

# REPRESENTATION AND CHARACTER THEORY OF FINITE CATEGORICAL GROUPS

NORA GANTER AND ROBERT USHER

ABSTRACT. We study the gerbal representations of a finite group  $G$  or, equivalently, module categories over Ostrik’s category  $\text{Vec}_G^\alpha$  for a 3-cocycle  $\alpha$ . We adapt Bartlett’s string diagram formalism to this situation to prove that the categorical character of a gerbal representation is a representation of the inertia groupoid of a categorical group. We interpret such a representation as a module over the twisted Drinfeld double  $D^\alpha(G)$ .

## 1. Introduction

Let  $\mathbf{k}$  be a field. In classical representation theory, there are several equivalent definitions of the notion of a projective representation of a finite group  $G$  on a  $\mathbf{k}$ -vector space  $V$ :

- (i) a group homomorphism  $\varrho : G \longrightarrow \text{PGL}(V)$ ,
- (ii) a map  $\varrho : G \longrightarrow \text{GL}(V)$  with 2-cocycle  $\theta : G \times G \longrightarrow \mathbf{k}^\times$  such that

$$\varrho(g) \cdot \varrho(h) = \theta(g, h) \cdot \varrho(gh)$$

- (iii) a group homomorphism  $\varrho : \tilde{G} \longrightarrow \text{GL}(V)$ , for  $\tilde{G}$  a central extension of  $G$  by  $\mathbf{k}^\times$ ,
- (iv) a module over the twisted group algebra  $\mathbf{k}^\theta[G]$  for some 2-cocycle  $\theta : G \times G \longrightarrow \mathbf{k}^\times$ .

In this work we consider the situation where  $V$  is replaced by a  $\mathbf{k}$ -linear category  $\mathcal{V}$  or, more generally, by an object of a  $\mathbf{k}$ -linear strict 2-category. In [FZ12], Frenkel and Zhu categorified points (i) to (iii) as follows<sup>1</sup>

- (i) a homomorphism of groups  $G \longrightarrow \pi_0(\text{GL}(\mathcal{V}))$ , see [FZ12, Definition 2.8],
- (ii) a projective 2-representation of  $G$  on  $\mathcal{V}$  for some 3-cocycle  $\alpha : G \times G \times G \longrightarrow \mathbf{k}^\times$ , see [FZ12, Remark 2.9],

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<sup>1</sup>We have slightly modified the context of their definitions to suit our purposes, demanding  $\mathbf{k}$ -linearity, while allowing ourselves to work in the 2-categorical setup.

- (iii) a homomorphism of categorical groups  $\mathcal{G} \rightarrow \mathrm{GL}(\mathcal{V})$  where  $\mathcal{G}$  is a 2-group extension of  $G$  by  $[\mathrm{pt}/\mathbf{k}^\times]$ , see [FZ12, Definition 2.6].

They prove that these three notions specify the same data and coin the term *gerbal representation* of  $G$  to describe any of these categorifications (we will also use the term *projective 2-representation*). To be more precise, the objects described in (ii) and (iii) are in an obvious manner organised into bicategories, and the argument in [FZ12, Theorem 2.10] sketches an equivalence of bicategories. The objects in (i) classify the objects of either of these bicategories up to equivalence. We review the work of Frenkel and Zhu in Section 4. A special case of [Ost03b] yields a categorification of the last point:

- (iv) a module category over the categorified twisted group algebra  $\mathrm{Vect}_{\mathbf{k}}^\alpha[G]$  or, in Ostrik’s notation,  $\mathrm{Vec}_{\mathcal{G}}^\alpha$ .

There is an equivalence of bicategories between this formulation and those of (i)–(iii), see Section 5.1.

An important class of examples of projective 2-representations are braid group actions, which can be read about in a paper of Khovanov and Thomas [KT07] building on work of Deligne [Del97]. Projective 2-representations also expected to play a role in TQFT applications; in [FHLT10] the authors argue that the categorified twisted group algebra determines a 3-dimensional extended TQFT whose value at the point is  $\mathrm{Vect}_{\mathbf{k}}^\alpha[G]$ .

The goal of the present work is to describe the characters of projective 2-representations. The special case where  $\alpha = 0$  was treated in [GK08] and [Bar08], where the character is defined using the *categorical trace*

$$X_\varrho(g) = \mathrm{Tr}(\varrho(g)) = 2\text{-Hom}(1_V, \varrho(g)).$$

The *categorical character* of  $\varrho$  then consists of the  $X_\varrho(g)$  together with a family of isomorphisms

$$\beta_{g,h} : X(g) \rightarrow X(hgh^{-1})$$

(compare Definition 4.14). We generalise these definitions to the projective case and arrive at the following theorem.

**1.1. MAIN THEOREM.** *Let  $\mathcal{G}$  be a finite categorical group, let  $V$  be an object of a  $\mathbf{k}$ -linear strict 2-category and let*

$$\varrho : \mathcal{G} \rightarrow \mathrm{GL}(V)$$

*be a linear representation of  $\mathcal{G}$  on  $V$ . Then the categorical character of  $\varrho$  is a representation of the inertia groupoid  $\Pi_1\Lambda\mathcal{G}$  of  $\mathcal{G}$ . (see Theorem 4.17)*

Using the work of Willerton [Wil08], we further show Corollary 4.18: The category of representations of  $\Pi_1\Lambda\mathcal{G}$  is equivalent to that of modules over the twisted Drinfeld double  $D^\omega(G)$ .

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## 2. Background

2.1. **CATEGORICAL GROUPS.** By a *categorical group* or *2-group* we mean a monoidal groupoid  $(\mathcal{G}, \bullet, \mathbf{1})$  where each object is weakly invertible. For a detailed introduction to the subject, we refer the reader to [BL04], where the term *weak 2-group* is used.

2.2. **EXAMPLE.** [Symmetry 2-groups] Let  $\mathcal{V}$  be a category. Then the autoequivalences of  $\mathcal{V}$  and the natural isomorphisms between them form a categorical group. More generally, let  $V$  be an object in a bicategory. Then the weakly invertible 1-morphisms of  $V$  and the 2-isomorphisms between them form the categorical group  $1\text{Aut}(V)$ . If  $V$  is a  $\mathbf{k}$ -linear category, we may restrict ourselves to linear functors and natural transformations and write  $\text{GL}(V)$ . We will also use this notation in general  $\mathbf{k}$ -linear 2-categories.

2.3. **EXAMPLE.** [Skeletal categorical groups] Let  $\mathcal{G}$  be a skeletal 2-group, i.e., assume that each isomorphism class in  $\mathcal{G}$  contains exactly one object. Then the objects of  $\mathcal{G}$  form a group  $G := \text{ob}(\mathcal{G})$ , and the automorphisms of  $\mathbf{1}$  form an abelian group  $A := \text{aut}_{\mathcal{G}}(\mathbf{1})$ . The group  $G$  acts on  $A$  by conjugation

$$a \longmapsto g \bullet a \bullet g^{-1}$$

(unambiguous, because  $\mathcal{G}$  is skeletal). We will denote this action by

$$a^g := g \bullet a \bullet g^{-1}.$$

We make the assumption that  $\mathcal{G}$  is *special*, i.e. that the unit isomorphisms are identities, i.e.,

$$\mathbf{1} \bullet g = g = g \bullet \mathbf{1}.$$

Then  $\mathcal{G}$  is completely determined by the data above together with the 3-cocycle

$$\alpha : G \times G \times G \longrightarrow A,$$

encoding the associators

$$\alpha(g, h, k) \bullet ghk \in \text{aut}_{\mathcal{G}}(ghk).$$

Every finite categorical group is equivalent to one of this form, and there is the following result of Sinh.

2.4. THEOREM. [see [Sin75] and [BL04, §8.3]] *Let  $\mathcal{G}$  be a finite categorical group. Then  $\mathcal{G}$  is determined up to equivalence by the data of*

- (i) *a group  $G$ ,*
- (ii) *a  $G$ -module  $A$ , and*
- (iii) *an element  $[\alpha]$  of the group cohomology  $H^3(G, A)$ .*

Without loss of generality, we may assume the cocycle  $\alpha$  to be normalised. We will be particularly interested in the case where  $A = \mathbf{k}^\times$ . Cocycles of this form are key to our understanding of a variety of different topics, ranging from Chern-Simons theory ([DW90], [FQ93]) to generalised and Mathieu moonshine ([Gan09], [GPRV12]) to line bundles on Moduli spaces and twisted sectors of vertex operator algebras. In the physics literature, evidence of such cocycles typically turns up in the form of so called *phase factors*.

2.5. DEFINITION. *Let  $\mathcal{G}$  and  $\mathcal{H}$  be categorical groups. By a homomorphism from  $\mathcal{G}$  to  $\mathcal{H}$  we mean a (strong) monoidal functor*

$$\mathcal{G} \longrightarrow \mathcal{H}.$$

*A linear representation of a categorical group  $\mathcal{G}$  with centre  $\text{End}_{\mathcal{G}}(\mathbf{1}) = \mathbf{k}^\times$  is a homomorphism*

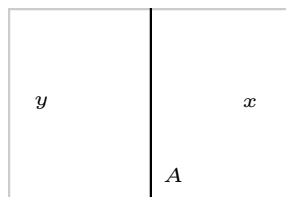
$$\varrho : \mathcal{G} \longrightarrow \text{GL}(V)$$

*where  $V$  is an object of a strict ( $\mathbf{k}$ -linear) 2-category, and  $\mathbf{k}^\times$  is required to act by multiplication with scalars.*

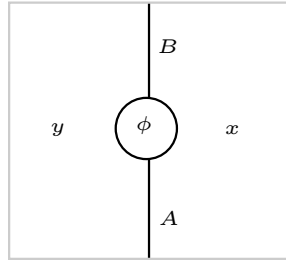
We will study such linear representations for skeletal  $\mathcal{G}$ . Note that the condition on the action of  $\mathbf{k}^\times$  implies that the action of  $G = \text{ob}(\mathcal{G})$  on  $\mathbf{k}^\times$  is trivial, restricting us to those skeletal 2-groups that are classified by  $[\alpha] \in H^3(G; \mathbf{k}^\times)$  where  $G$  acts trivially on  $\mathbf{k}^\times$ .

