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COMPOSITION OF MODULES FOR LAX FUNCTORS

ROBERT PARÉ

ABSTRACT. We study the composition of modules between lax functors of weak double categories. We adapt the bicategorical notion of local cocompleteness to weak double categories, which the codomain of our lax functors will be assumed to satisfy. We introduce a notion of factorization of cells, which most weak double categories of interest possess, and which is sufficient to guarantee the strong representability of composites of modules between lax functors whose domain satisfies it.

Introduction

We are interested in lax functors between weak double categories. There are several reasons for this. One is that representables for weak double categories are lax functors [15]. Another is that **V**-categories can be viewed as lax functors in various ways [17], [3]. Monads are lax functors defined on **1**. In fact in the first paper on bicategories [1], lax functors were called "morphisms of bicategories", thus underlining their fundamental nature.

The notion of transformation between lax functors is a little less obvious. In [1], lax and oplax transformations were defined but these present some problems. For one thing, no notion of transformation made bicategories with lax functors and transformations into a bicategory or a tricategory, a feature which seemed desirable. It also seemed a bit mysterious that there was no preferred choice between lax and oplax transformation. Over the years various other notions of transformation have appeared, for example the ICONs of [12], and the modules of [5]. Here we will see how all these notions fit together to form a virtual double category [6], and under the hypothesis of our main theorem 4.0.1 an actual weak double category, and this not just for bicategories but for weak double categories too.

In the context of double categories it is natural to look for horizontal and vertical transformations. There is a natural notion of horizontal morphism between lax functors, which we call simply natural transformations (of lax functors, if confusion is possible). The theory presents no problem. For 2-categories considered as horizontal double categories (i.e. vertical arrows are identities), natural transformations are exactly 2-natural transformations. For bicategories considered as vertical weak double categories, they correspond to (dual) ICONs.

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Vertical morphisms of lax functors are a little trickier but there is still a natural candidate, a kind of multiobject profunctor, which in fact reduces to profunctors in the case where the domain is $\mathbf{1}$ and the codomain $\mathbb{S}\mathrm{et}$ or, more generally, $\mathbf{V}$-profunctors for $\mathbf{V}$-$\mathbb{S}\mathrm{et}$. This was introduced in [5] for bicategories, and called "module". The definition given there is easily adapted to weak double categories. A notion of 2-cell, called "modulation", is given in [5], which is also easily lifted to the weak double category setting.

These vertical morphisms come up in the study of double limits and colimits where it is important to understand their dependence on vertical morphisms of diagrams. This is already suggested by the two dimensional property of limits introduced in [8].

Another context where vertical morphisms can be interesting is in model theory. A first order theory can be viewed as a weak double category in a way completely analogous to Lawvere theories, with terms giving horizontal morphisms and formulas the vertical ones [14]. Then a model is a product preserving functor into $\mathbb{S}\mathrm{et}$, so modules give important extra structure which seems to have been ignored so far.

The composition of modules is however not straightforward, and this is something which must be addressed if the theory is to proceed. Part of the problem already appears for $\mathbf{V}$-profunctors where completeness conditions on $\mathbf{V}$ are required, and the composite is given by a coend formula. So it is clear that certain cocompleteness properties of the recipient category will be needed.

But this is just part of the problem. A variation on this coend formula is given in [5] but it does not apply, as is, to the weak double category case. The problem is that their formula is not horizontally functorial on general double cells, although for special cells it is, and for bicategories, all cells are special. We isolate a certain factorization property of double cells which holds in many important cases, among which are bicategories, and for which a modification of the formula from [5] does indeed give composition of modules. We consider this to be the major contribution of the present work.

This property, which we call AFP, is only one of many similar ones which have come up in our work on double categories, e.g. double Kan extensions. What it means to factor an arrow, being a one dimensional entity, is clear. It is a composite of two arrows, an epi-like one followed by a mono-like one. By contrast, cells being two dimensional, a whole panorama of factorization schemes such as our AFP is opened up. We foresee many other kinds of factorizations playing a central role in double category theory.

