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CENTRALITY AND NORMALITY IN PROTOMODULAR CATEGORIES

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Abstract. We analyse the classical property of centrality of equivalence relations in terms of normal monomorphisms. For this purpose, the internal structure of connector is introduced, allowing to clarify classical results in Maltsev categories and to prove new ones in protomodular categories. This approach allows to work in the general context of finitely complete categories, without requiring the usual Barr exactness assumption.

1. Introduction

In this paper we investigate the relationship between the classical notion of centrality of equivalence relations [22] and the abstract notion of normal monomorphism introduced in [4].

A first important result in this direction was proved in the context of protomodular categories, where the abelian objects were characterized by a normality condition [4]. More precisely, an object Z in a protomodular category is abelian if and only if the diagonal $Z \rightarrowtail Z \times Z$ is a normal monomorphism. This neat result in the description of the property of centrality for the largest equivalence relation on a given object gave rise to the question whether it was always possible to express the centrality of two arbitrary equivalence relations on the same object in terms of normal monomorphisms. The aim of this paper consists in giving a positive answer to this question.

The most important notion in order to understand the precise link between centrality and normality is the structure of *connector*. If R and S are two equivalence relations on the same object Z, we denote by $R \underset{Z}{\times} S$ the pullback

$$\begin{array}{ccc} R \underset{Z}{\times} S & \xrightarrow{p_1} & S \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ R & \xrightarrow[d_1]{} & Z. \end{array}$$

A connector between R and S is an arrow $p : R \underset{Z}{\times} S \to Z$ such that

1. $xSp(x, y, z)$ 1.* $zRp(x, y, z)$

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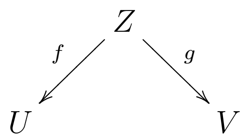
2. $p(x, x, y) = y$ 2.* $p(x, y, y) = x$

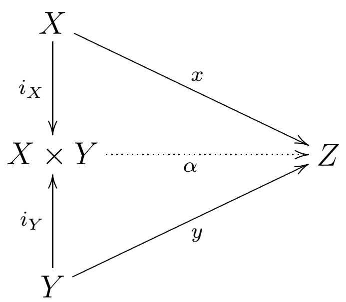
3. $p(x, y, p(y, u, v)) = p(x, u, v)$ 3.* $p(p(x, y, u), u, v) = p(x, y, v)$

Several very similar notions already exist in the literature in works by Kock (under the name of pregroupoid [16] [17] [18]), by Johnstone [15] (under the name of herdoid) and by Janelidze [13]; some of these notions have been later used also by Carboni, Pedicchio and others [10], [19], [20] and [21]. The difference between the notion of pregroupoid introduced by Kock and the notion of connector between R and S consists in the fact that his notion is defined globally as a special case of double relation Λ on an object Z, while ours emphasizes the role of the underlying link between the two extremal subrelations R and S of Λ. The difference between the notion of herdoid introduced by Johnstone and ours is that, for him, the equivalence relations R and S are effective (i.e. they are the kernel pairs of the projections f and g of a span). Once again, a herdoid appears to be a global structure, but on a span.



The diversity of the terminology for very similar notions was problematic for us. We propose here the new name of connector to emphasize the fact that there actually are three independent data, namely the relations R and S and the map which connects them.

After introducing the structure of connector and giving a few examples, we then study its properties in several significant contexts. We first consider the case when the basic category $\mathcal{C}$ is pointed. In this situation any equivalence relation R on Z determines the equivalence class $X \rightarrowtail Z$ of the canonical point of Z. This class is called the normal subobject associated with R. The presence of a connector between R and S determines a centrality property, which is materialized by the existence of a factorisation $\alpha\colon X \times Y \to Z$ of the two normal subobjects $X \rightarrowtail Z$ and $Y \rightarrowtail Z$ associated with R and S, respectively.



The richer the structure of the category, the stronger the meaning of the notion of connector becomes. When $\mathcal{C}$ is Maltsev [9], namely when any reflexive relation is an equivalence relation, we know that there is at most one connector between two relations R and S [10]. If there is such a connector, we then say that R and S are connected. If $\mathcal{C}$

is also Barr exact and has coequalizers, we can use the categorical notion of commutator introduced by Pedicchio [19] and one of her pioneering results in order to assert that R and S are connected if and only if the commutator is trivial, in other words $[R, S] = \Delta_Z$ (where Δ_Z is the smallest equivalence relation on Z). Several stability properties of connected relations in Maltsev categories have been studied also in [7].

