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A PROPERTY OF EFFECTIVIZATION AND ITS USES IN CATEGORICAL LOGIC

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Abstract. We show that a fully faithful and covering regular functor between regular categories induces a fully faithful (and covering) functor between their respective effectivizations. Such a functor between effective categories is known to be an equivalence. We exploit this result in order to give a constructive proof of conceptual completeness for regular logic. We also use it in analyzing what it means for a morphism between effective categories to be a quotient in the 2-category of effective categories and regular functors between them.

1. Introduction

In this note we contribute to the study of the effectivization process for regular categories, aiming at clarifying a couple of standard results in first-order categorical logic. More precisely we present a property of the 2-functor of effectivization, i.e the left 2-adjoint to the inclusion of the 2-category EFF of (small) effective (or Barr-exact) categories, regular functors and natural transformations into REG, the 2-category of (small) regular categories, regular functors and natural transformations. We show that the effectivization process carries a fully faithful and covering regular functor between regular categories to a fully faithful and covering functor (hence an equivalence of categories) between their respective effectivizations.

The initial motivation for proving this property of effectivization comes from the study of conceptual completeness in categorical logic: It is known since [Makkai and Reyes 1977] that whenever a morphism of pretoposes $F\colon \mathcal{C} \to \mathcal{D}$ induces an equivalence

$$F^{*}\colon \mathrm{PRETOP}(\mathcal{D}, \mathrm{Set}) \to \mathrm{PRETOP}(\mathcal{C}, \mathrm{Set})$$

between their respective categories of (Set-valued) models, then F itself is an equivalence. The proof was model-theoretic, by an argument involving the compactness theorem and the method of diagrams. A. Pitts improved on that, [Pitts 1987], giving a categorical proof, valid over any base topos with a natural number object when allowing models to take values in a suitable class of toposes and equivalence to mean a fully faithful functor which is essentially surjective on objects. His argument involved the topos of

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filters construction and the calculus of relations inside a topos. When passing to the doctrine of regular logic a similar result holds: If a functor as above between effective categories induces an equivalence between the respective categories of models then it is itself an equivalence. This is an immediate consequence of the main result in [Makkai 1990]. The argument is again model-theoretic and non-constructive. A purely categorical argument, one that possibly exploits the result of Pitts on pretoposes, is hitherto missing. We present such an argument in the third section of this note.

In the final section we use our basic lemma in order to give a purely category-theoretic justification why quotient morphisms between both pretoposes and effective categories are *defined* to be morphisms in the respective 2-categories that are covering and full on subobjects. Usually justification for such a definition is provided by logical considerations, [Makkai 1985], [Pitts 1989], [Breiner 2014], i.e by analyzing the behavior of the functor induced between the pretopos (effective category) associated with a coherent (respectively, regular) theory and the one associated to an extension of the theory by adding new axioms but no new logical symbols. If instead we define a quotient morphism as a 1-cell in the relevant 2-category that universally (with respect to the 2-category) inverts a set of morphisms, then such morphisms are characterized as the covering ones that are full on subobjects.

The present work has become possible because of the progress that has taken place in connection to the study of the various completion processes of categories in the decades that followed [Makkai and Reyes 1977], [Makkai 1985], [Pitts 1987], [Makkai 1990], starting with [Carboni and Magno]. We use results from [Lack 1999], [Lack and Vitale 2001]. In particular we rely on the description, given in [Lack 1999], of the effectivization of a regular category as a full subcategory of the category of sheaves for the regular epi coverage. This particular description allows us to argue using "generalized elements". Further developments around such completions appear in [Maietti and Rosolini 2013], [Maietti and Rosolini 2015].