The subject of this paper presents an expository challenge. The proof of the main theorem requires checking a large number of details, each of which is more or less straightforward once everything has been properly set up. Conventional mathematical writing style would probably suggest they be omitted, but doing so certainly leaves a lingering doubt that something important has been missed. So we have written down all the details, which are there for anyone to check. Notationally, this is non-trivial, and we consider this to be the secondary contribution of the paper. Our notation, though not perfect, is a vast improvement over previous drafts.

Section 1 recalls the necessary background. We give the relevant definitions listing the conditions in point form for easy reference later on.

In section 2 we begin the construction of the composition of modules.

Section 3 contains our main definition, AFP, and completes the construction.

A complete proof of the main theorem is given in section 4.

The paper ends with a section tying up some loose ends. We show that identities are always strongly representable, which together with theorem 4.0.1, shows that modules provide the vertical structure for a weak double category of lax functors (under the hypotheses of the theorem of course). We show that our construction reduces to that of [5] in the case of bicategories, although not in a completely straightforward way. This validates their claim that composition of modules is associative.

1. Preliminaries

1.1. Lax Functors, Modules, Modulations We summarize here the basic definitions for easy reference. The two dimensional cell diagrams are given fully in [15], to which the reader is referred for more detail and examples.

1.1.1. Convention. By "double category" we always, unless otherwise specified, "*weak* double category".

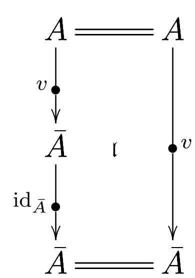
To say that a double category is *weak* means that vertical composition is associative and unitary up to coherent special isomorphisms

$$\mathfrak{a} = \mathfrak{a}_{v_3,v_2,v_1} : v_3 \cdot (v_2 \cdot v_1) \Rightarrow (v_3 \cdot v_2) \cdot v_1$$

$$\mathfrak{r} = \mathfrak{r}_v : v \cdot \mathrm{id}_A \Rightarrow v$$

$$\mathfrak{l} = \mathfrak{l}_v : \mathrm{id}_{\bar{A}} \cdot v \Rightarrow v$$

satisfying the same coherence conditions as bicategories, i.e. pentagon, etc. The double arrow indicates that we are talking about a *special cell*, i.e. one whose vertical domain and codomain are identities. For example $\mathfrak{l}$ looks like



In what follows we will omit the associativity isomorphisms and treat them as equalities, otherwise the already complicated formulas would risk becoming incomprehensible. They are easily added in for anyone wishing to do so. We sometimes also omit the unit isomorphisms when we feel it helps, although it's usually better not to.

1.1.2. Definition. A *lax functor* $F : \mathbb{A} \longrightarrow \mathbb{X}$ consists of the following data:

(LF1) Functions, all denoted F, taking objects, arrows and cells of $\mathbb{A}$ to similar ones in $\mathbb{X}$, respecting boundaries (domains and codomains);

(LF2) (Laxity cells) for every object A of $\mathbb{A}$ a special cell $\phi A : \mathrm{id}_{FA} \Rightarrow F(\mathrm{id}_A)$ and for every pair of vertical arrows $A \xrightarrow{v} \bar{A} \xrightarrow{\bar{v}} \tilde{A}$ a special cell $\phi(\bar{v}, v) : F(\bar{v}) \cdot F(v) \Rightarrow F(\bar{v} \cdot v)$.

They are required to satisfy:

(LF3) (Horizontal functoriality) $F(1_A) = 1_{FA}$, $F(1_v) = 1_{Fv}$, $F(f'f) = F(f')F(f)$, $F(\alpha'\alpha) = F(\alpha')F(\alpha)$;

(LF4) (Naturality of ϕ) $F(\mathrm{id}_f)(\phi A) = (\phi A')(\mathrm{id}_{Ff})$, $F(\bar{\alpha} \cdot \alpha)\phi(\bar{v}, v) = \phi(\bar{v}, v)(F\bar{\alpha} \cdot F\alpha)$;

(LF5) (Unit and associativity laws for ϕ) $\phi(\mathrm{id}_{\bar{A}}, v)(\phi F\bar{A} \cdot Fv) = \mathrm{can}$, $\phi(v, \mathrm{id}_A)(Fv \cdot \phi A) = \mathrm{can}$, $\phi(\tilde{v}, \bar{v} \cdot v)(F\tilde{v} \cdot \phi(\bar{v}, v)) = \phi(\tilde{v}, \cdot\bar{v}, v)(\phi(\tilde{v}, \bar{v}) \cdot Fv)$.