2.6. STRING DIAGRAMS FOR STRICT 2-CATEGORIES. We recall the string diagram formalism from [CW10, §1.1] and [Bar08, Chapter 4] (our diagrams are upside down in comparison to those in [Bar08]). This already turns up in [Pen71] and in the work of Joyal and Street [JS91], with a reference to [KL80].

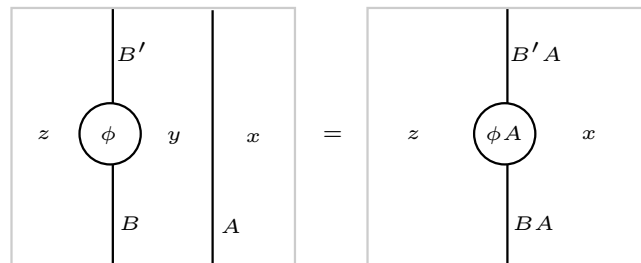
Let  $\mathcal{C}$  be a strict 2-category, i.e. a category enriched over the category of small categories. Let  $x, y$  be objects in  $\mathcal{C}$ , and let  $A$  be a 1-morphism from  $x$  to  $y$ . In string diagram notation,  $A$  is drawn



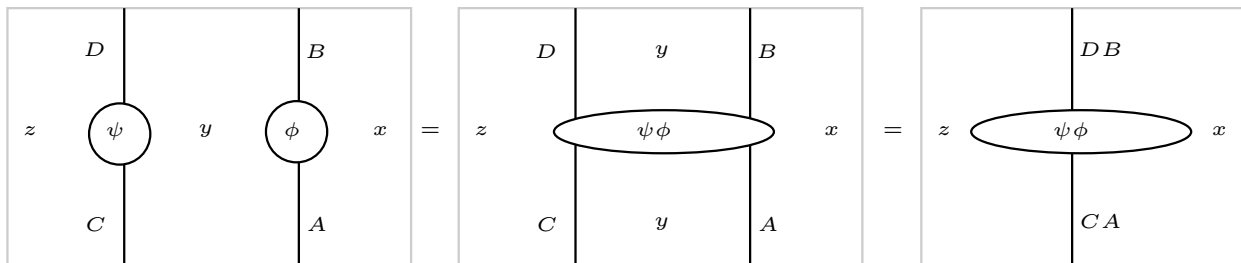
Given  $A, B \in 1\text{-Hom}_{\mathcal{C}}(x, y)$ , let  $\phi : A \Rightarrow B$  be a 2-morphism. In string diagram notation,  $\phi$  is drawn



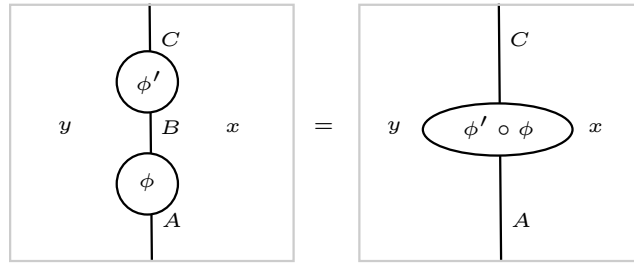
So, our string diagrams are read from right to left and from bottom to top. Horizontal and vertical composition are represented by the respective concatenations of string diagrams. For example, if  $A \in 1\text{-Hom}_{\mathcal{C}}(x, y)$ , and  $\Phi : B \Rightarrow B'$  is a 2-morphism between  $B, B' \in 1\text{-Hom}_{\mathcal{C}}(y, z)$ , then the horizontal composition of  $A$  with  $\Phi$  is represented by either of the diagrams



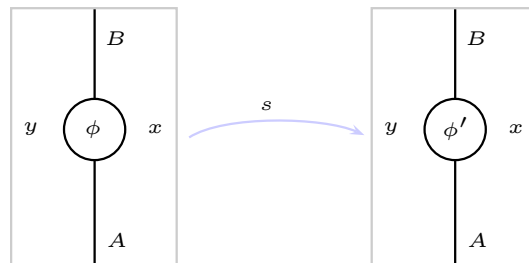
The equals sign in this figure indicates that both string diagrams refer to the same 2-morphism. Given  $C, D \in 1\text{-Hom}_{\mathcal{C}}(y, z)$ , let  $\psi : C \Rightarrow D$  be a 2-morphism, then the horizontal composition of  $\phi$  with  $\psi$  is represented by



If  $\phi : A \Rightarrow B$  and  $\phi' : B \Rightarrow C$  are composable 2-morphisms, their vertical composition is represented by



We will often work with  $\mathbf{k}$ -linear 2-categories. There each  $2\text{-Hom}_{\mathcal{C}}(A, B)$  is a  $\mathbf{k}$ -vector space and vertical composition is  $\mathbf{k}$ -bilinear. If  $\phi, \phi' \in 2\text{-Hom}_{\mathcal{C}}(A, B)$  are 2-morphisms related by a scalar  $s \in \mathbf{k}$  (i.e.  $s\phi = \phi'$ ), then we draw



We will occasionally omit borders and labels of diagrams where the context is clear.

2.7. REMARK. Given a categorical group  $\mathcal{G}$ , one could use a strictification result, as suggested in [Bar08], to make use of string diagram notation. Rather than doing this, we note that string diagrams for skeletal categorical groups are also unambiguous. The string diagrams appearing in the next section are similar to the ones above, but differ in that we now need to keep track of associators.

### 3. Categorized conjugacy classes

The goal of this section is to describe the inertia groupoid of a skeletal categorical group and interpret modules over  $D^\alpha(G)$  as representations of that inertia groupoid.

3.1. HOMOMORPHISMS OF SKELETAL CATEGORICAL GROUPS. A categorical group  $\mathcal{G}$  may be viewed as one-object bicategory. We will denote this bicategory  $\bullet // \mathcal{G}$ , i.e., the object is  $\bullet$  and  $1\text{-Hom}(\bullet, \bullet) = \mathcal{G}$ . In this section we study the bicategory of bifunctors

$$\text{Bicat}(\bullet // \mathcal{H}, \bullet // \mathcal{G}),$$

where  $\mathcal{H}$  and  $\mathcal{G}$  are skeletal categorical groups, classified by cocycles

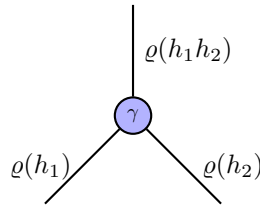
$$\alpha: G \times G \times G \longrightarrow A,$$

and

$$\beta: H \times H \times H \longrightarrow B,$$

as in Example 2.3. One may think of this as the bicategory of representations of  $\mathcal{H}$  in  $\mathcal{G}$ , but this time the target is not a strict 2-category.

3.1.1. THE OBJECTS. Objects of  $\text{Bicat}(\bullet//\mathcal{H}, \bullet//\mathcal{G})$  are homomorphisms of categorical groups, a.k.a. strong monoidal functors, from  $\mathcal{H}$  to  $\mathcal{G}$ . Such a homomorphism is determined by the following data: a group homomorphism  $\varrho: H \rightarrow G$ , an  $H$ -equivariant homomorphism  $f: B \rightarrow A$ , and a 2-cochain  $\gamma: H \times H \rightarrow A$ . For  $h_1, h_2 \in H$ , we draw  $\gamma(h_1, h_2) \bullet \varrho(h_1 h_2)$  as



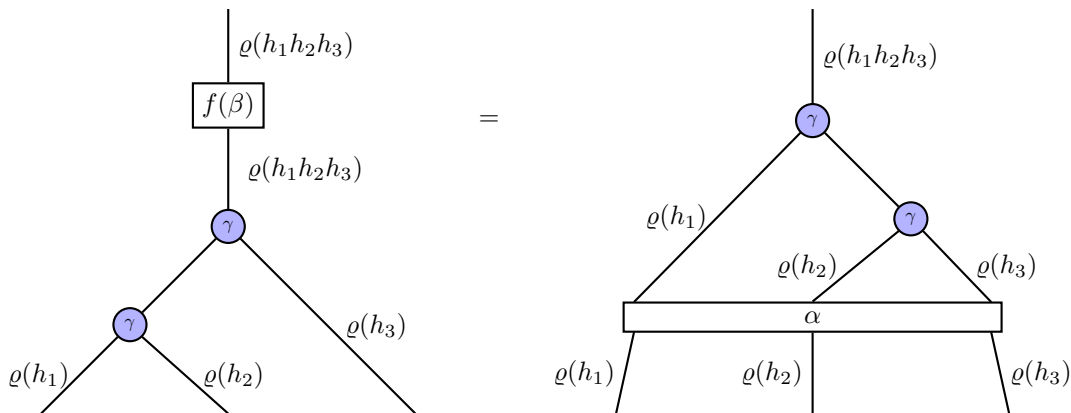
If we let  $H$  act on  $A$  via  $\varrho$  then  $\gamma$  has to satisfy

$$d\gamma = \frac{\varrho^* \alpha}{f_* \beta},$$

i.e.,

$$\frac{\gamma(h_1 h_2, h_3) \cdot \gamma(h_1, h_2)}{\gamma(h_1, h_2 h_3) \cdot \gamma(h_2, h_3) \varrho(h_1)} = \frac{\alpha(\varrho(h_1), \varrho(h_2), \varrho(h_3))}{f(\beta(h_1, h_2, h_3))}. \tag{1}$$

for all  $h_1, h_2, h_3 \in H$ . The hexagon equation (1) is drawn in string diagram notation as



A priori, one expects one more piece of data, namely an arrow

$$a: \varrho(1) \rightarrow 1,$$

i.e, an element  $a \in A$ , satisfying<sup>2</sup>

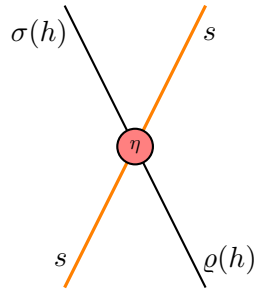
$$\begin{aligned} \gamma(1, h) &= a \\ \gamma(h, 1) &= a^{\varrho(h)} \end{aligned}$$

for all  $h \in H$ . Since  $\alpha$  and  $\beta$  are normalised, this is automatic from (1). Indeed, set  $a = \gamma(1, 1)$  and apply (1) to the triples  $(1, 1, h)$  and  $(h, 1, 1)$ .

