_i. If C is an arc, then it is either *vertical*, if A_i is an annulus and C connects the two components of ∂A_i, or it and part of the boundary of A_i bound a disk.

If a circle of $A_1 \cap A_2$ is non-horizontal in A_1 (say), then some such circle C is *innermost*, meaning that it bounds a naked disk D in A_1, as in Figure 2(a). In this case let A_2' and A_2'' be the connected components of the compression of A_2 along the disk D, as in Figure 2(b). Both A_2' and A_2'' are admissible, and at least one is essential, for otherwise A_2 would not be. If A_2' (say) is essential, then it intersects A_1 less than A_2 does, and it does not intersect A_2 at all. But since A_1 and A_2 intersect least among descent-inequivalent pairs of essential surfaces, it would follow that A_1 and A_2 are both descent-equivalent to A_2', a contradiction.

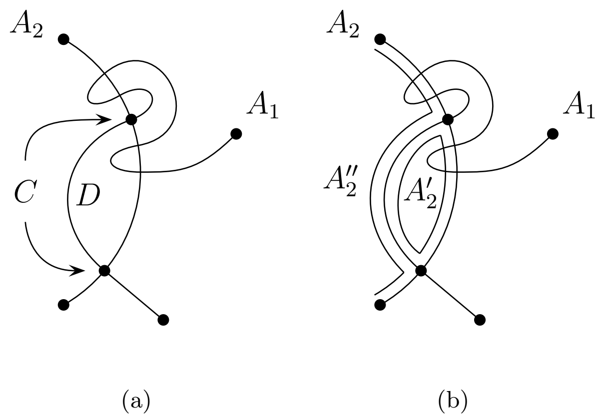

Figure 2: Compressing A_2 along the disk D to simplify $A_1 \cap A_2$ (side view).

The same argument applies if $A_1 \cap A_2$ has a non-vertical arc in A_1, and of course also to non-horizontal circles and non-vertical arcs in A_2. Thus $A_1 \cap A_2$ consists entirely of vertical segments or horizontal circles in both A_1 and A_2. In particular, both A_1 and A_2 are vertical annuli and not disks or spheres.

Suppose that $A_1 \cap A_2$ consists of horizontal circles. We can assume that none of the four circles of ∂A_1 and ∂A_2 bounds a disk in $S \times \partial I$. If, say, $C \subset A_1$ bounds a naked disk $D \subset S \times \partial I$, then we replace A_1 by the disk $D \cup A_1$ and reduce to a previous case without worsening $A_1 \cap A_2$. Otherwise let $C \subset A_1 \cap A_2$ be an *outermost* circle, meaning that it and one component of ∂A_1 bound a naked annulus $A \subset A_1$. The circle C divides A_2 into two annuli A_2' and A_2'', one of which, say A_2', makes a vertical annulus together with A. The annulus $A \cup A_2'$ is necessarily essential since its boundary circles do not bound disks in $S \times \partial I$. But after displacement, $A \cup A_2'$ intersects A_1 and A_2 less than they do each other. It is therefore descent equivalent to both.

Finally suppose that $A_1 \cap A_2$ consists of vertical arcs. The boundary of a regular neighborhood of $A_1 \cup A_2$ consists of vertical annuli $B_1, B_2, \ldots, B_n$. Each B_i is disjoint from both A_1 and A_2, so if any of them is essential, it is descent equivalent to both A_1 and A_2, a contradiction. But if they are all inessential, then one of them, say B_1, separates A_1 and A_2 from the link L and bounds a ball that contains A_1 and A_2. This contradicts the hypothesis that A_1 and A_2 are essential. □

Theorem 1 implies that if two links are equivalent as virtual links, then they are equivalent as links. It also generalizes to oriented virtual links, to colored virtual links, and even to virtual tangled graphs. The proof in each case is the same.

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Department of Mathematics, University of California
Davis, CA 95616, USA

Email: `greg@math.ucdavis.edu`

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