Now, when $\mathcal{C}$ is protomodular [2], there is a perfect correspondence between centrality and normality: the existence of a connector between two equivalence relations can be expressed, as we expected, in terms of normality conditions. Indeed, the following conditions are proved to be equivalent:

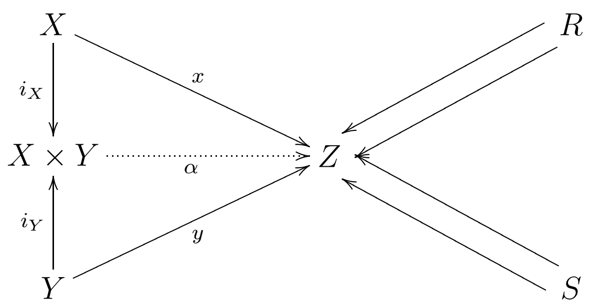
1. the two equivalence relations R and S on Z are connected

2. the inclusion of $s_0 \circ x_R \colon X_R \to Z \to S$ is normal in $\mathcal{C}$ (where $x_R \colon X_R \to Z$ is a normal subobject associated with R and $s_0 \colon Z \to S$ is the subdiagonal giving the reflectivity of the relation S)

3. the map $s\colon R \to R \underset{Z}{\times} S$ is normal in the protomodular fibre $\mathrm{Pt}_Z(\mathcal{C})$ (where $\mathrm{Pt}_Z(\mathcal{C})$ is the category of split epimorphisms with codomain Z, and $s(x, y) = (x, y, y)$)

The property of connectors culminates when the basic category $\mathcal{C}$ is pointed and strongly protomodular [5], as in the case of the varieties of groups and rings. In this situation, we have the converse of our first observation about connectors in pointed categories, without requiring any further assumption.

Indeed, the following result can be proved: let $x\colon X \to Z$ and $y\colon Y \to Z$ be the normal subobjects associated with the equivalence relations R and S on Z. Then R and S are connected if and only if there is a (unique) factorisation $\alpha\colon X \times Y \to Z$ such that $\alpha \circ i_X = x$ and $\alpha \circ i_Y = y$.



This paper is structured in five sections: 2. Connector
3. Pointed categories
4. Maltsev categories
5. Protomodular categories
6. Strongly protomodular categories

2. Connector

Let $\mathcal{C}$ be a finitely complete category. The kernel pair of a map $f\colon X \to Y$ is denoted by $R[f]$; if R and S are two equivalence relations on Z, we denote by $R \underset{Z}{\times} S$ the pullback

$$\begin{array}{ccc} R \underset{Z}{\times} S & \xrightarrow{p_1} & S \\ {\scriptstyle p_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ R & \xrightarrow[d_1]{} & Z. \end{array} \tag{1}$$

2.1. Definition. *A left action of R on S is a map $p : R \underset{Z}{\times} S \to Z$ such that*

1. $xSp(x, y, z)$
2. $p(x, x, y) = y$
3. $p(x, y, p(y, u, v)) = p(x, u, v)$

2.2. Definition. *A left action of R on S is a connector between R and S when the map $p : R \underset{Z}{\times} S \to Z$ also satisfies the symmetric properties*

1. $zRp(x, y, z)$
2. $p(x, y, y) = x$
3. $p(p(x, y, u), u, v) = p(x, y, v)$

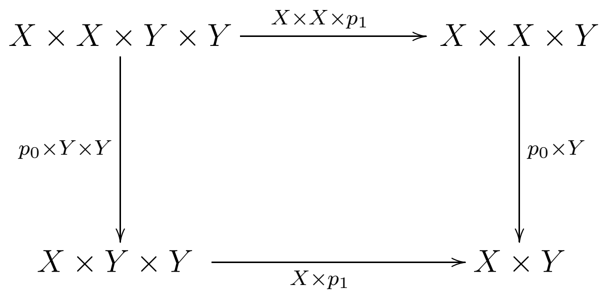
Thinking of the equivalence relation $(d_0, d_1)\colon R \rightarrowtail Z \times Z$ as a special kind of internal groupoid, a left action is equivalent to an action of the groupoid $(d_0, d_1)\colon R \rightarrowtail Z \times Z$ on $d_0\colon S \to Z$, thanks to the arrow $\pi_0\colon R \underset{Z}{\times} S \to S$

$$\begin{array}{ccccc} R \underset{Z}{\times} S & \overset{p_1}{\underset{\pi_0}{\rightrightarrows}} & S & \xrightarrow{d_1} & Z \\ {\scriptstyle s_0}\uparrow\;\downarrow{\scriptstyle p_0} & & {\scriptstyle s_0}\uparrow\;\downarrow{\scriptstyle d_0} & & \\ R & \overset{d_1}{\underset{d_0}{\rightrightarrows}} & Z & & \end{array}$$

where $\pi_0(x, y, z) = (x, p(x, y, z))$ (and consequently $p = d_1 \circ \pi_0$). This action is a connector when the map $\pi_1(x, y, z) = (p(x, y, z), z)$ also defines an action of $(d_0, d_1)\colon S \rightarrowtail Z \times Z$ on $d_1\colon R \to Z$. Remark that all the commutative squares in the diagram above are pullbacks.