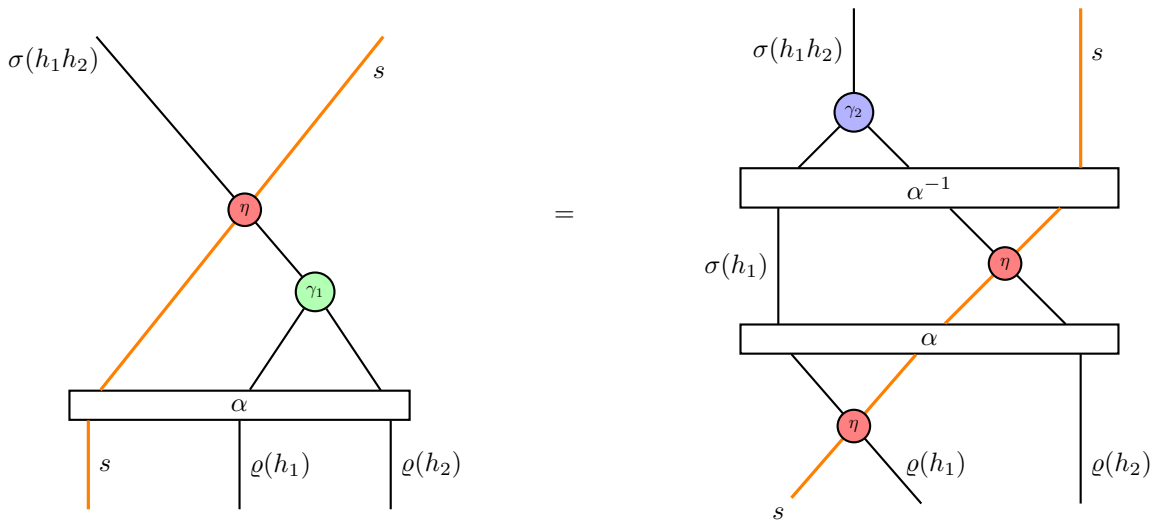
**3.1.2. THE 1-MORPHISMS.** Let  $(\varrho, f_1, \gamma_1)$  and  $(\sigma, f_2, \gamma_2)$  be homomorphisms from  $\mathcal{H}$  to  $\mathcal{G}$ . Then the 1-morphisms between them are transformations from  $(\varrho, f_1, \gamma_1)$  to  $(\sigma, f_2, \gamma_2)$ . We will follow the conventions in [GPS95]. A transformation then amounts to the data of an element  $s \in G$ , together with a 1-cochain  $\eta: H \rightarrow A$  satisfying

$$d_\sigma \eta(h_1, h_2) := \frac{\eta(h_2)^{\sigma(h_1)} \cdot \eta(h_1)}{\eta(h_1 h_2)} = \frac{\gamma_1(h_1, h_2)^s}{\gamma_2(h_1, h_2)} \cdot \frac{\alpha(\sigma(h_1), \sigma(h_2), s) \cdot \alpha(s, \varrho(h_1), \varrho(h_2))}{\alpha(\sigma(h_1), s, \varrho(h_2))}. \tag{2}$$

for all  $h_1, h_2 \in H$ . For  $h \in H$ , we draw  $\eta(h) \bullet \sigma(h) \bullet s$  as



and so the eight-term equation (2) is drawn in string diagram notation as



<sup>2</sup>Note that the axioms in [Lei98] are formulated in terms of  $a^{-1}$ .



The second condition spells out to

$$\eta(1) = \frac{\gamma_1(1, 1)^s}{\gamma_2(1, 1)}$$

which we do not postulate, since in our situation it is automatic from (2). Indeed, it is obtained from the formula for  $d\eta(1, 1)$ , because  $\alpha$  is normalised.

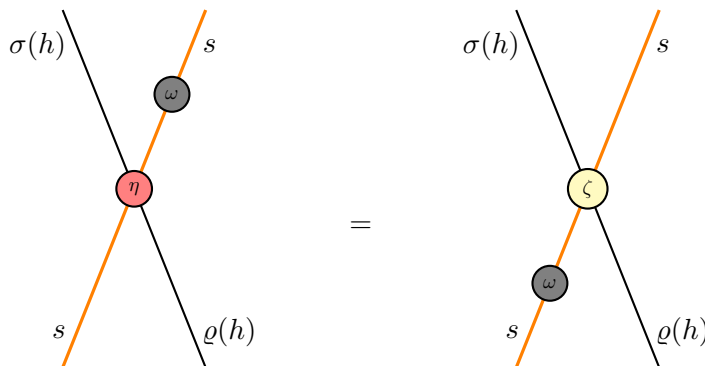
3.1.3. THE 2-MORPHISMS. A modification from  $(s, \eta)$  to  $(t, \zeta)$  requires  $s = t$  and amounts to a 0-cochain  $\omega$  (i.e. an element  $\omega \in A$ ) satisfying

$$\frac{\omega^{\sigma(h)}}{\omega} = \frac{\zeta(h)}{\eta(h)} \tag{3}$$

for all  $h \in H$ . We draw  $\omega \bullet s$  as



and so condition (3) is drawn in string diagram notation as



3.2. EXAMPLE. [Group extensions] Let  $G$  be a group, and let  $A$  be an abelian group. Then the bicategory of bifunctors from  $\bullet // G$  to  $\bullet // \bullet // A$  has as objects 2-cocycles on  $G$  with values in  $A$ , viewed as a trivial  $G$ -module. A 1-morphism from  $\gamma_1$  to  $\gamma_2$  is a 1-cochain  $\eta$  with  $d\eta = \gamma_2/\gamma_1$ . All the 2-morphisms are 2-automorphisms, and each 2-automorphism group is isomorphic to  $A$ . If we truncate at the level of 2-automorphisms, then this is the category of central extensions of  $G$  by  $A$  and their isomorphisms over  $G$ .

3.3. INERTIA (2)GROUPOIDS.

3.4. DEFINITION. We define the inertia 2-groupoid of a categorical group  $\mathcal{G}$  as the 2-groupoid

$$\Lambda\mathcal{G} = \text{Bicat}(\bullet//\mathbb{Z}, \bullet//\mathcal{G}),$$

where the integers are viewed as a discrete 2-group (only identity morphisms).

3.5. EXAMPLE. If  $\mathcal{G} = G$  is a finite group, viewed as a categorical group with only identity morphisms, then  $\Lambda G$  is the usual inertia groupoid with objects  $g \in G$  and arrows  $g \rightarrow sgs^{-1}$ .

In general, let  $\mathcal{G}$  be a special skeletal 2-group with objects  $\text{ob}(\mathcal{G}) = G$ . Then the canonical 2-group homomorphism

$$p: \mathcal{G} \longrightarrow G$$

induces a morphism of 2-groupoids

$$\Lambda p: \Lambda\mathcal{G} \longrightarrow \Lambda G$$

3.6. LEMMA. The map  $\Lambda(p)$  is surjective on objects and full.

PROOF. The proof relies on the knowledge of the group cohomology of the integers, see for instance [Bro10, Exa. 3.1]. The objects of  $\Lambda\mathcal{G}$  are identified with pairs  $(g, \gamma)$ , where  $g$  is an element of  $G$  (namely  $g = \varrho(1)$ ) and

$$\gamma: \mathbb{Z} \times \mathbb{Z} \longrightarrow A$$

is a 2-cochain with boundary

$$d_g \gamma(l, m, n) := \frac{\gamma(l+m, n) \cdot \gamma(l, m)}{\gamma(l, m+n) \cdot \gamma(m, n)^{g^l}} = \alpha(g^l, g^m, g^n).$$

The map  $\Lambda p$  sends  $(g, \gamma)$  to  $g$ . Since

$$H^3(\mathbb{Z}, A) = 0$$

for any  $\mathbb{Z}$ -action on  $A$ , we may conclude that  $\Lambda p$  is surjective on objects. Let now  $(g, \gamma)$  and  $(f, \phi)$  be two objects of  $\Lambda\mathcal{G}$ , and assume that we are given an arrow from  $g$  to  $f$  in  $\Lambda G$ . Such an arrow amounts to an element  $s$  of  $G$  with

$$sgs^{-1} = f.$$

Applying the cocycle condition for  $\alpha$  four times, namely

$$\begin{aligned} d\alpha(s, g^l, g^m, g^n) &= 0, \\ d\alpha(f^l, s, g^m, g^n) &= 0, \\ d\alpha(f^l, f^m, s, g^n) &= 0, \\ d\alpha(f^l, f^m, f^n, s) &= 0, \end{aligned}$$

we obtain that the 2-cochain

$$(m, n) \longmapsto \frac{\phi(m, n)}{\gamma(m, n)^s} \cdot \frac{\alpha(f^m, s, g^n)}{\alpha(f^m, f^n, s) \cdot \alpha(s, g^m, g^n)}$$

is a 2-cocycle for the  $\mathbb{Z}$ -action on  $A$  induced by  $f$ . Since

$$H^2(\mathbb{Z}, A) = 0,$$

we may conclude that  $\Lambda p$  is surjective on 1-morphisms. ■

Let  $\mathbf{G}$  be a groupoid, and let  $A$  be an abelian group. We recall from [Wil08, p.17] how an  $A$ -valued 2-cocycle  $\theta$  on  $\mathbf{G}$  defines a central extension  $\widetilde{\mathbf{G}}$  of  $\mathbf{G}$ . The objects of  $\widetilde{\mathbf{G}}$  are the same as those of  $\mathbf{G}$ . The arrows are

$$\text{Hom}_{\widetilde{\mathbf{G}}}(g, h) = A \times \text{Hom}_{\mathbf{G}}(g, h)$$

with composition

$$(a_1, g_1)(a_2, g_2) := (\theta(g_1, g_2)a_1a_2, g_1g_2).$$

Let  $\mathcal{G}$  be the 2-group defined by  $\alpha$  as above, and assume that the  $G$ -action on  $A$  is trivial. Then all the 2-morphisms in  $\Lambda\mathcal{G}$  are 2-automorphisms. In this case, we may view  $\Lambda\mathcal{G}$  as a groupoid, forgetting the 2-morphisms. Let us denote by  $\Pi_1\Lambda\mathcal{G}$  the groupoid obtained by truncating  $\Lambda\mathcal{G}$  to forget 2-arrows.