1.1.3. Remark. The first condition of (LF4) is vacuous for bicategories.

1.1.4. Remark. As much as possible we use a Greek letter corresponding to the name of the functor to represent the laxity cells.

1.1.5. Definition. If $F, G : \mathbb{A} \longrightarrow \mathbb{X}$ are lax functors, a *natural transformation* $t : F \longrightarrow G$ consists of the following data:

(NT1) Functions, both denoted t, taking objects A of $\mathbb{A}$ to horizontal arrows $tA : FA \longrightarrow GA$ in $\mathbb{X}$, and taking vertical arrows v of $\mathbb{A}$ to cells

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ {\scriptstyle Fv}\big\downarrow\bullet & tv & \bullet\big\downarrow{\scriptstyle Gv} \\ F\bar{A} & \xrightarrow[t\bar{A}]{} & G\bar{A} \end{array}$$

in $\mathbb{X}$.

These are required to satisfy:

(NT2) (Horizontal naturality)

$$\begin{array}{ccc} FA & \xrightarrow{tA} & GA \\ {\scriptstyle Ff}\big\downarrow & & \big\downarrow{\scriptstyle Gf} \\ FB & \xrightarrow[tB]{} & GB \end{array} \quad \text{and} \quad \begin{array}{ccc} Fv & \xrightarrow{tv} & Gv \\ {\scriptstyle F\alpha}\big\downarrow & & \big\downarrow{\scriptstyle G\alpha} \\ Fw & \xrightarrow[tw]{} & Gw \end{array}$$

commute for all horizontal arrows f and cells α;

(NT3) (Vertical functoriality)

$$
\begin{array}{ccc}
\mathrm{id}_{FA} & \xrightarrow{\phi A} & F(\mathrm{id}_A) \\
{\scriptstyle \mathrm{id}_{tA}}\downarrow & & \downarrow{\scriptstyle t(\mathrm{id}_A)} \\
\mathrm{id}_{GA} & \xrightarrow[\gamma A]{} & G(\mathrm{id}_A)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
F\bar{v}\cdot Fv & \xrightarrow{\phi(\bar{v},v)} & F(\bar{v}\cdot v) \\
{\scriptstyle t\bar{v}\cdot tv}\downarrow & & \downarrow{\scriptstyle t(\bar{v}\cdot v)} \\
G\bar{v}\cdot Gv & \xrightarrow[\gamma(\bar{v},v)]{} & G(\bar{v}\cdot v)
\end{array}
$$

commute for all objects A and composable pairs of vertical arrows v, $\bar{v}$.

Natural transformations compose in the obvious way and give a category. They are the horizontal arrows between lax functors and admit as special cases 2-natural transformations, when $\mathbb{A}$ and $\mathbb{X}$ are 2-categories (i.e. double categories whose vertical arrows are identities), and (the dual of) ICONs when $\mathbb{A}$ and $\mathbb{X}$ are bicategories (i.e. double categories whose horizontal arrows are identities).

Our main concern will be with the vertical morphisms between lax functors. This is more difficult. They were studied in some detail in [15] and in fact in [5] for bicategories before that and for poly-bicategories in [4] even before that. We are referring, of course, to modules, a kind of "multiobject profunctor".