2.3. Example. An associative Maltsev operation $p\colon X \times X \times X \to X$ is precisely a connector between ∇_X and ∇_X, where ∇_X is the largest equivalence relation on X (the kernel equivalence of the terminal arrow $X \to 1$).

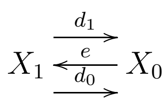
2.4. Example. **The canonical connector underlying the product**. Given two objects X and Y, there is a canonical connector between $R[p_X]$ and $R[p_Y]$ (where the arrows $p_X\colon X \times Y \to X$ and $p_Y\colon X \times Y \to Y$ are the product projections). If we consider the following pullback



then the canonical connector $p\colon X \times X \times Y \times Y \to X \times Y$ is defined by

$$p(x, x', y, y') = (x', y).$$

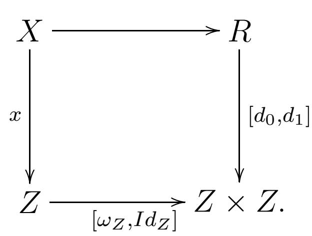
2.5. Example. Given a reflexive graph



the connectors between $R[d_0]$ and $R[d_1]$ are in bijection with the groupoid structures on this reflexive graph [10]. The natural groupoid structure of the equivalence relation ∇_X on an object X corresponds exactly to the connector underlying the product $X \times X$ (example 2.4).

3. Pointed categories

When $\mathcal{C}$ is a pointed category, any equivalence relation R on Z gives rise to a specific equivalence class $x\colon X \rightarrowtail Z$, namely that of the canonical point. If $\omega_Z\colon Z \to Z$ denotes the zero arrow, the arrow $x\colon X \rightarrowtail Z$ can be obtained by the pullback



Let $\mathrm{Eq}(\mathcal{C})$ denote the category whose objects are the equivalence relations and whose maps between (R, X) and (R', X') are morphisms $f\colon X \to X'$ in $\mathcal{C}$ such that there is a (unique) factorisation $\tilde{f}$ making the following diagram commutative

$$\begin{array}{ccc} R & \xrightarrow{\tilde{f}} & R' \\ {\scriptstyle [d_0,d_1]}\downarrow & & \downarrow{\scriptstyle [d_0,d_1]} \\ X \times X & \xrightarrow[f\times f]{} & X' \times X'. \end{array}$$

If the previous square is a pullback, we denote R by $f^{-1}(R')$ and say that R is the inverse image of R' by f. We say that a morphism $f\colon (R, X) \to (R', X')$ in $\mathrm{Eq}(\mathcal{C})$ is fibrant if the following square is a pullback:

$$\begin{array}{ccc} R & \xrightarrow{\tilde{f}} & R' \\ {\scriptstyle d_0}\downarrow & & \downarrow{\scriptstyle d_0} \\ X & \xrightarrow[f]{} & X'. \end{array}$$

3.1. Definition. [4] *An arrow $f\colon X \to X'$ is said normal to R' when*

1. $f^{-1}(R') = \nabla_X$
2. *the arrow $f\colon (\nabla_X, X) \to (R', X')$ in $\mathrm{Eq}(\mathcal{C})$ is fibrant:*

$$\begin{array}{ccc} X \times X & \xrightarrow{\tilde{f}} & R' \\ {\scriptstyle d_1}\downarrow\downarrow{\scriptstyle d_0} & & {\scriptstyle d_1}\downarrow\downarrow{\scriptstyle d_0} \\ X & \xrightarrow[f]{} & X' \end{array}$$

It can be proved [4] that a normal arrow is necessarily a monomorphism, and that normal arrows are stable by pullbacks. In the following, we shall write $\omega_{X,Y}$ for the zero arrow from X to Y, $i_X\colon X \to X \times Y$ for the arrow $[Id_X, \omega_{X,Y}]$ and $i_Y\colon Y \to X \times Y$ for the arrow $[\omega_{Y,X}, Id_Y]$.