Notational convention: We denote by $\zeta_{\mathcal{C}}\colon \mathcal{C} \to \mathcal{C}_{ef}$ the inclusion of $\mathcal{C}$ into its effectivization (restriction of the Yoneda embedding) and we omit it from our notation when it acts on morphisms coming from $\mathcal{C}$. We denote by $F^* = F_{ef} : \mathcal{C}_{ef} \to \mathcal{D}_{ef}$ the action of effectivization on a regular functor $F\colon \mathcal{C} \to \mathcal{D}$ (so that $F^* \cdot \zeta_{\mathcal{C}} \cong \zeta_{\mathcal{D}} \cdot F$). We use F^* only when acting on objects and arrows that do not come from $\mathcal{C}$. Abusively we may write composites such as $F^*q \cdot Fu$, relying on the latter isomorphism and consistently omitting $\zeta_{\mathcal{C}}$.

2. A property of effectivization

The effectivization of a regular category was described in [Lack 1999] as a full subcategory of a category of sheaves. In particular, if $\mathcal{C}$ is regular, J is the (subcanonical) Grothendieck topology on it whose generating covering families are singleton families consisting of regular epimorphisms, then its effectivization $\mathcal{C}_{ef}$ has as objects quotients in $Sh(\mathcal{C}, J)$ of equivalence relations coming from $\mathcal{C}$. Recall that a regular functor $F\colon \mathcal{C} \to \mathcal{D}$ between regular categories is one that preserves finite limits and regular epimorphisms and that such a functor is covering when for every object $D \in \mathcal{D}$ there is an object $C \in \mathcal{C}$ and a regular epimorphism $FC \to D$.

2.1. LEMMA. *If* $F: \mathcal{C} \to \mathcal{D}$ *is a full and faithful regular functor then* $F^* = F_{ef} : \mathcal{C}_{ef} \to \mathcal{D}_{ef}$ *is faithful.*

PROOF. Let $f_1,\ f_2 : X \to Y$ be morphisms in $\mathcal{C}_{ef}$, such that $F^* f_1 = F^* f_2$ and

$$\begin{tikzcd} \zeta_{\mathcal{C}} H' \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & \zeta_{\mathcal{C}} H \arrow[r, "q_Y"] & Y \end{tikzcd}$$

be a presentation of Y as a quotient of an equivalence relation.

X occurs as a quotient $q_X: \zeta_{\mathcal{C}} E \to X$ and q_Y is an epimorphism so for each $i = 1, 2$, there is a covering $e: C \to E$ and a factorization of $f_i \cdot q_X \cdot e$ through q_Y, $f_i \cdot q_X \cdot e = q_Y \cdot \alpha_i$. Notice that if C_i, $i = 1, 2$ are two different domains for the arrows α_i induced, respectively, by the f_i, then we can pullback e_1 along e_2 and get a common domain C so that, if p_i are the projections of the pullback, we consider instead as α_i the composites $\alpha_i \cdot p_i$ and as q the composite of regular epimorphisms $q_X \cdot e_i \cdot p_i$ (for either i). The following diagram

$$\begin{tikzcd} & \zeta_{\mathcal{C}} C \arrow[r, "q"] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\alpha_2"] & X \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] \\ \zeta_{\mathcal{C}} H' \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & \zeta_{\mathcal{C}} H \arrow[r, "q_Y"] & Y \end{tikzcd}$$

induces through the action of F^* a commutative diagram

$$\begin{tikzcd} & \zeta_{\mathcal{D}} FC \arrow[r, "F^* q"] \arrow[d, shift right, "F\alpha_1"'] \arrow[d, shift left, "F\alpha_2"] \arrow[dl] & F^* X \arrow[d, "F^* f_1 = F^* f_2"] \\ \zeta_{\mathcal{D}} FH' \arrow[r, shift left, "Fh_1"] \arrow[r, shift right, "Fh_2"'] & \zeta_{\mathcal{D}} FH \arrow[r, "q_{F^* Y}"] & F^* Y \end{tikzcd}$$

Since (Fh_1, Fh_2) is an equivalence relation, hence the kernel - pair of its coequalizer, and $q_{F^*Y} \cdot F\alpha_1 = F^* f_1 \cdot F^* q = F^* f_2 \cdot F^* q = q_{F^*Y} \cdot F\alpha_2$, by the universal property of the kernel pair, there is a diagonal arrow $FC \to FH'$, of the form Fh by fullness of F, satisfying $Fh_i \cdot u = F\alpha_i$, $i = 1, 2$.