**3.7. PROPOSITION.** *The groupoid  $\Pi_1\Lambda\mathcal{G}$  is equivalent to the central extension of  $\mathcal{G}$  defined by the transgression of  $\alpha$ ,*

$$\tau(\alpha) \left( g \xrightarrow{s} h \xrightarrow{t} k \right) = \frac{\alpha(t, s, g) \cdot \alpha(k, t, s)}{\alpha(t, h, s)}.$$

**PROOF.** For each  $g \in G$ , fix an object  $(g, \gamma)$  of  $\Pi_1\Lambda\mathcal{G}$  mapping to  $g$  under  $\Lambda p$ . Since  $\Lambda p$  is surjective on arrows, the full subgroupoid  $\Pi_1\Lambda\mathcal{G}'$  of  $\Pi_1\Lambda\mathcal{G}$  with the objects we just fixed is equivalent to  $\Pi_1\Lambda\mathcal{G}$ . Since  $G$  (and hence  $\mathbb{Z}$ ) acts trivially on  $A$ , the  $A$ -valued one-cocycles on  $\mathbb{Z}$  are just group homomorphisms from  $\mathbb{Z}$  to  $A$ . Hence, for any 2-cocycle  $\xi \in Z^2(\mathbb{Z}, A)$ ,

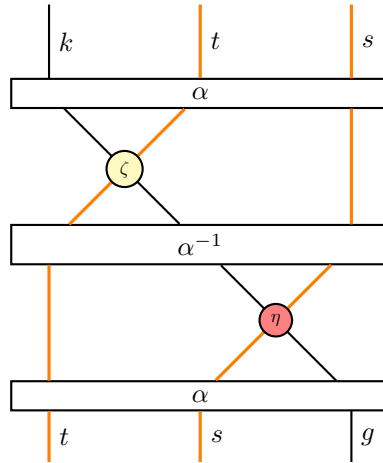
we have a bijection

$$\begin{aligned} \{\eta \mid d\eta = \xi\} &\longrightarrow A \\ \eta &\longmapsto \eta(1). \end{aligned}$$

Let now  $s$  be an element of  $G$ , and let

$$h = sgs^{-1}.$$

Inserting the right-hand side of (2) for  $\xi$ , allows us identify the set of arrows in  $\Pi_1\Lambda\mathcal{G}'$  mapping to  $s$  with  $A$ . Let now  $t$  be another element of  $G$  and let  $k = tht^{-1}$ . The following string diagram illustrates the composition of arrows  $(s, \eta)$  and  $(t, \zeta)$  in  $\Pi_1\Lambda\mathcal{G}'$ .



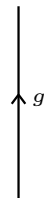
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#### 4. Projective 2-representations

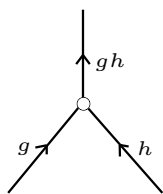
The following is a  $\mathbf{k}$ -linear version of [FZ12, Definition 2.8].

4.1. DEFINITION. *Let  $G$  be a finite group, and  $\mathcal{C}$  a strict  $\mathbf{k}$ -linear 2-category. A projective 2-representation of  $G$  on  $\mathcal{C}$  consists of the following data*

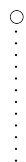
- (a) *an object  $V$  of  $\mathcal{C}$*
- (b) *for each  $g \in G$ , a 1-automorphism  $\varrho(g) : V \longrightarrow V$ , drawn as*



(c) for every pair  $g, h \in G$ , a 2-isomorphism  $\psi_{g,h} : \varrho(g)\varrho(h) \xrightarrow{\cong} \varrho(gh)$ , drawn as



(d) a 2-isomorphism  $\psi_1 : \varrho(1) \xrightarrow{\cong} \text{id}_V$ , drawn as

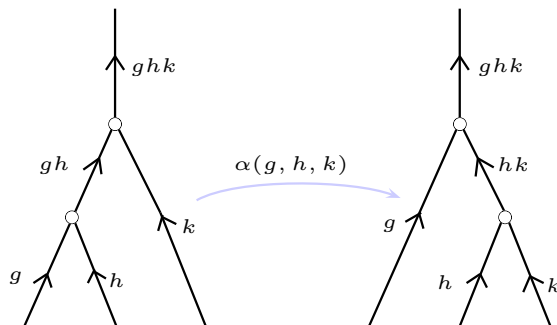


such that the following conditions hold

(i) for any  $g, h, k \in G$ , we have

$$\psi_{g,hk}(\varrho(g)\psi_{h,k}) = \alpha(g, h, k)\psi_{gh,k}(\psi_{g,h}\varrho(k)),$$

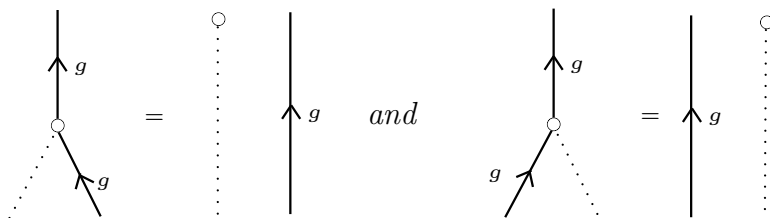
where  $\alpha(g, h, k) \in \mathbf{k}^\times$ . In string diagram notation, we draw this as



(ii) for any  $g \in G$ , we have

$$\psi_{1,g} = \psi_1\varrho(g) \text{ and } \psi_{g,1} = \varrho(g)\psi_1.$$

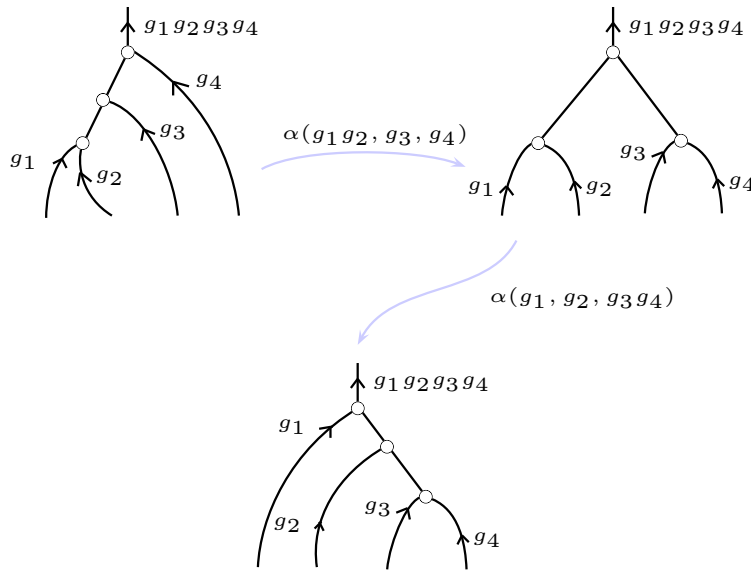
In string diagram notation, we draw these as



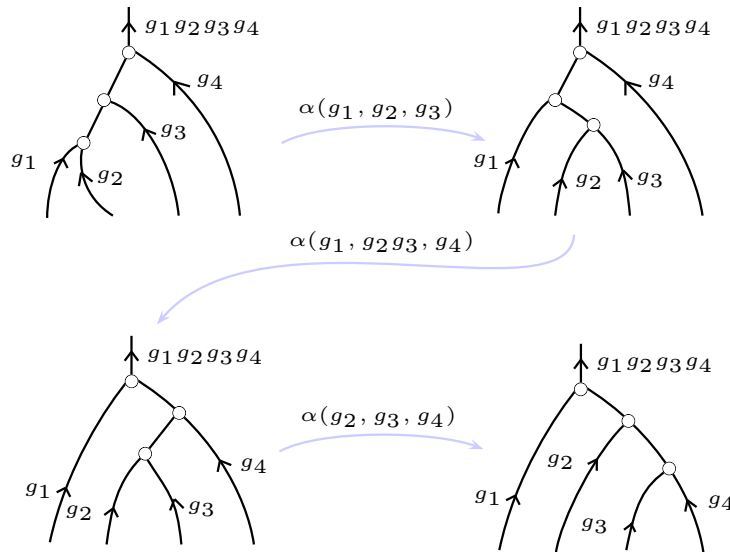
4.2. THE 3-COCYCLE CONDITION.