3.2. Proposition. *Let $\mathcal{C}$ be a finitely complete pointed category. If $x\colon X \to Z$ and $y\colon Y \to Z$ denote the normal monomorphisms associated with R and S, a connector between R and S determines a factorisation $\alpha\colon X \times Y \to Z$, such that $\alpha \circ i_X = x$ and $\alpha \circ i_Y = y$.*

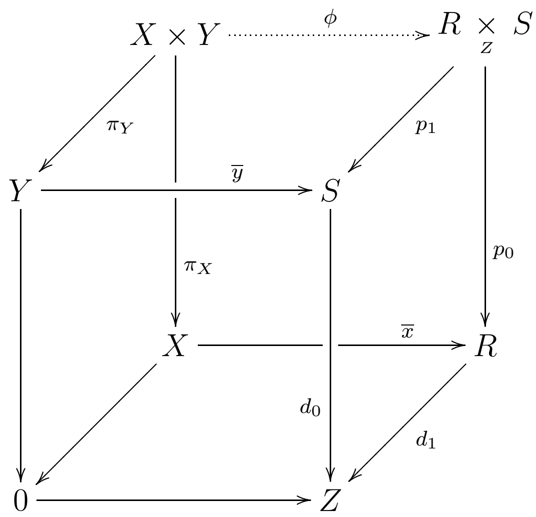
Proof. When $\mathcal{C}$ is pointed, any equivalence relation determines a unique normal monomorphism. Let us consider the diagram

$$\begin{array}{ccccc} X & \xrightarrow{l_X} & X \times X & \xrightarrow{\tilde{x}} & R \\ \downarrow & & {\scriptstyle p_1}\downarrow\downarrow{\scriptstyle p_0} & & {\scriptstyle d_1}\downarrow\downarrow{\scriptstyle d_0} \\ 0 & \longrightarrow & X & \xrightarrow[x]{} & Z, \end{array}$$

where $l_X = [Id_X, \omega_X]$ and $\tilde{x}$ is the unique arrow making the right hand squares commutative. Then the arrow $\overline{x} = \tilde{x} \circ l_X$ is clearly the kernel of d_1. Similarly in the diagram

$$\begin{array}{ccccc} Y & \xrightarrow{r_Y} & Y \times Y & \xrightarrow{\tilde{y}} & S \\ \downarrow & & {\scriptstyle p_1}\downarrow\downarrow{\scriptstyle p_0} & & {\scriptstyle d_1}\downarrow\downarrow{\scriptstyle d_0} \\ 0 & \longrightarrow & Y & \xrightarrow[y]{} & Z \end{array}$$

$\overline{y} = \tilde{y} \circ r_Y$ is the kernel of d_0. It follows that there is a unique factorisation $\phi \colon X \times Y \to R \underset{Z}{\times} S$ as in the diagram



If $p \colon R \underset{Z}{\times} S \to Z$ is a connector between R and S, the composite $\alpha = p \circ \phi \colon X \times Y \to Z$ gives the desired factorisation. Using the axioms 2 and 2^* in the definition of the connector, one can check that the arrow α has the property that $\alpha \circ i_X = x$ and $\alpha \circ i_Y = y$. ■

3.3. Remark. The assumption which is actually needed in order to prove Proposition 3.2 is slightly weaker than the existence of a connector between R and S: indeed, the existence of a partial Maltsev operation $p \colon R \underset{Z}{\times} S \to Z$ [10] suffices (i.e. axioms 2 and 2^*).

4. Maltsev categories

A category is Maltsev if any internal reflexive relation is an equivalence relation [9]. This property is equivalent to many other interesting ones [8], among which we would like to recall the property of difunctionality. A relation $R \rightarrowtail A \times B$ from A to B is difunctional if, whenever xRy, zRy and zRu, then xRu (for any $x, z \in A$ and $y, u \in B$). A finitely complete category is Maltsev if and only if any relation is difunctional.

If $\mathcal{C}$ denotes a Maltsev category with finite limits, the definition of a connector between two equivalence relations R and S can be highly simplified, since conditions $1, 1^*, 3$ and 3^* can be dropped. Moreover, when there exists a connector between R and S, it is necessarily unique (see also [10]).

4.1. Proposition. *Let $\mathcal{C}$ be a Maltsev category, R and S two equivalence relations on $Z \in \mathcal{C}$.*

1. *an arrow $p\colon R \underset{Z}{\times} S \to Z$ is a connector if and only if $p(x, x, y) = y$ and $p(x, y, y) = x$*
2. *a connector between R and S is unique, when it exists*

Proof. Suppose that $p\colon R \underset{Z}{\times} S \to Z$ satisfies the Maltsev conditions. Define a relation $H \to (Z \times Z) \times Z$ as follows: $(x, y)Hz$ if and only if

1. xRy and ySz
2. $xSp(x, y, z)$

For any $(x, y, z) \in R \underset{Z}{\times} S$ one obviously has $(x, y)Hy, (y, y)Hy$ and $(y, y)Hz$. By difunctionality it follows $(x, y)Hz$ and $xSp(x, y, z)$. Similarly one proves that $zRp(x, y, z)$. By difunctionality one can also prove that we have the two associativity conditions in the definition of connector, and hence the mixed associativity $p(x, y, p(z, u, v)) = p(p(x, y, z), u, v)$. It classically follows that if p' is another connector between R and S, we have $p = p'$ [10]. ∎

An important consequence of the proposition above is that for two equivalence relations R and S in a Maltsev category, to have a connector becomes a property. If there is such a connector, then we shall say that R and S are connected.