1.1.6. Definition. Let $F, G : \mathbb{A} \longrightarrow \mathbb{X}$ be lax functors. A *module* $m : F \longrightarrow\!\!\!\!\!\bullet\,\, G$ consists of the following data:

(M1) Functions, both denoted m, taking vertical arrows $v : A \longrightarrow\!\!\!\!\!\bullet\,\, \bar{A}$ in $\mathbb{A}$ to vertical arrows $mv : FA \longrightarrow\!\!\!\!\!\bullet\,\, G\bar{A}$, and cells $\sigma : v \longrightarrow w$ in $\mathbb{A}$ to cells $m\sigma : Fv \longrightarrow Gw$;

(M2) (Left and right actions) For every pair of vertical arrows $A \xrightarrow{v} \bar{A} \xrightarrow{\bar{v}} \tilde{A}$, special cells

$$\lambda(\bar{v}, v) : G\bar{v} \cdot mv \Rightarrow m(\bar{v} \cdot v)$$

$$\rho(\bar{v}, v) : m\bar{v} \cdot Fv \Rightarrow m(\bar{v} \cdot v).$$

These are required to satisfy:

(M3) (Horizontal functoriality) $m(1_v) = 1_{m(v)}$, $m(\alpha'\alpha) = m(\alpha')m(\alpha)$;

(M4) (Naturality of λ and ρ)

$$
\begin{array}{ccc}
G\bar{v}\cdot mv & \xrightarrow{\lambda(\bar{v},v)} & m(\bar{v}\cdot v) \\
{\scriptstyle G\bar{\alpha}\cdot m\alpha}\downarrow & & \downarrow{\scriptstyle m(\bar{\alpha}\cdot\alpha)} \\
G\bar{w}\cdot mw & \xrightarrow[\lambda(\bar{w},w)]{} & m(\bar{w}\cdot w)
\end{array}
\qquad \text{and} \qquad
\begin{array}{ccc}
m\bar{v}\cdot Fv & \xrightarrow{\rho(\bar{v},v)} & m(\bar{v}\cdot v) \\
{\scriptstyle m\bar{\alpha}\cdot F\alpha}\downarrow & & \downarrow{\scriptstyle m(\bar{\alpha}\cdot\alpha)} \\
m\bar{w}\cdot Fw & \xrightarrow[\rho(\bar{w},w)]{} & m(\bar{w}\cdot w)
\end{array}
$$

commute.

(M5) (Unit laws)

$$\begin{array}{ccc} \mathrm{id}_{G(\bar{A})} \cdot m(v) & \xrightarrow{\gamma\bar{A}\cdot m(v)} & G(\mathrm{id}_{\bar{A}}) \cdot m(v) \\ & \searrow^{\cong} & \downarrow{\scriptstyle \lambda(\mathrm{id}_{\bar{A}},v)} \\ & & m(\mathrm{id}_{\bar{A}} \cdot v) \end{array} \quad \text{and} \quad \begin{array}{ccc} m(v) \cdot \mathrm{id}_{FA} & \xrightarrow{m(v)\cdot\phi A} & m(v) \cdot F(\mathrm{id}_A) \\ & \searrow^{\cong} & \downarrow{\scriptstyle \rho(v,\mathrm{id}_A)} \\ & & m(v \cdot \mathrm{id}_A) \end{array}$$

commute.

(M6) (Associativity)

$$\begin{array}{ccc} G\tilde{v} \cdot G\bar{v} \cdot mv & \xrightarrow{G\tilde{v}\cdot\lambda(\bar{v},v)} & G\tilde{v} \cdot m(\bar{v} \cdot v) \\ {\scriptstyle \gamma(\tilde{v},\bar{v})\cdot mv}\downarrow & & \downarrow{\scriptstyle \lambda(\tilde{v},\bar{v}\cdot v)} \\ G(\tilde{v} \cdot \bar{v}) \cdot mv & \xrightarrow[\lambda(\tilde{v}\cdot\bar{v},v)]{} & m(\tilde{v} \cdot \bar{v} \cdot v) \end{array} \qquad \begin{array}{ccc} G\tilde{v} \cdot m\bar{v} \cdot Fv & \xrightarrow{\lambda(\tilde{v},\bar{v})\cdot Fv} & m(\tilde{v} \cdot \bar{v}) \cdot Fv \\ {\scriptstyle G\tilde{v}\cdot\rho(\bar{v},v)}\downarrow & & \downarrow{\scriptstyle \rho(\tilde{v}\cdot\bar{v},v)} \\ G\tilde{v} \cdot m(\bar{v} \cdot v) & \xrightarrow[\lambda(\tilde{v},\bar{v}\cdot v)]{} & m(\tilde{v} \cdot \bar{v} \cdot v) \end{array}$$