Since F and $\zeta_{\mathcal{C}}$ are faithful it fits in a commutative diagram

$$\begin{tikzcd}
 & \zeta_{\mathcal{C}}C \arrow[r, "q"] \arrow[dl, "u"'] \arrow[d, shift right, "\alpha_1"'] \arrow[d, shift left, "\alpha_2"] & X \arrow[d, shift right, "f_1"'] \arrow[d, shift left, "f_2"] \\
\zeta_{\mathcal{C}}H' \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & \zeta_{\mathcal{C}}H \arrow[r, "q_Y"] & Y
\end{tikzcd}$$

with $h_i \cdot u = \alpha_i$, $i = 1, 2$. Hence $f_1 \cdot q = q_Y \cdot \alpha_1 = q_Y \cdot h_1 \cdot u = q_Y \cdot h_2 \cdot u = q_Y \cdot \alpha_2 = f_2 \cdot q$. But q is an epimorphism so $f_1 = f_2$ ∎

2.2. Lemma. *If a functor G is fully faithful and covering then it reflects regular epimorphisms.*

Proof. Let $Gq\colon GY \to GX$ be the coequalizer of $W \underset{r_2}{\overset{r_1}{\rightrightarrows}} GY$. Cover W by $e\colon GZ \to W$. The composites $r_i \cdot e\colon GZ \to GX$ are of the form $r_i \cdot e = Gs_i$, $i = 1, 2$. By faithfulness of G we get $q \cdot s_1 = q \cdot s_2$. If for some $t\colon Y \to T$ we have $t \cdot s_1 = t \cdot s_2$, then we have $Gt \cdot r_1 \cdot e = Gt \cdot r_2 \cdot e$ and canceling out the epimorphism e we find that $Gt \cdot r_1 = Gt \cdot r_2$. The universal property of the coequalizer gives a unique $u\colon GX \to GT$ such that $u \cdot Gq = Gt$ and the full faithfulness of G gives $v\colon X \to T$ such that $u = Gv$ and $v \cdot q = t$. ∎

2.3. Lemma. *If $F\colon \mathcal{C} \to \mathcal{D}$ is a covering, full and faithful regular functor then $F^* = F_{ef} : \mathcal{C}_{ef} \to \mathcal{D}_{ef}$ is full.*

Proof. Given $g\colon F^*X \to F^*Y$ we consider presentations $\zeta_{\mathcal{D}}E' \underset{e_2}{\overset{e_1}{\rightrightarrows}} \zeta_{\mathcal{C}}E \xrightarrow{q_X} X$ and $\zeta_{\mathcal{C}}H' \underset{h_2}{\overset{h_1}{\rightrightarrows}} \zeta_{\mathcal{C}}H \xrightarrow{q_Y} Y$ of X and Y, respectively. Since $\zeta_{\mathcal{D}}FH \to F^*Y$ is epimorphic the generalized element $g \cdot F^*q_X$ is locally in $\zeta_{\mathcal{D}}FH$, meaning that there is a covering $d\colon D \to FE$ and a factorization of $g \cdot F^*q_X \cdot F^*e$ through F^*q_Y. Since F is covering we may instead consider a factorization

$$\begin{tikzcd}
\zeta_{\mathcal{D}}FC \arrow[r, "e"] \arrow[d, "Fw"] & \zeta_{\mathcal{D}}D \arrow[r, "d"] & \zeta_{\mathcal{D}}FE \arrow[r, "F^*q_X"] & F^*X \arrow[d, "g"] \\
\zeta_{\mathcal{D}}FH \arrow[rrr, "F^*q_Y"] & & & F^*Y
\end{tikzcd}$$

where both $\zeta_{\mathcal{D}}FC \to D \to \zeta_{\mathcal{D}}FE$ and $\zeta_{\mathcal{D}}FC \to \zeta_{\mathcal{D}}FH$ are in the image of F by fullness of F, denote the first one by Fc and the second one by Fw.