When $\mathcal{C}$ is exact Maltsev with coequalizers Pedicchio defined in [19] a notion of commutator of equivalence relations generalizing the classical notion of commutator used in universal algebra. In this context connectedness is characterized by the triviality of the commutator.

4.2. Proposition. [20] *Let $\mathcal{C}$ be an exact Maltsev category with coequalizers. Then two equivalence relations R and S on an object Z are connected if and only if $[R, S] = \Delta_Z$ (where Δ_Z is the smallest equivalence relation on Z).*

Proof. This follows from the fact that in an exact Maltsev category with coequalizers two equivalence relations R and S on Z are such that $[R, S] = \Delta_Z$ if and only if the span

$$\begin{array}{ccccc} & & Z & & \\ & \swarrow^{q_R} & & \searrow^{q_S} & \\ \frac{Z}{R} & & & & \frac{Z}{S} \end{array}$$

has an internal herdoid structure (where q_R and q_S are the canonical quotients) [20]. By the remark concerning herdoids in the introduction, this is equivalent to the fact that the kernel equivalence relations R of q_R and S of q_S are connected. ■

5. Protomodular categories

Let $\mathcal{C}$ be a finitely complete category. We denote by $\mathrm{Pt}(\mathcal{C})$ the category whose objects are the split epimorphisms with a given splitting and arrows the commutative squares between these data. We denote by $\pi\colon \mathrm{Pt}(\mathcal{C}) \to \mathcal{C}$ the functor associating its codomain with any split epimorphism; this functor π is a fibration, which is called the *fibration of pointed objects*. A protomodular category $\mathcal{C}$ is a left exact category such that every change of base functor with respect to the fibration π is conservative. If $f\colon X \to Y$ is an arrow in $\mathcal{C}$, we denote by $f^*\colon \mathrm{Pt}_Y(\mathcal{C}) \to \mathrm{Pt}_X(\mathcal{C})$ the change of base functor along f, where $\mathrm{Pt}_X(\mathcal{C})$ and $\mathrm{Pt}_Y(\mathcal{C})$ are the fibres over X and Y respectively. Any protomodular category is a Maltsev category [3]. Important examples of protomodular categories are given by several varieties of classical algebraic structures, such as the categories of groups, rings, associative and Lie algebras, Ω-groups [11]. The categories of internal algebraic structures of this kind in any finitely complete category $\mathcal{C}$ are protomodular. This is also the case for any fibre $\mathrm{Grpd}_Z(\mathcal{C})$ of the fibration $()_0\colon \mathrm{Grpd}(\mathcal{C}) \to \mathcal{C}$ associating its object of objects with any internal groupoid. Finally, since the category of Heyting algebras is protomodular, so is the dual category of any elementary topos [3]. Let us consider the following useful

5.1. Lemma. *Let (R, X) and (R', X') be two equivalence relations in a protomodular category $\mathcal{C}$. When the arrow $f\colon (R, X) \to (R', X')$ is fibrant in $\mathrm{Eq}(\mathcal{C})$, then it is cocartesian with respect to the fibration $U\colon \mathrm{Eq}(\mathcal{C}) \to \mathcal{C}$, where $U(R, X) = X$.*

Proof. Consider a fibrant map $f\colon (R, X) \to (R', X')$ in $\mathrm{Eq}(\mathcal{C})$. Assume that we have a map $g\colon (R, X) \to (S, Y)$ in $\mathrm{Eq}(\mathcal{C})$ and a map $h\colon X' \to Y$ in $\mathcal{C}$ such that $h \circ f = g$. We have to construct a map $h\colon (R', X') \to (S, Y)$, and, for this, it suffices to show that $R' \subset h^{-1}(S)$. But we have $R \subset g^{-1}(S) = f^{-1}[h^{-1}(S)]$. Consider the following diagram in $\mathrm{Eq}(\mathcal{C})$

$$\begin{array}{ccccc} R \cap g^{-1}(S) & \xrightarrow{f'} & R' \cap h^{-1}(S) & \longrightarrow & h^{-1}(S) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\alpha} & & \downarrow \\ R & \xrightarrow[f]{} & R' & \longrightarrow & \nabla_{X'}. \end{array}$$

The left hand square is a pullback since the right hand square and the outer rectangle are pullbacks. Now f is fibrant, so f' is fibrant. But β is an iso since $R \subset g^{-1}(S)$, and $\mathcal{C}$ being protomodular, α is an isomorphism, whence $R' \subset h^{-1}(S)$. ∎

A given arrow in a category can be normal to different equivalence relations (it is the case, for instance, in the category of *Sets*). A remarkable property of protomodular categories comes from the fact that an arrow can be normal to at most one equivalence relation [4]; *accordingly, this defines normality as a property.* In the following theorem we show that in the case of protomodular categories the connectedness of two equivalence relations is characterized by a normality condition. If S is any equivalence relation on Z, we denote by $s_0 \colon Z \to S$ the subdiagonal giving the reflectivity of the equivalence relation S.