4.3. PROPOSITION. [Compare [FZ12, Theorem 2.10]] *Let  $\rho$  be a projective 2-representation of a group  $G$ . Then the map  $\alpha : G \times G \times G \rightarrow \mathbf{k}^\times$  appearing in condition (i) is a normalised 3-cocycle for the trivial  $G$ -action on  $\mathbf{k}^\times$ .*

PROOF. We use Definition 4.1 (i) for all steps of our proof. Consider



On the other hand, we have



Comparing these diagrams, we find

$$\alpha(g_1g_2, g_3, g_4) \cdot \alpha(g_1, g_2, g_3g_4) = \alpha(g_2, g_3, g_4) \cdot \alpha(g_1, g_2g_3, g_4) \cdot \alpha(g_1, g_2, g_3),$$

so  $\alpha$  is indeed a 3-cocycle. ■

4.4. COROLLARY. *A projective 2-representation of  $G$  with cocycle  $\alpha$  is precisely a linear representation of the 2-group  $\mathcal{G}$  classified by  $\alpha$  (see Definition 2.5).*

PROOF. Indeed, Condition (i) of Definition 4.1 amounts to the hexagon diagram for a strong monoidal functor, and Condition (ii) translates to the unit diagrams. ■

4.5. EXAMPLE. [Compare [GK08, §5.1]] Let  $G$  be a finite group, and let  $\theta$  be a normalised 2-cochain. Let  $\alpha = d\theta$  be the coboundary of  $\theta$ , i.e.,

$$\alpha(g, h, k) = \frac{\theta(gh, k) \cdot \theta(g, h)}{\theta(g, hk) \cdot \theta(h, k)}.$$

Let  $\text{Vect}_{\mathbf{k}}$  be the category of finite dimensional  $\mathbf{k}$ -vector spaces. Then we define a projective 2-representation of  $G$  on  $\text{Vect}_{\mathbf{k}}$  with corresponding 3-cocycle  $\alpha$  as follows : for  $g \in G$ , we let

$$\varrho(g) = \text{id} : \text{Vect}_{\mathbf{k}} \longrightarrow \text{Vect}_{\mathbf{k}}$$

be the identity functor on  $\text{Vect}_{\mathbf{k}}$ . For  $g, h \in G$  we let

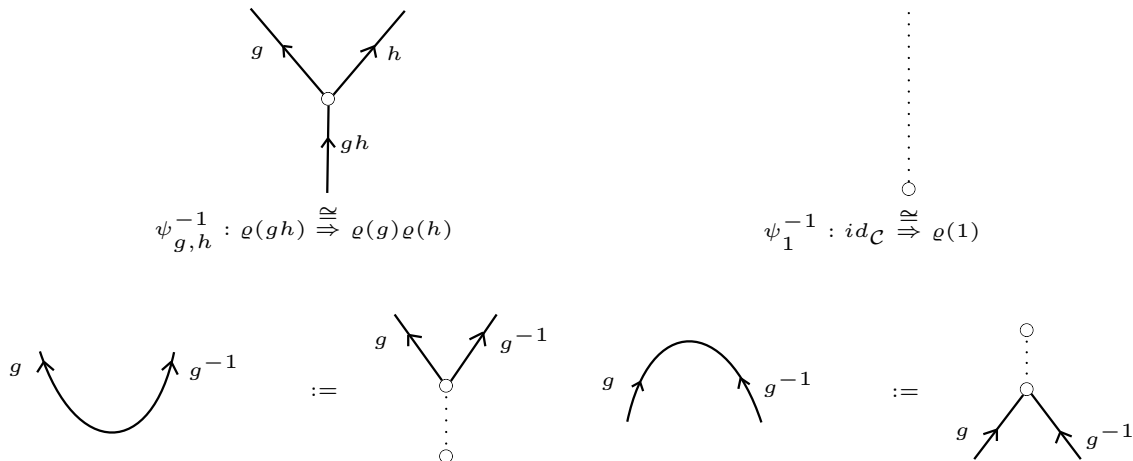
$$\psi_{g,h} : \text{id} \circ \text{id} \xrightarrow{\cong} \text{id}$$

be given by multiplication by  $\theta(g, h)$ . Further,

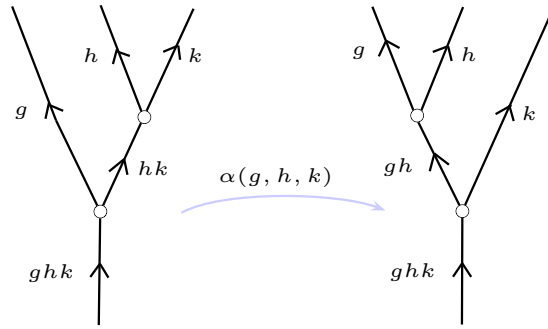
$$\psi_1 : \varrho(1) \xrightarrow{\cong} \text{id}$$

is the identity natural transformation.

We recall some further notation from [Bar08].



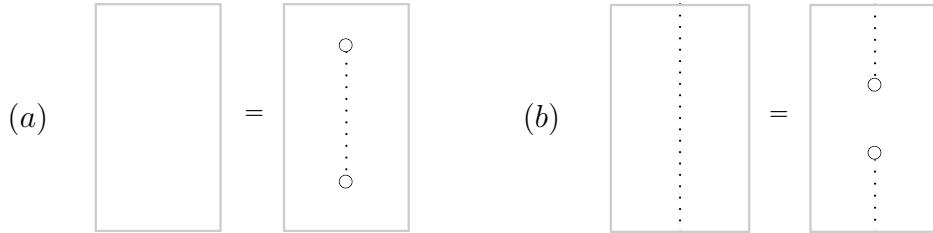
4.6. REMARK. For future reference, we present the following tautological string diagram equations, as in [Bar08, §7.1.1]. By inverting condition (i) of Definition 4.1, we get



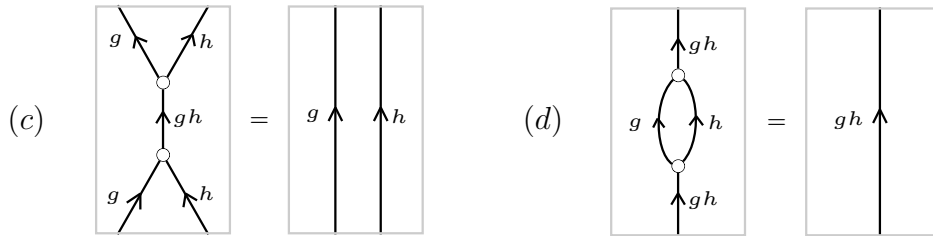
that is,

$$(\psi_{g,h}^{-1} \varrho(k)) \psi_{gh,k}^{-1} = \alpha(g, h, k) (\varrho(g) \psi_{h,k}^{-1}) \psi_{g,ghk}^{-1} \tag{4}$$

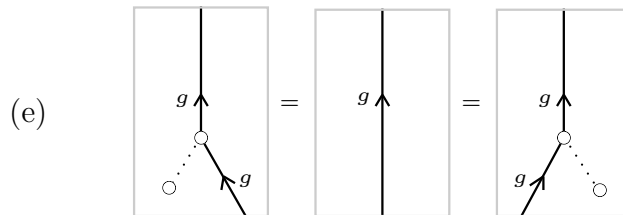
We have  $\psi_1 \circ \psi_1^{-1} = \text{id}_C$  and  $\psi_1^{-1} \circ \psi_1 = \varrho(1)$ , drawn as



Similarly,  $\psi_{g,h}^{-1} \circ \psi_{g,h} = \varrho(g) \varrho(h)$  and  $\psi_{g,h} \circ \psi_{g,h}^{-1} = \varrho(gh)$  for all  $g, h \in G$ , drawn as



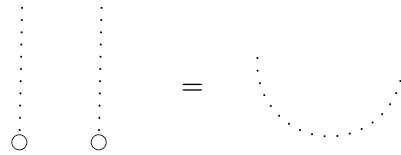
Finally,  $\psi_{1,g}(\psi_1^{-1} \varrho(g)) = \varrho(g) = \psi_{g,1}(\varrho(g) \psi_1^{-1})$  for all  $g \in G$ , drawn as



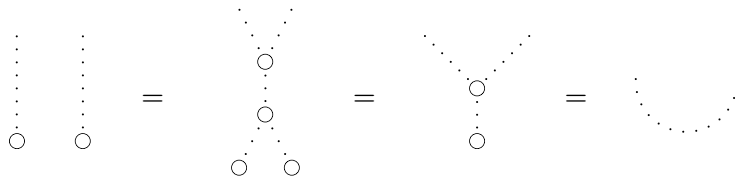
Some less tautological graphical equations for projective 2-representations are given by the following results.



4.7. LEMMA. [Bar08, Lemma 7.3 (ii)] *The following graphical equation holds*

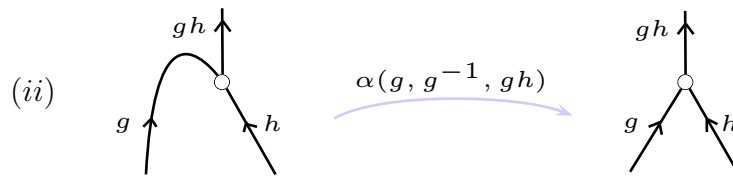
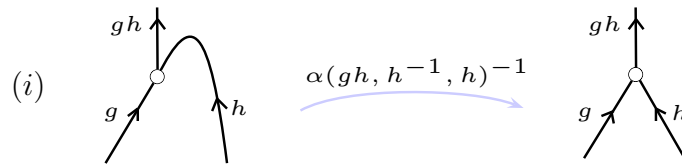


PROOF.