Fc is a regular epimorphism, being the composite of regular epimorphisms in an effective category. Moreover, being in the image of the fully faithful and covering functor $\zeta_{\mathcal{D}}F$ we get from the previous Lemma that c is itself a regular epimorphism. Hence c will be the coequalizer of its kernel pair $C' \underset{c_2}{\overset{c_1}{\rightrightarrows}} C$. Such a coequalizer is preserved by the regular functor $\zeta_{\mathcal{D}}F$. (The embedding $\zeta_{\mathcal{C}}$ is regular: If $t: \zeta_{\mathcal{C}}C \to T$ is such that $t \cdot c_1 = t \cdot c_2$, it means that we have a singleton family which is compatible with respect to the topology J, as it restricts to the same element in the pullback of the covering $E \to C$ along itself. But T is a J-sheaf, so there is an extension $\zeta_{\mathcal{C}}E \to T$.)

Obviously in $\mathcal{D}_{ef}$ we have $F^*q_Y \cdot Fw \cdot Fc_1 = F^*q_Y \cdot Fw \cdot Fc_2$ hence from the universal property of the kernel pair (Fh_1, Fh_2) we get a factorization $Fv: \zeta_{\mathcal{D}}FC' \to \zeta_{\mathcal{D}}FH'$ (using the fullness of F), such that $Fw \cdot Fc_i = Fh_i \cdot Fv$, $i = 1, 2$. By the faithfulness of F these equations are reflected in $\mathcal{C}_{ef}$.

In the diagram

$$\begin{tikzcd}
\zeta_{\mathcal{C}}C' \arrow[r, shift left, "c_1"] \arrow[r, shift right, "c_2"'] \arrow[d, "v"'] & \zeta_{\mathcal{C}}C \arrow[r, "c"] \arrow[d, "w"] & \zeta_{\mathcal{C}}E \\
\zeta_{\mathcal{C}}H' \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & \zeta_{\mathcal{C}}H \arrow[r, "q_Y"] & Y
\end{tikzcd}$$

we have

$$q_Y \cdot w \cdot c_1 = q_Y \cdot h_1 \cdot v = q_Y \cdot h_2 \cdot u = q_Y \cdot w \cdot c_2$$

so we obtain a $u: \zeta_{\mathcal{C}}E \to Y$ such that $u \cdot c = q_Y \cdot w$. Passing via F^* to $\mathcal{D}_{ef}$ we get

$$\begin{tikzcd}
\zeta_{\mathcal{D}}FC' \arrow[r, shift left, "Fc_1"] \arrow[r, shift right, "Fc_2"'] \arrow[d, "Fv"'] & \zeta_{\mathcal{D}}FC \arrow[r, "Fc"] \arrow[d, "Fw"] & \zeta_{\mathcal{D}}FE \arrow[r, "F^*q_X"] \arrow[dr, "F^*u"] & F^*X \arrow[d, "g"] \\
\zeta_{\mathcal{D}}FH' \arrow[r, shift left, "Fh_1"] \arrow[r, shift right, "Fh_2"'] & \zeta_{\mathcal{D}}FH \arrow[rr, "F^*q_Y"] & & F^*Y
\end{tikzcd}$$

We have now that $F^*u \cdot Fc = F^*q_Y Fw = g \cdot F^*q_X \cdot Fc$ and canceling out the epimorphism Fc, $F^*u = g \cdot F^*q_X$. Eventually, recalling the presentation of X we get

$$F^*u \cdot Fe_1 = g \cdot F^*q_X \cdot Fe_1 = g \cdot F^*q_X \cdot Fe_2 = F^*u \cdot Fe_2$$