5.2. Theorem. *Let $\mathcal{C}$ be a protomodular category, R and S two equivalence relations on Z, $x \colon X \to Z$ and $y \colon Y \to Z$ two normal subobjects of Z associated with R and S. Then R and S are connected if and only if the map $s_0 \circ x \colon X \to Z \to S$ is normal in $\mathcal{C}$.*

Proof. Let the arrow $s_0 \circ x \colon X \to Z \to S$ be a normal monomorphism, and let

$$\begin{array}{ccc} & \xrightarrow{d_1} & \\ T & \longleftarrow & S \\ & \xrightarrow[d_0]{} & \end{array}$$

be the equivalence relation associated with $s_0 \circ x$. We thus have a fibrant map $s_0 \circ x \colon (\nabla_X, X) \to (T, S)$. We also have the map $x \colon (\nabla_X, X) \to (R, Z)$ in $\mathrm{Eq}(\mathcal{C})$, and the map $d_0 \colon S \to Z$ such that $d_0 \circ (s_0 \circ x) = x$. According to the previous lemma there is a map $d_0 \colon (T, S) \to (R, Z)$ in $\mathrm{Eq}(\mathcal{C})$, which is fibrant since $x \colon (\nabla_X, X) \to (R, Z)$ is fibrant:

$$\begin{array}{ccc} T & \xrightarrow{\overline{d_0}} & R \\ {\scriptstyle d_1}\downarrow\downarrow{\scriptstyle d_0} & & {\scriptstyle d_1}\downarrow\downarrow{\scriptstyle d_0} \\ S & \xrightarrow[d_0]{} & Z. \end{array}$$

Consequently T is $R \underset{Z}{\times} S$, and the connector is given by

$$d_1 \circ d_0 \colon R \underset{Z}{\times} S \to S \to Z.$$

Conversely, let $p \colon R \underset{Z}{\times} S \to Z$ be the connector between R and S. The pair of arrows (π_0, p_1), where $\pi_0(x, y, z) = (x, p(x, y, z))$ and $p_1(x, y, z) = (y, z)$, defines a relation on S:

$$R \underset{Z}{\times} S \underset{p_1}{\overset{\pi_0}{\rightrightarrows}} S.$$

We call this relation the Chasles relation $\mathrm{Ch}[p]$ associated with p, where $(x, t)\mathrm{Ch}[p](y, z)$ if and only if $(x, y, z) \in R \underset{Z}{\times} S$ and $t = p(x, y, z)$. It is reflexive since $p(x, x, y) = y$, hence an equivalence relation since any protomodular category is Maltsev [3]. The fact that $p(x, y, y) = x$ and the normality of $x\colon X \to Z$ imply that $s_0 \circ x\colon X \to Z \to S$ is normal to the equivalence relation $\mathrm{Ch}[p]$. ∎

The previous theorem uses the normal monomorphisms as a data. In any quasi pointed category (i.e. when there is an initial object 0 and the unique arrow $0 \rightarrowtail 1$ is a monomorphism) the normal monomorphisms can be always produced from the equivalence relations [6]. This gives a criterion to check whether an internal reflexive graph is an internal groupoid. Indeed, if

$$X_1 \; \overset{d_1}{\underset{d_0}{\overset{\xleftarrow{\;e\;}}{\rightrightarrows}}} \; X_0$$

is a reflexive graph, then it is an internal groupoid if and only if $R[d_0]$ and $R[d_1]$ are connected, and consequently if and only if the arrow

$$s_0 \circ \mathrm{Ker}(d_1)\colon K[d_1] \to X_1 \to R[d_0]$$

is a normal monomorphism. Of course, by the previous results, this happens if and only if the arrow

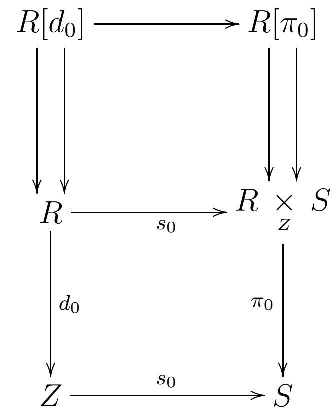
$$s_0 \circ \mathrm{Ker}(d_0)\colon K[d_0] \to X_1 \to R[d_1]$$

is a normal monomorphism.