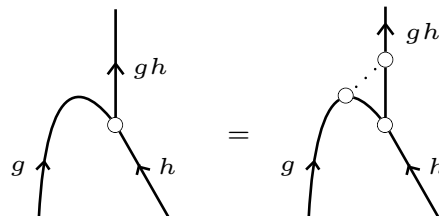


The first equality follows from 4.6 (c), the second from 4.6 (e), with the final following by definition. ■

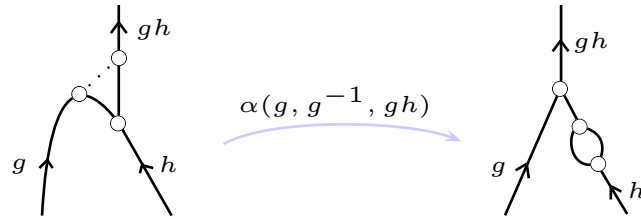
4.8. LEMMA. [Compare [Bar08, Lemma 7.3 (iii)]] *The following graphical equations hold*



PROOF. We will prove (ii); the proof of (i) is almost identical. By combining 4.6 (c) and (e), we obtain

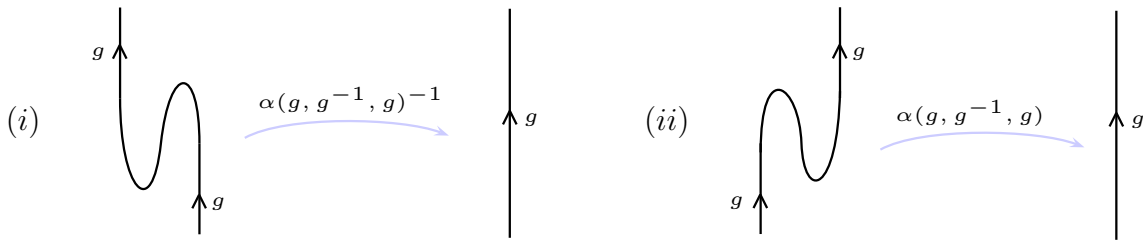


Next, by 4.1 (i), we obtain

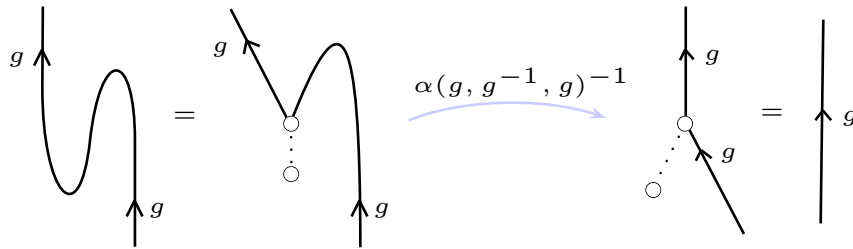


A final application of 4.6 (d) gives us the desired result. ■

4.9. COROLLARY. [Compare [Bar08, Lemma 7.3 (i)]] *The following graphical equations hold*

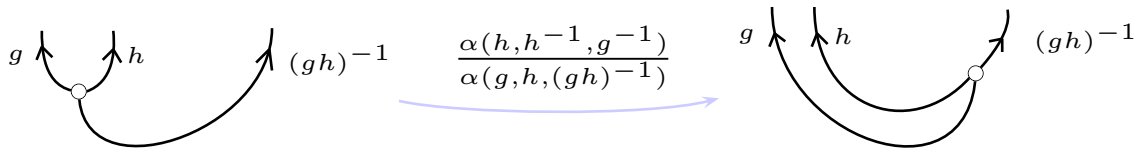


PROOF. We will prove (i); the proof of (ii) is almost identical. By applying 4.8 and then 4.6 (e), we have

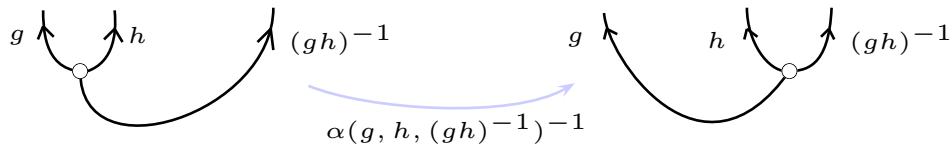


as required. ■

4.10. COROLLARY. [Compare [Bar08, Lemma 7.3 (iv)]] *The following graphical equation holds*



PROOF. Applying 4. we get



Inverting the equation derived in part (ii) of 4.8 gives the desired result. ■

4.11. THE CHARACTER OF A PROJECTIVE 2-REPRESENTATION. Recall that the character of a classical representation  $\varrho$  is the map  $\chi : G \rightarrow \mathbf{k}$  defined by  $\chi(g) = \text{tr}(\varrho(g))$ . This motivates the following definition of [GK08] and [Bar08].

4.12. DEFINITION. [GK08, Definition 3.1] and [Bar08, Definition 7.8] *Let  $\mathcal{C}$  be a 2-category,  $x \in \text{ob}(\mathcal{C})$  and  $A \in 1\text{-Hom}_{\mathcal{C}}(x, x)$  a 1-endomorphism of  $x$ . The categorical trace of  $A$  is defined to be*

$$\text{Tr}(A) = 2\text{-Hom}_{\mathcal{C}}(1_x, A)$$

where  $1_x$  is the identity 1-morphism of  $x$ .

4.13. REMARK. If  $\mathcal{C}$  is a  $\mathbf{k}$ -linear 2-category, then the categorical trace of a 1-endomorphism  $A : x \rightarrow x$  is a  $\mathbf{k}$ -vector space.

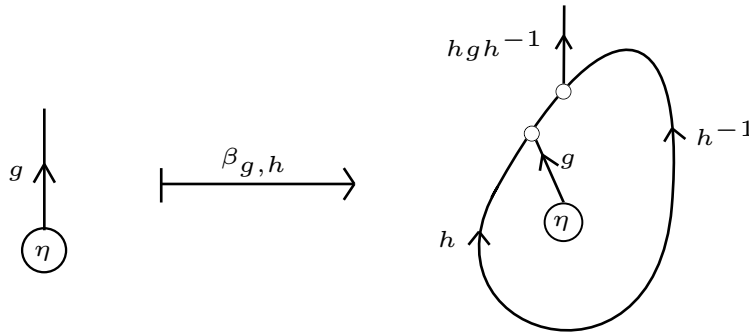
4.14. DEFINITION. [Compare [GK08, Definition 4.8] and, in particular, [Bar08, Definition 7.9]] *Let  $\varrho$  be a projective 2-representation of a finite group  $G$ . The character of  $\varrho$  is the assignment*

$$g \mapsto \text{Tr}(\varrho(g)) =: X_{\varrho}(g)$$

for each  $g \in G$ , and the collection of isomorphisms

$$\beta_{g,h} : X_{\varrho}(g) \rightarrow X_{\varrho}(hgh^{-1})$$

defined in terms of string diagrams



for each  $g, h \in G$ . That the  $\beta_{g,h}$  are isomorphisms is a consequence of Theorem 4.17.

We note that the definitions in [GK08] and [Bar08] are the special case  $\alpha = 1$ , although they look a bit different at first sight. There are several thinkable generalisations of those definitions. This definition was chosen based on the discussion in Section 3.

4.15. DEFINITION. [GK08, Definition 4.12] *Let  $\varrho$  be a projective 2-representation of a finite group  $G$  on a  $\mathbf{k}$ -linear 2-category. If  $g, h \in G$  is a pair of commuting elements, then  $\beta_{g,h}$  is an automorphism of  $X_{\varrho}(g)$ . Assuming  $\beta_{g,h}$  to be of trace class, we have the joint trace of  $g$  and  $h$ ,*

$$\chi_{\varrho}(g, h) := \text{tr}(\beta_{g,h}).$$

If the joint trace is defined for all commuting  $g, h \in G$ , we refer to  $\chi_{\varrho}$  as the 2-character of  $\varrho$ .

4.16. **EXAMPLE.** As in [GK08, §5.1], let us consider the categorical character and 2-character of the projective 2-representation defined in Example 4.5. For  $g \in G$ , we have

$$X_\varrho(g) = \text{Tr}(\text{id}_{\text{Vec}_{\mathbf{k}}}) = \mathbf{k}$$

Let  $g, h \in G$  be commuting elements. Then it follows from Definition 4.14 that the joint trace  $\chi_\varrho(g, h)$  is given by multiplication by the transgression of  $\theta$

$$\frac{\theta(h, g)}{\theta(hgh^{-1}, h)} = \frac{\theta(h, g)}{\theta(g, h)}.$$

We now present our main result.

4.17. **THEOREM.** Let  $\mathcal{G}$  be a finite categorical group, let  $V$  be an object of a  $\mathbf{k}$ -linear strict 2-category and let

$$\varrho: \mathcal{G} \longrightarrow \text{GL}(V)$$

be a linear representation of  $\mathcal{G}$  on  $V$ . Then the categorical character of  $\varrho$  is a representation of the inertia groupoid  $\Pi_1 \Lambda \mathcal{G}$  of  $\mathcal{G}$ .

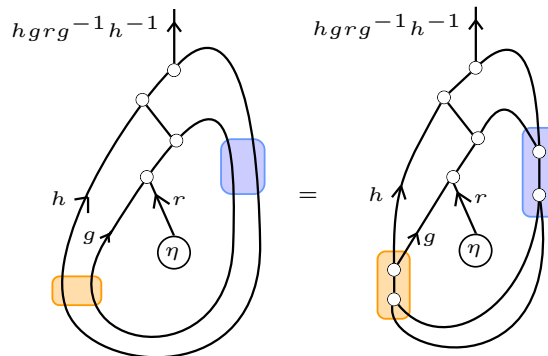
**PROOF.** Recall that  $\mathcal{G}$  is determined by a finite group  $G$  together with a 3-cocycle  $\alpha$  on  $G$  with values in  $\mathbf{k}^\times$ . By Proposition 3.7, we must show that the diagram

$$\begin{array}{ccc} X_\varrho(r) & \xrightarrow{\beta_{r,hg}} & X_\varrho(hgrg^{-1}h^{-1}) \\ & \searrow \beta_{r,g} & \nearrow \beta_{grg^{-1},h} \\ & X_\varrho(grg^{-1}) & \end{array}$$