Both composites are in the image of F^* so Lemma 2.1 gives us $u \cdot e_1 = u \cdot e_2$. The universal property of X as a coequalizer of e_1, e_2 implies eventually the existence of an $f: X \to Y$ such that $f \cdot q_X = u$. The universal property of F^*X as a coequalizer of Fe_1, Fe_2 gives finally that $g = F^*f$. ∎

Observe that when $F: \mathcal{C} \to \mathcal{D}$ is covering then $F_{ef} : \mathcal{C}_{ef} \to \mathcal{D}_{ef}$ is covering: Every $X \in \mathcal{D}_{ef}$ is a quotient $\zeta_{\mathcal{D}}D \to X$, $D \in \mathcal{D}$ is a quotient $FC \to D$ and $\zeta_{\mathcal{D}}$ preserves regular epimorphisms, hence we have a quotient $\zeta_{\mathcal{D}}FC \cong F_{ef}\zeta_{\mathcal{C}} \to X$.

Summarizing the content of this section and taking into account [Makkai and Reyes 1977], Lemma 1.4.9, we obtain

2.4. Theorem. *If $F: \mathcal{C} \to \mathcal{D}$ is a covering, full and faithful regular functor then*

$$F_{ef} : \mathcal{C}_{ef} \to \mathcal{D}_{ef}$$

is covering, full and faithful, hence an equivalence of categories.

Remarks: 1. The Theorem gives as corollaries two well-known instances of effectivization: The category of abelian groups is the effectivization of the full subcategory of torsion-free abelian groups, since every group is a quotient of a free one. The category of compact Hausdorff topological spaces is the effectivization of the full subcategory of Stone spaces, since every compact Hausdorff space is covered by a Stone space (e.g the Gleason cover of such a space).

2. Several significant effective categories arise as free completions of a different kind. Namely, as free effective categories over categories with (possibly weak) finite limits, [Carboni and Vitale 1998]. In this case the objects in the image of the completion $\eta_{\mathcal{C}}$: $\mathcal{C} \to \mathcal{C}_{ex/lex}$ (the "ex/lex" completion) are projective with respect to regular epimorphisms. A straightforward modification of the above arguments (using projectivity of objects in the image of η instead of the covering assumption) gives that, if the cartesian functor $F: \mathcal{C} \to \mathcal{D}$ (between categories with finite limits) is fully faithful, then so is the induced regular functor $F_{ex/lex}$: $\mathcal{C}_{ex/lex} \to \mathcal{D}_{ex/lex}$ between their completions.

3. Conceptual completeness for regular logic

We denote by $P(\mathcal{C})$ the pretopos associated with a regular category $\mathcal{C}$, i.e the pretopos enjoying the universal property that, for any pretopos $\mathcal{V}$, we have an equivalence of categories $\mathrm{REG}(\mathcal{C}, \mathcal{V}) \simeq \mathrm{PRETOP}(P(\mathcal{C}), \mathcal{V})$, and by $\eta_{\mathcal{C}}: \mathcal{C} \to P(\mathcal{C})$ the fully faithful inclusion (universal functor to pretoposes).

3.1. Lemma. *If $F: \mathcal{C} \to \mathcal{D}$ is a regular functor such that the induced functor between the associated pretoposes*

$$P(F): P(\mathcal{C}) \to P(\mathcal{D})$$

is covering, then F itself is covering.

Proof. An object $D \in \mathcal{D}$ is taken by $\eta_{\mathcal{D}}$ to an object that is covered by an object in the image of $P(F)$, $P(F)X \to \eta_{\mathcal{D}} D$. Such an X is constructed by objects in the image of $\eta_{\mathcal{C}}$ by taking finite suprema of subobjects of objects in $\mathcal{C}$, coproducts and quotients of such objects. The functor $P(F)$ preserves such constructions, hence $\eta_{\mathcal{D}} D$ is covered by an object that is coproduct of finite suprema of subobjects of objects in the image of $P(F) \cdot \eta_{\mathcal{C}} \cong \eta_{\mathcal{D}} \cdot F$. But objects in the image of $\eta_{\mathcal{D}}$ are super-compact ([Johnstone 2002] D3.3.11), hence $\eta_{\mathcal{D}} D$ is covered by one of them, $\eta_{\mathcal{D}} FC$, i.e there is a regular epimorphism $FC \to D$. ∎