There is also an intrinsic characterization of connectedness in the protomodular fibre $\mathrm{Pt}_Z(\mathcal{C})$ above an object Z.

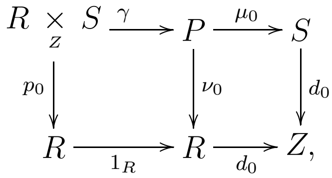
5.3. Proposition. *Let $\mathcal{C}$ be a protomodular category, R and S two equivalence relations on Z. Then R and S are connected if and only if the map s_0 in $\mathrm{Pt}_Z(\mathcal{C})$ from (R, d_0, s_0) to $(R \underset{Z}{\times} S, d_0 \circ p_0, s_0 \circ s_0)$ is normal in $\mathrm{Pt}_Z(\mathcal{C})$.*

Proof. Suppose that R and S are connected, and let π_0 denote the associated action of R on $d_0\colon S \to Z$. Consider the diagram



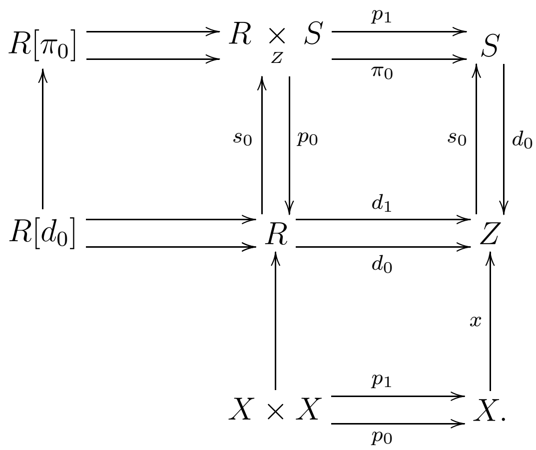
The two upper squares are pullbacks, since the lower square is a pullback. This shows that the arrow $s_0\colon R \to R \underset{Z}{\times} S$ is normal to the equivalence relation $R[\pi_0]$ in $\mathrm{Pt}_Z(\mathcal{C})$.

Conversely, let us assume that $s_0 \colon R \to R \underset{Z}{\times} S$ is a normal monomorphism in the pointed protomodular category $\mathrm{Pt}_Z(\mathcal{C})$. The arrow $p_0 \colon R \underset{Z}{\times} S \to R$ is an epimorphism split by the normal monomorphism s_0. By Proposition 12 in [4], we know that there is a canonical isomorphism $\gamma \colon R \underset{Z}{\times} S \to P$ in $\mathrm{Pt}_Z(\mathcal{C})$, where $(P, \nu_0 \colon P \to R, \mu_0 \colon P \to S)$ is the product $(R, d_0, s_0) \times \mathrm{Ker}(p_0) = (R, d_0, s_0) \times (S, d_0, s_0)$ in $\mathrm{Pt}_Z(\mathcal{C})$. We consider the commutative diagram



where the right hand square is a pullback in $\mathcal{C}$ (and a product in $\mathrm{Pt}_Z(\mathcal{C})$). We then set $p = d_1 \circ \mu_0 \circ \gamma$, and to prove that p is a connector between R and S it is enough to check that it verifies the Maltsev identities by Proposition 4.1. ∎

The two results above can be summarized by a double interpretation of the following diagram



The horizontal rectangle specifies that $s_0 \colon (R, d_0, s_0) \to (R \underset{Z}{\times} S, d_0 \circ p_0, s_0 \circ s_0)$ is normal in $\mathrm{Pt}_Z(\mathcal{C})$. The vertical rectangle specifies that $s_0 \circ x \colon X \to S$ is normal in $\mathcal{C}$.

6. Strongly protomodular categories

A finite limit preserving functor $F \colon \mathcal{C} \to \mathcal{D}$ between finitely complete categories is normal when it is conservative and reflects normal monomorphisms [5]. This latter condition means that when $j \colon X \to Y$ is an arrow in $\mathcal{C}$ such that $F(j) \colon F(X) \to F(Y)$ is normal to some equivalence relation S on $F(Y)$, then there exists an equivalence relation R on Y such that j is normal to R and $F(R) = S$. A finitely complete category is said to be strongly protomodular when the change of base functors with respect to $\pi \colon \mathrm{Pt}(\mathcal{C}) \to \mathcal{C}$ are normal. There are many interesting examples of strongly protomodular categories:

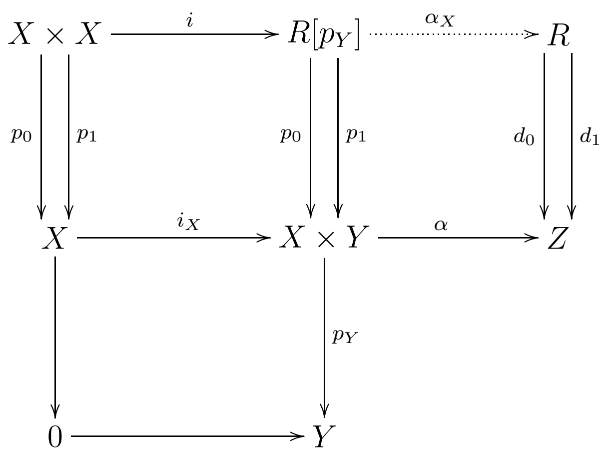
the categories of groups and the category of rings, the category of internal groups or rings in any category with finite limits, the fibres $\mathrm{Grpd}_Z(\mathcal{C})$, any presheaf category of groups $\mathrm{Grp}^{\mathcal{E}}$ or rings $\mathrm{Rng}^{\mathcal{E}}$, any protomodular naturally Maltsev category. Not all protomodular categories have their change of base functors normal: a counter-example is given by the category of the digroups [5], which are a particular kind of Ω-groups.

In pointed strongly protomodular categories we shall have the converse of the property given in section 3. This gives a description of centrality as it works in the category of groups without requiring the usual Barr exactness assumption on the basic category. In the following theorem we use the same notations as in section 3.:

6.1. Theorem. *Let $\mathcal{C}$ be a pointed strongly protomodular category. Given two normal subobjects $x\colon X \to Z$ and $y\colon Y \to Z$ of Z associated with the equivalence relations R and S, then R and S are connected if and only if there is a (unique) factorisation $\alpha\colon X\times Y \to Z$ such that $\alpha \circ i_X = x$ and $\alpha \circ i_Y = y$.*

Proof. If R and S are connected, the existence of the factorisation $\alpha\colon X \times Y \to Z$ has been proved in Proposition 3.2.

Let us then assume that there is such a factorisation α with $\alpha \circ i_X = x$ and $\alpha \circ i_Y = y$. In the diagram



the left upper squares are pullbacks by construction. In any protomodular category a fibrant arrow in $\mathrm{Eq}(\mathcal{C})$ is cocartesian for the fibration $U\colon \mathrm{Eq}(\mathcal{C}) \to \mathcal{C}$, so that there exists an arrow $\alpha_X\colon R[p_Y] \to R$ making the right hand squares commutative and such that $\alpha_X \circ i = \tilde{x}$, since $\alpha \circ i_X = x$. The upper rectangles are pullbacks by the normality of $x\colon X \to Z$ and then the right hand squares are pullbacks by the protomodularity assumption, making the arrow $\alpha\colon (R[p_Y], X\times Y) \to (R, Z)$ fibrant in $\mathrm{Eq}(\mathcal{C})$. There is a similar diagram involving the normal monomorphism $y\colon Y \to Z$, producing an arrow $\alpha_Y\colon R[p_X] \to S$. But $\alpha^*\colon \mathrm{Pt}_Z(\mathcal{C}) \to \mathrm{Pt}_{X\times Y}(\mathcal{C})$ reflects the normal monomorphisms; it is then sufficient to prove that $\alpha^*(s_0)$ is normal in $\mathrm{Pt}_{X\times Y}(\mathcal{C})$ (where $s_0\colon (R, d_0, s_0) \to (R \underset{Z}{\times} S, d_0 \circ p_0, s_0 \circ s_0)$ in $\mathrm{Pt}_Z(\mathcal{C})$). This is the case, according to the canonical connector underlying the product $X \times Y$ (see example 2.4 in the second section), and to the fact that $\alpha^*(R, d_0, s_0) = (R[p_Y], p_0, s_0)$ and $\alpha^*(R \underset{Z}{\times} S, d_0 \circ p_0, s_0 \circ s_0) = (R[p_Y] \underset{X\times Y}{\times} R[p_X], p_0 \circ p_0, s_0 \circ s_0)$. Let us finally recall that in any pointed protomodular category the inclusions i_X and i_Y are jointly epimorphic [2], so that any factorisation α as above is necessarily unique, when it exists. ∎

We then can say that two normal subobjects $x\colon X \to Z$ and $y\colon Y \to Z$ of Z are connected if they satisfy the assumptions of Theorem 6.1 (see also the definition of commuting morphisms in [12]). On the other hand, this theorem allows to measure the difference between protomodular and strongly protomodular categories.

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