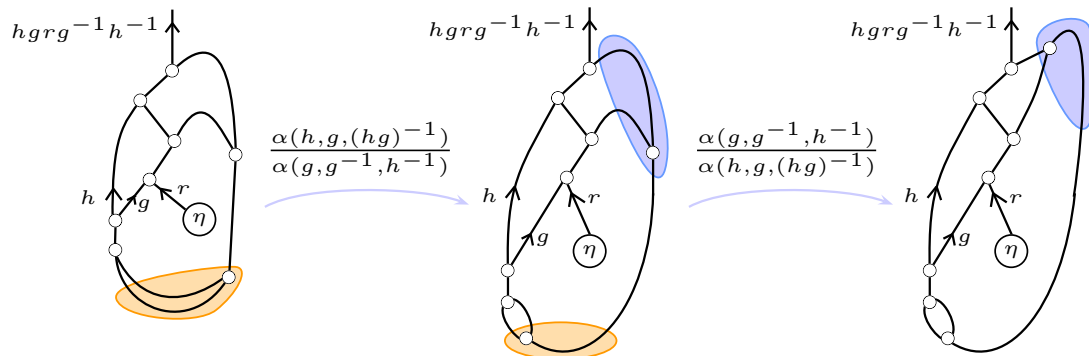
commutes up to a scalar, i.e.

$$\frac{\alpha(h, grg^{-1}, g)}{\alpha(hgrg^{-1}h^{-1}, h, g) \cdot \alpha(h, g, r)} \cdot \beta_{grg^{-1},h} \circ \beta_{r,g} = \beta_{r,hg} \tag{5}$$

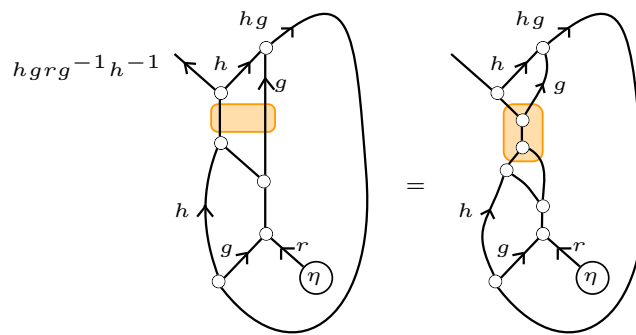
for all  $r, g, h \in G$ . Fix elements  $r, g, h \in G$  and  $\eta \in X(r)$ . By applying 4.6 (d) twice, we find



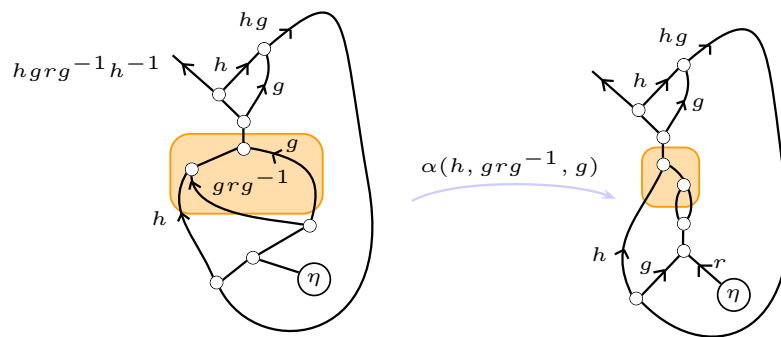
Applying 4.10 twice, we have



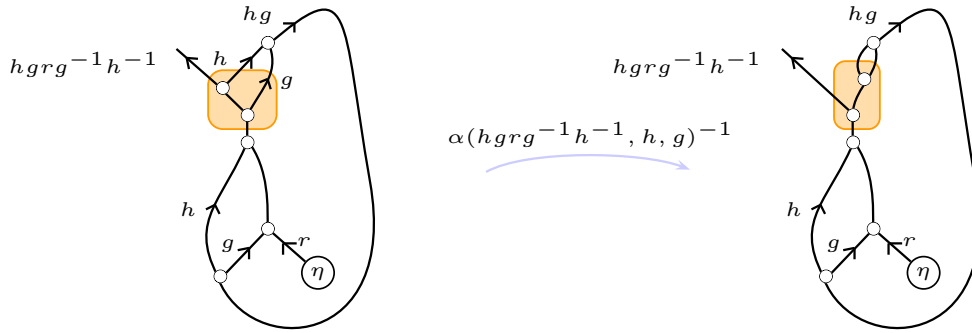
These two factors cancel, so the first and last diagram in this figure are equal. We redraw this diagram by removing the loop (as per 4.6 (d)), then apply 4.6 (c) to get



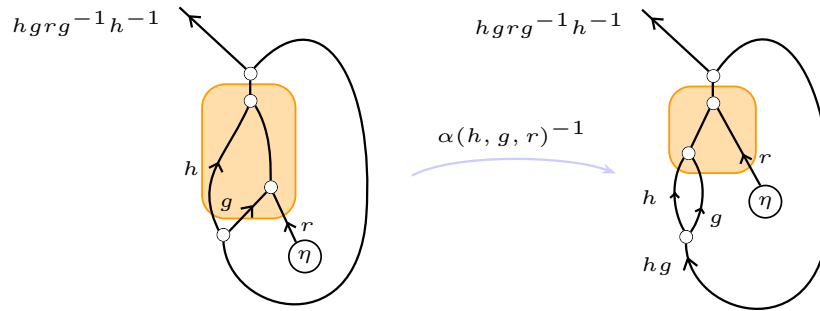
Next, we apply 4.1 (i) to obtain



By removing the loop and applying 4, we get



Finally, we remove this loop then apply 4.1 (i) to compute



After removing the loop we recognise this final diagram as representing  $\beta_{r,hg}(\eta)$ . We have therefore shown that

$$\frac{\alpha(h, grg^{-1}, g)}{\alpha(hgrg^{-1}h^{-1}, h, g) \cdot \alpha(h, g, r)} \cdot \beta_{grg^{-1},h} \circ \beta_{r,g}(\eta) = \beta_{r,hg}(\eta),$$

as required. ■

The main result of [Wil08] identifies the twisted Drinfeld module of  $G$  for  $\alpha$  with the twisted groupoid algebra

$$D^\alpha(G) \cong \mathbf{k}^{\tau(\alpha)}[\Lambda G],$$

and so we get the following corollary.

4.18. COROLLARY. [compare [KP09, Theorem 5.8]] *Let  $G$  be a finite group,  $\alpha$  a 3-cocycle on  $G$  with values in  $\mathbf{k}^\times$ , and  $\mathcal{G}$  the corresponding categorical group. Then representations of the inertia groupoid  $\Pi_1\Lambda\mathcal{G}$  are modules over the twisted Drinfeld double  $D^\alpha(G)$ .*

## 5. Module categories and induction

5.1. PROJECTIVE 2-REPRESENTATIONS AS MODULE CATEGORIES. Let  $\mathbf{k}$  be a field, and let

$$\theta : G \times G \longrightarrow \mathbf{k}^\times$$

be a 2-cocycle. Then there is an equivalence of categories

$$\text{Rep}_{\mathbf{k}}^{\theta}(G) \simeq \mathbf{k}^{\theta}[G]\text{-mod} \tag{6}$$

from the projective  $G$ -representations with cocycle  $\theta$  to modules over the twisted group algebra  $\mathbf{k}^{\theta}[G]$ . In the context of 2-representations,  $\mathbf{k}$  is replaced by the one-dimensional 2-vector space  $\text{Vect}_{\mathbf{k}}$ . The *categorified twisted group algebra*  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$  is the monoidal category of  $G$ -graded finite dimensional  $\mathbf{k}$ -vector spaces, where the monoidal structure consists of the graded tensor product, with associators twisted by  $\alpha$  (see [Ost03b], where  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$  is denoted  $\text{Vec}_G^{\alpha}$ ).

5.2. DEFINITION. *Let  $\mathcal{C}$  be a strict  $\mathbf{k}$ -linear 2-category, and let  $\mathcal{G}$  be the skeletal 2-group classified by the (normalised) 3-cocycle  $\alpha : G \times G \times G \rightarrow \mathbf{k}^{\times}$ , as in Example 2.3. We write*

$$2\text{Rep}_{\mathcal{C}}^{\alpha}(G) := \text{Bicat}(\bullet // \mathcal{G}, \mathcal{C}) \tag{7}$$

for the 2-category of 2-representations as in [Bar08, Definition 7.1].<sup>3</sup>

In the case where  $\mathcal{C}$  is the 2-category of finite dimensional Kapranov-Voevodsky 2-vector spaces<sup>4</sup>, we will use the notation  $2\text{Rep}_{\text{Vect}_{\mathbf{k}}}^{\alpha}(G)$ . The 2-categorical analogue of Equation 6 is then

$$2\text{Rep}_{\text{Vect}_{\mathbf{k}}}^{\alpha}(G) \simeq \text{Vect}_{\mathbf{k}}^{\alpha}[G]\text{-mod}$$

We will switch freely between the points of view of module categories and projective 2-representations.

5.3. EXAMPLE. Let  $\theta$  be a 2-cochain on  $G$  with boundary  $d\theta = \alpha$ . Then

$$\mathbf{k}^{\theta}[G] = \bigoplus_{g \in G} \mathbf{k}$$

with multiplication twisted by  $\theta$  is an algebra object in  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$ . Note that is it *not* an algebra. The  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$ -module category

$$\text{Vect}_{\mathbf{k}}^{\alpha}[G] - \mathbf{k}^{\theta}[G]$$

of right  $\mathbf{k}^{\theta}[G]$ -modules in  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$  is the basic example of a module category in [Ost03a, §3.1]. It translates into our Example 4.5 via the equivalence

$$\begin{aligned} F : \text{Vect}_{\mathbf{k}}^{\alpha}[G] - \mathbf{k}^{\theta}[G] &\longrightarrow \text{Vect}_{\mathbf{k}} \\ \bigoplus_{g \in G} M_g &\longmapsto M_1. \end{aligned}$$

Indeed, if we equip  $\text{Vect}_{\mathbf{k}}$  with the module structure of Example 4.5, then  $F$  can be

<sup>3</sup>This is not the same as the category  $\text{Hom}_{2\text{-Grp}}(\mathcal{G}, \text{Aut}(\mathcal{C}))$  in [FZ12] after Definition 2.6.