$$\text{and} \qquad \begin{array}{ccc} m\tilde{v} \cdot F\bar{v} \cdot Fv & \xrightarrow{\rho(\tilde{v},\bar{v})\cdot Fv} & m(\tilde{v} \cdot \bar{v}) \cdot Fv \\ {\scriptstyle m\tilde{v}\cdot\phi(\bar{v},v)}\downarrow & & \downarrow{\scriptstyle \rho(\tilde{v}\cdot\bar{v},v)} \\ m\tilde{v} \cdot F(\bar{v} \cdot v) & \xrightarrow[\rho(\tilde{v},\bar{v}\cdot v)]{} & m(\tilde{v} \cdot \bar{v} \cdot v) \end{array}$$

commute.

There is also a notion of cell in [5], modulation, which we adapt to our situation.

1.1.7. Definition. Let $t : F \longrightarrow F'$ and $s : G \longrightarrow G'$ be natural transformations and $m : F \longrightarrow\!\!\!\!\!\bullet\;\, G$ and $m' : F' \longrightarrow\!\!\!\!\!\bullet\;\, G'$ be modules. A *modulation* μ with boundary

$$\begin{array}{ccc} F & \xrightarrow{t} & F' \\ {\scriptstyle m}\downarrow\!\bullet & & \bullet\!\downarrow{\scriptstyle m'} \\ G & \xrightarrow[s]{} & G' \end{array}$$

consists of the following data:

(m1) For vertical arrows $v : A \longrightarrow\!\!\!\!\!\bullet\;\, \bar{A}$ we are given cells

$$\begin{array}{ccc} FA & \xrightarrow{tA} & F'A \\ {\scriptstyle mv}\downarrow\!\bullet & \mu v & \bullet\!\downarrow{\scriptstyle m'v} \\ G\bar{A} & \xrightarrow[s\bar{A}]{} & G'\bar{A} \end{array}$$

which are required to satisfy

(m2) (Horizontal naturality) For every cell $\alpha : v \longrightarrow v'$

$$\begin{array}{ccc} mv & \xrightarrow{\mu v} & m'v \\ {\scriptstyle m\alpha}\downarrow & & \downarrow{\scriptstyle m'\alpha} \\ mv' & \xrightarrow[\mu v']{} & m'v' \end{array}$$

commutes.

(m3) (Equivariance) For all pairs $A \xrightarrow{v} \bar{A} \xrightarrow{\bar{v}} \tilde{A}$

$$\begin{array}{ccc} G\bar{v} \cdot mv & \xrightarrow{s\bar{v}\cdot\mu v} & G'\bar{v} \cdot m'v \\ {\scriptstyle \lambda(\bar{v},v)}\downarrow & & \downarrow{\scriptstyle \lambda'(\bar{v},v)} \\ m(\bar{v} \cdot v) & \xrightarrow[\mu(\bar{v}\cdot v)]{} & m'(\bar{v} \cdot v) \end{array} \quad \text{and} \quad \begin{array}{ccc} m\bar{v} \cdot Fv & \xrightarrow{\mu\bar{v}\cdot tv} & m'\bar{v} \cdot F'v \\ {\scriptstyle \rho(\bar{v},v)}\downarrow & & \downarrow{\scriptstyle \rho'(\bar{v},v)} \\ m(\bar{v} \cdot v) & \xrightarrow[\mu(\bar{v}\cdot v)]{} & m'(\bar{v} \cdot v) \end{array}$$

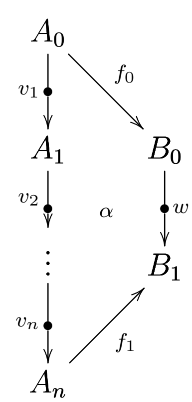
commute.

If $\mathbb{A}$ and $\mathbb{X}$ are bicategories (only identity horizontal arrows) then our definition of module is the same as that of [5]. If furthermore s and t are identities, then our definition of modulation is also the same as theirs. Note, however, that even for bicategories there can be non-identity s and t (dual ICONs) so our definition allows for more modulations.