3.2. Theorem. *(Conceptual Completeness for Regular Logic) Let $F: \mathcal{C} \to \mathcal{D}$ be a regular functor such that, for all toposes $\mathcal{V}$ in a sufficient class (in the sense of [Pitts 1987], Definition 2.3), the induced functors between the categories of models*

$$- \cdot F: \mathrm{REG}(\mathcal{D}, \mathcal{V}) \to \mathrm{REG}(\mathcal{C}, \mathcal{V})$$

are equivalences. Then

$$F_{ef}: \mathcal{C}_{ef} \to \mathcal{D}_{ef}$$

is an equivalence of categories.

Proof. Using the equivalence $\mathrm{REG}(\mathcal{C}, \mathcal{V}) \simeq \mathrm{PRETOP}(P(\mathcal{C}), \mathcal{V})$ and similarly for $\mathcal{D}$, we have an equivalence for models of pretoposes

$$- \cdot P(F): \mathrm{PRETOP}(P(\mathcal{D}), \mathcal{V}) \simeq \mathrm{PRETOP}(P(\mathcal{C}), \mathcal{V}),$$

which from the conceptual completeness result for pretoposes of A. Pitts ([Pitts 1987], Theorem 2.13) gives an equivalence $P(F): P(\mathcal{C}) \to P(\mathcal{D})$. Now $P(F)$ is fully faithful, hence from the square

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "F"] \arrow[d, "\eta_{\mathcal{C}}"'] & \mathcal{D} \arrow[d, "\eta_{\mathcal{D}}"'] \\
P(\mathcal{C}) \arrow[r, "P(F)"] & P(\mathcal{D})
\end{tikzcd}$$

which commutes up to natural isomorphism, we get that F is fully faithful. By the previous Lemma it is also covering, hence by the results of the previous section it induces the required equivalence. ∎

4. Quotient morphisms between effective categories and pretoposes

Quotient morphisms in 2-categories that are relevant to categorical logic (in particular in the categories of regular categories, effective categories and pretoposes) play an important role in the development of the subject. The notion of quotient that appears in the standard literature on first-order categorical logic is defined in an ad hoc manner. At the same time there have been several attempts to give a unifying notion of quotient (1-cell) inside a 2-category, usually as part of attempts to introduce 2-categorical notions of regularity and exactness. One such, more suitable for our purposes, is [Benabou 1989]. That work, along with many other attempts, falls into the much more general framework of [Bourke and Garner 2014] (see the references therein for several notions of 2-regularity that have been considered by various authors). Adapting the general definition of [Bourke and Garner 2014], let us define a 1-cell $F: \mathcal{C} \to \mathcal{D}$ in the 2-category REG to be a *quotient* if there is a diagram

$$\begin{tikzcd} \mathcal{K} \arrow[r, bend left, "R_0"] \arrow[r, bend right, "R_1"'] \arrow[r, phantom, "\Downarrow\sigma"] & \mathcal{C} \arrow[r, "F"] & \mathcal{D} \end{tikzcd}$$

so that $F\sigma$ is invertible and F is universal with this property. Recall from [Benabou 1989] that every regular morphism factors in REG as a quotient morphism followed by a conservative one. The factorization has as interposing object the regular category $\mathcal{C}[\ker F^{-1}]$ obtained by universally inverting the arrows in $\mathcal{C}$ which are taken by F to an isomorphism. The latter collection of arrows is a "*regular congruence*" in the terminology of [Benabou 1989], meaning a set of arrows Σ in a regular category $\mathcal{C}$ such that the localization functor $P_\sigma : \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ is in REG. Such sets of arrows were characterized there (Theorem 2.2.2 in loc. cit.) by elementary conditions that extend those giving a calculus of right fractions on Σ.