<sup>4</sup>A 2-vector space is a semisimple  $\text{Vect}_{\mathbf{k}}$ -module category with finitely many simple objects, see [KV94]

made a module functor as follows: given a  $\mathbf{k}^\theta[G]$ -module object  $M$  in  $\text{Vect}_{\mathbf{k}}^\alpha$  with action

$$M \otimes \mathbf{k}^\theta[G] \xrightarrow[\cong]{s} M,$$

we choose the isomorphism

$$M_{g^{-1}} = F(\mathbf{k}_g \otimes M) \longrightarrow \mathbf{k}_g \cdot F(M) = M_1$$

to be the map

$$M_{g^{-1}} \otimes \mathbf{k}_g \hookrightarrow F(M \otimes \mathbf{k}^\theta[G]) \xrightarrow{F(s)} F(M).$$

5.4. INDUCED 2-REPRESENTATIONS. Let  $H \subset G$  be finite groups, and let  $\alpha$  be a normalised 3-cocycle on  $G$ . Let  $\varrho$  be a projective 2-representation of  $H$  on  $W \in \text{ob}(\mathcal{C})$  with cocycle  $\alpha|_H$ .

5.5. DEFINITION. *The induced 2-representation of  $W$ , if it exists, is characterised by the universal property of a left-adjoint. More precisely, an object  $\text{ind}_H^G W$ , together with a projective 2-representation  $\text{ind}_H^G \varrho$  with cocycle  $\alpha$  and a 1-morphism*

$$j : \varrho \longrightarrow \text{ind}_H^G \varrho$$

in  $2\text{Rep}_{\text{Vect}_{\mathbf{k}}}^{\alpha|_H}(H)$  is called induced by  $\varrho$ , if for any projective  $G$ -2-representation  $\pi$  on  $V \in \text{ob}(\mathcal{C})$  with cocycle  $\alpha$  and any 1-morphism of projective  $H$ -2-representations (for  $\alpha|_H$ )

$$F : \varrho \longrightarrow \pi$$

there exists a 1-morphism of projective  $G$ -2-representations (for  $\alpha$ )

$$\bar{F} : \text{ind}_H^G \varrho \longrightarrow \pi$$

and a 2-isomorphism  $\Phi$  fitting in the commuting diagram of  $H$ -maps

$$\begin{array}{ccc}
 \varrho & \xrightarrow{j} & \text{ind}_H^G \varrho \\
 \searrow F & \nearrow \Phi & \searrow \bar{F} \\
 & \pi & 
 \end{array}$$

such that  $(\bar{F}, \Phi)$  is determined uniquely up to unique 2-isomorphism. Here “unique” means that given two such pairs  $(\bar{F}_1, \Phi_1)$  and  $(\bar{F}_2, \Phi_2)$ , there is a unique 2-isomorphism

$$\eta : \bar{F}_1 \xrightarrow{\cong} \bar{F}_2$$



satisfying

$$(\eta j) \circ \Phi_1 = \Phi_2.$$

If it exists, the induced projective 2-representation of  $W$  is determined uniquely up to a 1-equivalence in  $2\text{Rep}_{\text{Vect}_{\mathbf{k}}}^{\alpha}(G)$ , which is unique up to canonical 2-isomorphism.

In the following, we abbreviate  $\alpha|_H$  with  $\alpha$ .

5.6. PROPOSITION. Let  $\mathcal{M}$  be a  $\mathbf{k}$ -linear left  $\text{Vect}_{\mathbf{k}}^{\alpha}[H]$ -module category. Then the induced projective 2-representation of  $\mathcal{M}$  exists, and is given by the tensor product of  $\text{Vect}_{\mathbf{k}}^{\alpha}[H]$ -module categories

$$\text{ind}_H^G \mathcal{M} = \text{Vect}_{\mathbf{k}}^{\alpha}[G] \boxtimes_{\text{Vect}_{\mathbf{k}}^{\alpha}[H]} \mathcal{M}$$

defined in [ENO10, Definition 3.3].

PROOF. Using the universal property of  $-\boxtimes_{\text{Vect}_{\mathbf{k}}^{\alpha}[H]} -$ , one equips  $\text{Vect}_{\mathbf{k}}^{\alpha}[G] \boxtimes_{\text{Vect}_{\mathbf{k}}^{\alpha}[H]} \mathcal{M}$  with the structure of a left  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$ -module category. Using the universal property of  $-\boxtimes_{\text{Vect}_{\mathbf{k}}^{\alpha}[H]} -$  again, we deduce the universal property for  $\text{ind}_H^G \mathcal{M}$ . ■

5.7. PROPOSITION. Let  $A$  be an algebra object in  $\text{Vect}_{\mathbf{k}}^{\alpha}[H]$ , and let  $\mathcal{M} = \text{Vect}_{\mathbf{k}}^{\alpha}[H]-A$  be the category of right  $A$ -module objects in  $\text{Vect}_{\mathbf{k}}^{\alpha}[H]$ . Then we have

$$\text{ind}_H^G \mathcal{M} = \text{Vect}_{\mathbf{k}}^{\alpha}[G]-A,$$

and the map  $j$  is the canonical inclusion

$$j : \text{Vect}_{\mathbf{k}}^{\alpha}[H]-A \longrightarrow \text{Vect}_{\mathbf{k}}^{\alpha}[G]-A$$

i.e.  $j(M)|_H = M$  and  $j(M)|_g = 0$  for  $g \notin H$ .

PROOF. Let  $M = \bigoplus_{g \in G} M_g$  be a right  $A$ -module object in  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$ . Then  $M$  is the direct sum of  $A$ -module objects

$$M = \bigoplus_{G/H} M|_{rH}$$

where

$$(M|_{rH})_s = \begin{cases} M_s & s \in rH \\ 0 & \text{otherwise} \end{cases}$$

Fix a system  $\mathcal{R}$  of left coset representatives, and assume we are given a  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$ -module category  $\mathcal{N}$  together with a  $\text{Vect}_{\mathbf{k}}^{\alpha}[H]$ -module functor

$$F : \text{Vect}_{\mathbf{k}}^{\alpha}[H]-A \longrightarrow \mathcal{N}.$$

We define the  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$ -module functor

$$\begin{aligned} \bar{F} &: \text{Vect}_{\mathbf{k}}^{\alpha}[G] - A \longrightarrow \mathcal{N} \\ M &\longmapsto \bigoplus_{r \in \mathcal{R}} \mathbf{k}_r \cdot F(\mathbf{k}_{r^{-1}} \cdot M|_{rH}) \end{aligned}$$

Then  $\Phi$  is the inclusion of the summand  $F(M)$  in  $\bar{F}(j(M))$ . This  $\Phi$  is an isomorphism, because the other summands of  $\bar{F}j(M)$  are (canonically) zero.

Let  $(\bar{F}_2, \Phi_2)$  be a second pair fitting into the diagram on page 565, for instance, from a different choice of coset representatives. Then the isomorphism  $\eta : \bar{F} \implies \bar{F}_2$  is the inverse of the composition

$$\bar{F}_2(M) \cong \bigoplus_{r \in \mathcal{R}} \bar{F}_2(M|_{rH}) \cong \bigoplus_{r \in \mathcal{R}} \mathbf{k}_r \cdot \bar{F}_2(\mathbf{k}_{r^{-1}} \otimes M|_{rH}) \stackrel{\Phi_2}{\cong} \bar{F}(M)$$

■

5.8. COROLLARY. *As right  $\text{Vect}_{\mathbf{k}}^{\alpha}[H]$ -modules*

$$\text{Vect}_{\mathbf{k}}^{\alpha}[G] \cong \bigoplus_{\substack{r \in \mathcal{R} \\ G/H}} \text{Vect}_{\mathbf{k}}^{\alpha}[H]$$

5.9. COMPARISON OF CLASSIFICATIONS. In [GK08, Proposition 7.3], the finite dimensional 2-representations are classified. In [Ost03b, Example 2.1], the indecomposable module categories over  $\text{Vect}_{\mathbf{k}}^{\alpha}[G]$  are classified. In [GK08], a comparison with Ostrik’s work was attempted, but the dictionary established in the previous section appears to be more suitable, as it translates directly between these two results. Indeed, for trivial  $\alpha$ , the following corollary specialises to [GK08, Proposition 7.3].

5.10. COROLLARY. *Let  $\varrho$  be a projective 2-representation of a group  $G$  with 3-cocycle  $\alpha$  on a semisimple  $\mathbf{k}$ -linear abelian category  $\mathcal{V}$  with finitely many simple objects. Then*

$$\mathcal{V} \cong \bigoplus_{i=1}^m \text{ind}_{H_i}^G \varrho_{\theta_i}$$

where the  $H_i$  are subgroups of  $G$ ,  $\theta_i$  is a 2-cochain on  $H_i$  such that  $d\theta_i = \alpha|_{H_i}$ , and  $\varrho_{\theta_i}$  is the projective 2-representation corresponding to the pair  $(H_i, \theta_i)$  described in Examples 4.5 and 5.3.

PROOF. Ostrik's result [Ost03b, Example 2.1] yields a decomposition

$$\begin{aligned} \mathcal{V} &\simeq \bigoplus_{i=1}^m \text{Vect}_{\mathbf{k}}^{\alpha}[G] - \mathbf{k}^{\theta_i}[H_i] \\ &\simeq \bigoplus_{i=1}^m \text{ind}_{H_i}^G(\text{Vect}_{\mathbf{k}}^{\alpha}[H_i] - \mathbf{k}^{\theta_i}[H_i]) \\ &\simeq \bigoplus_{i=1}^m \text{ind}_{H_i}^G \varrho_{\theta_i} \end{aligned}$$

Here, the second equivalence is Proposition 5.7, the third is Example 5.3, and  $\varrho_{\theta_i}$  is as in Example 4.5. ■

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