1.2. Virtual Double Categories The problem which is the focus of this paper is the composition of modules. In order to properly formulate it, we must introduce virtual double categories.

These have been variously called **T**-catégories [2], "**fc**-multicategories" [13], "multicategories with several objects" [10], "multibicategories" [5] and "lax double categories" [7]. We adopt the name "virtual double category" from [6] to which we refer the reader for more details. There, an elegant and convincing case is made for the importance of this central concept. We simply sketch the relevant parts here.

A *virtual double category* has objects, horizontal and vertical arrows and *multicells* whose boundaries look like



which we will denote as $\alpha : v_n, v_{n-1}, \ldots, v_1 \longrightarrow w$ or, using vector notation, $\alpha : \mathbf{v} \longrightarrow w$. Horizontal arrows form a category but there is no composition of the vertical ones. We have a composition of multicells in the style of functions of several variables. Given a cell

$$\beta : w_m, \ldots, w_1 \longrightarrow x$$

and compatible cells $\alpha_i : \mathbf{v}_i \longrightarrow w_i$ we are given

$$\beta(\alpha_m, \ldots, \alpha_1) : \mathbf{v}_m, \mathbf{v}_{m-1}, \ldots, \mathbf{v}_1 \longrightarrow x.$$

This composite is required to be associative and unitary (i.e. for each v, $1_v : v \longrightarrow v$ is given, and $1_w \alpha = \alpha = \alpha(1_{v_m}, \ldots, 1_{v_1})$).

1.2.1. Definition. Let $A_0 \xrightarrow{v_1}\!\!\bullet A_1 \xrightarrow{v_2}\!\!\bullet \cdots \xrightarrow{v_n}\!\!\bullet A_n$ be a path of vertical arrows in a virtual double category $\mathbb{A}$. We say it has a *strongly representable composite* if there is an arrow $v : A_0 \longrightarrow\!\!\bullet A_n$ and a special multicell

$$\iota : v_n, v_{n-1}, \ldots, v_1 \Rightarrow v$$

such that for all compatible paths $\mathbf{x}$, $\mathbf{y}$ and multicells

$$\alpha : \mathbf{x}, \mathbf{v}, \mathbf{y} \longrightarrow z$$

there is a unique multicell $\bar{\alpha} : \mathbf{x}, v, \mathbf{y} \longrightarrow z$ such that

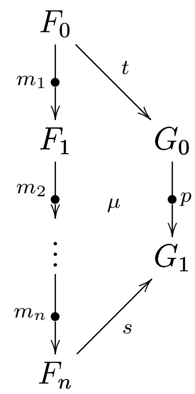
$$\begin{array}{ccc} \mathbf{x}, \mathbf{v}, \mathbf{y} & \xrightarrow{1_{\mathbf{x}}, \iota, 1_{\mathbf{y}}} & \mathbf{x}, v, \mathbf{y} \\ & \searrow^{\alpha} & \downarrow{\scriptstyle \bar{\alpha}} \\ & & z \end{array}$$

1.2.2. Remark. Mere representability, where the universal property is only required for empty $\mathbf{x}$ and $\mathbf{y}$, is of course enough to determine the composite up to special isomorphism, but strong representability is needed to get associativity of the composite. If $\mathbf{v}$ has a composite we use $[\mathbf{v}]$ to denote a chosen one.

Thus in order to properly state the problem of the composition of modules it will be sufficient to define a virtual double category structure on $\mathbb{L}\mathrm{ax}(\mathbb{A}, \mathbb{X})$. We gave a definition of multimodulation in [15] which we reproduce here. Our definition was an adaptation of the multimodulations of [5].