The passage from an exact category to a regular category of fractions does not necessarily preserve exactness: The coreflection to the inclusion of Stone spaces into the dual category of presheaves on finite Boolean algebras provides a counterexample: For the former it is well-known that it is regular but not exact. For the latter, we know that it is exact, being the dual of a topos. The fact that we have a coreflection allows us to present the category of Stone spaces as a category of right fractions of the dual category of presheaves.

Quotient morphisms in EFF are defined similarly to those in REG. Being a quotient morphism is a kind of colimit condition, hence the factorization of a 1-cell $F : \mathcal{C} \to \mathcal{D}$ in the 2-category EFF as a quotient functor followed by a conservative one arises via the effectivization of $\mathcal{C}[\ker F^{-1}]$ (which lives in REG but not necessarily in EFF).

Recall that a functor $F : \mathcal{C} \to \mathcal{D}$ is *full on subobjects* if whenever $Y \to FC$ represents a subobject of FC in $\mathcal{D}$, then there is $X \to C$, representing a subobject in $\mathcal{C}$, and an isomorphism $FX \cong Y$ in $\mathcal{D}$.

4.1. Proposition. *Let $F : \mathcal{C} \to \mathcal{D}$ be a regular functor between effective categories, which is covering and full on subobjects. Then the conservative part R: $\mathcal{C}[kerF^{-1}] \to \mathcal{D}$ of its quotient - conservative factorization in* REG *is full, faithful and covering, hence the induced regular functor (conservative part of its factorization in* EFF*) R_{ef}: $\mathcal{C}[kerF^{-1}]_{ef} \to \mathcal{D}_{ef} \simeq \mathcal{D}$ is an equivalence.*

Proof. Consider the factorization of F in EFF

$$\begin{tikzcd} \mathcal{C} \arrow[rr, "F"] \arrow[dr, "Q"'] & & \mathcal{D} \\ & \mathcal{C}[\ker F^{-1}] \arrow[ur, "R"'] \arrow[dr, "\zeta"'] & \\ & & \mathcal{C}[\ker F^{-1}]_{ef} \arrow[uu, "R_{ef}"'] \end{tikzcd}$$

where $F \cong R \cdot Q$ is its factorization in REG. It is obvious that R is faithful, as a conservative functor that preserves finite limits (equalizers in particular) and covering, because Q is the identity on objects. It is also full by the following standard argument (see [Benabou 1989] 2.5.4 as well as [Makkai 1985], Theorem 2.3.2 and the discussion preceding it for details): A morphism $g: FA \to FB$ in $\mathcal{D}$ gives rise to a subobject (its graph) $FA \to FA \times FB \cong F(A \times B)$, coming by assumption from a subobject $T \to A \times B$. Since F is faithful, the latter is a "functional subobject" of $A \times B$, hence it represents an arrow $A \to B$ in $\mathcal{C}$. Its image under F is g. Then one concludes that R_{ef} is an equivalence by our Theorem 2.4. ∎

4.2. Corollary. *A* 1*-cell* $F: \mathcal{C} \to \mathcal{D}$ *in* EFF *is a quotient if and only if it is covering and full on subobjects.*

One may now move on and extend this characterization to quotient morphisms in PRETOP. The following is an immediate corollary of more general results concerning reflexive coinverters given in [Kelly et al 1993].

4.3. Proposition. *Let $\mathcal{C}$ be a category with coproducts and Σ a class of morphisms of $\mathcal{C}$ admitting a calculus of right fractions. Assume that, for all $s: A \to B$, $t: C \to D$ in Σ, we have that $s \sqcup t: A \sqcup C \to B \sqcup D$ is in Σ. Then the category of fractions $\mathcal{C}[\Sigma^{-1}]$ has coproducts and the quotient functor $P_\Sigma: \mathcal{C} \to \mathcal{C}[\Sigma^{-1}]$ preserves them.*

Proof. Although, as said, the result is proved on the basis of much more general considerations, it may be worth noticing that given objects A, B in $\mathcal{C}[\Sigma^{-1}]$, their coproduct is given by $A \sqcup B$, because if $A \xleftarrow{s} I \xrightarrow{f} C$ and $B \xleftarrow{t} J \xrightarrow{g} C$ are two arrows from A to C and from B to C, respectively, in $\mathcal{C}[\Sigma^{-1}]$ (where the broken arrows denote morphisms in Σ), then $A \sqcup B \xleftarrow{s \sqcup t} I \sqcup J \xrightarrow{[f,g]} C$ is the required unique factorization through $A \sqcup B$ making the two triangles commutative. ∎

In particular we have

4.4. Corollary. *If $F: \mathcal{C} \to \mathcal{D}$ is a morphism of pretoposes then the class* $\ker F$*, of morphisms inverted by F, is a regular congruence that has the property of the above proposition. This means in particular that the category of fractions $\mathcal{C}[\ker F^{-1}]$ is regular extensive and the quotient functor $Q: \mathcal{C} \to \mathcal{C}[\ker F^{-1}]$ is regular.*

Proof. It follows from the above in combination with the fact that the effectivization process preserves extensivity, [Lack and Vitale 2001]. ∎

Recall further ([Makkai 1985], Proposition 2.4.7) that if $F: \mathcal{C} \to \mathcal{D}$ is a morphism of pretoposes then the localization functor $Q: \mathcal{C} \to \mathcal{C}[\ker F^{-1}]$ gives the quotient part of its factorization in the 2-category of coherent categories and that the quotient part of its factorization in PRETOP is obtained as

$$\mathcal{C} \to \mathcal{C}[\ker F^{-1}] \to P(\mathcal{C}[\ker F^{-1}]),$$

where the latter category denotes the pretopos freely generated by the coherent category. In turn, [Johnstone 2002], the pretopos freely generated by a coherent category is given by first obtaining its *positivization*, then its effectivization. But, as said, $\mathcal{C}[\mathrm{ker}F^{-1}]$ is already extensive (hence positive), so that the quotient part of the factorization of F in PRETOP arises as

$$\mathcal{C} \to \mathcal{C}[\mathrm{ker}F^{-1}] \to \mathcal{C}[\mathrm{ker}F^{-1}]_{ef}.$$

In particular $\mathcal{C}[\mathrm{ker}F^{-1}]_{ef}$ is extensive and $\zeta\colon \mathcal{C}[\mathrm{ker}F^{-1}] \to \mathcal{C}[\mathrm{ker}F^{-1}]_{ef}$ preserves coproducts by [Lack and Vitale 2001], Theorem 2.3. Also R_{ef} preserves coproducts because by [Lack and Vitale 2001], Proposition 2.5, when

$$\begin{tikzcd} \zeta E_i \arrow[r, shift left, "h_1^i"] \arrow[r, shift right, "h_2^i"'] & \zeta C_i \arrow[r, "q_i"] & X_i \end{tikzcd}$$

is a presentation of the objects X_i in $\mathcal{C}[\mathrm{ker}F^{-1}]_{ef}$, then

$$\begin{tikzcd} \zeta(E_1 \sqcup E_2) \arrow[r, shift left, "h_1"] \arrow[r, shift right, "h_2"'] & \zeta(C_1 \sqcup C_2) \arrow[r, "q"] & X_1 \sqcup X_2 \end{tikzcd}$$

is a presentation of their coproduct. Hence

$$R_{ef}(X_1 \sqcup X_2) \cong \mathrm{coeq}\big(\begin{tikzcd} RE_1 \sqcup RE_2 \arrow[r, shift left, "Rh_1"] \arrow[r, shift right, "Rh_2"'] & RC_1 \sqcup RC_2 \end{tikzcd} \big) \cong R_{ef}(X_1) \sqcup R_{ef}(X_2)$$

We conclude that

4.5. Corollary. *A* 1-*cell* $F\colon \mathcal{C} \to \mathcal{D}$ *in* PRETOP *is a quotient if and only if it is covering and full on subobjects.*

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