1.2.3. Definition. Let $F_0, F_1, \ldots, F_n, G_0, G_1$ be lax functors $\mathbb{A} \longrightarrow \mathbb{X}$, $m_1 : F_0 \longrightarrow\!\!\bullet F_1, m_2 : F_1 \longrightarrow\!\!\bullet F_2, \ldots, m_n : F_{n-1} \longrightarrow\!\!\bullet F_n$, $p : G_0 \longrightarrow\!\!\bullet G_1$ modules, and $t : F_0 \longrightarrow G_0$, $s : F_n \longrightarrow G_1$ natural transformations. A *multimodulation*

$$\mu : \mathbf{m} \longrightarrow p$$



consists of the following data:

(mm1) For every path $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \dots \xrightarrow{v_n} A_n$ a cell

$$\mu(v_n, \dots, v_1) : m_n v_n \cdot m_{n-1} v_{n-1} \cdot \dots \cdot m_1 v_1 \longrightarrow p(v_n \cdot \dots \cdot v_1), \quad \mu(\mathbf{v}) : [\mathbf{mv}] \longrightarrow p[\mathbf{v}]$$

These must satisfy the following conditions:

(mm2) (Horizontal naturality) For any n-path of cells $\alpha : \mathbf{v} \longrightarrow \mathbf{v}'$,

$$\begin{array}{ccc} [\mathbf{mv}] & \xrightarrow{[\mathbf{m}\alpha]} & [\mathbf{mv}'] \\ {\scriptstyle \mu(\mathbf{v})}\downarrow & & \downarrow{\scriptstyle \mu(\mathbf{v}')} \\ p[\mathbf{v}] & \xrightarrow[p[\alpha]]{} & p[\mathbf{v}'] \end{array}$$

commutes.

(mm3) (Inner equivariance) For any path of length $n+1$

$$y_n, \dots, y_{i+1}, v, x_i, \dots, x_1 \quad (0 < i < n)$$

$$\mathbf{y}, v, \mathbf{x}$$

we have that

$$\begin{array}{ccc} [\mathbf{my}] \cdot F_i v \cdot [\mathbf{mx}] & \xrightarrow{1 \cdot \lambda_i \cdot 1} & [\mathbf{my}] \cdot [\mathbf{m}(v \cdot \mathbf{x})] \\ {\scriptstyle 1 \cdot \rho_{i+1} \cdot 1}\downarrow & & \downarrow{\scriptstyle \mu(\mathbf{y}, v \cdot \mathbf{x})} \\ [\mathbf{m}(\mathbf{y} \cdot v)] \cdot [\mathbf{mx}] & \xrightarrow[\mu(\mathbf{y} \cdot v, \mathbf{x})]{} & p[\mathbf{y}, v\mathbf{x}] \end{array}$$

commutes.

($\mathrm{mm3}_l$) (Left equivariance)

$$\begin{array}{ccc} F_n v \cdot [\mathbf{mx}] & \xrightarrow{\lambda_n \cdot 1} & [\mathbf{m}(v \cdot \mathbf{x})] \\ {\scriptstyle sv \cdot \mu(\mathbf{x})}\downarrow & & \downarrow{\scriptstyle \mu(v \cdot \mathbf{x})} \\ G_1 v \cdot p[\mathbf{x}] & \xrightarrow[\lambda_p \cdot 1]{} & p[v, \mathbf{x}] \end{array}$$

commutes.

($\mathrm{mm3}_r$) (Right equivariance)

$$\begin{array}{ccc} [\mathbf{my}] \cdot F_o v & \xrightarrow{1 \cdot \rho_0} & [\mathbf{m}(\mathbf{y} \cdot v)] \\ {\scriptstyle \mu(\mathbf{y}) \cdot tv}\downarrow & & \downarrow{\scriptstyle \mu(\mathbf{y} \cdot v)} \\ p[\mathbf{y}] \cdot G_o v & \xrightarrow[\rho_p]{} & p[\mathbf{y}, v] \end{array}$$

commutes.

1.2.4. NOTATION. In this definition, we have introduced some notational conventions, without which the formulas would quickly become incomprehensible. We set them down here for clarity.

As mentioned before the definition, we use vector notation for paths

$$\mathbf{v} = (v_n, \ldots, v_1) = A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} \ldots \xrightarrow{v_n} A_n$$

and square brackets for a chosen composite

$$[\mathbf{v}] = v_n \cdot v_{n-1} \cdot \ldots \cdot v_1 = ((v_n \cdot v_{n-1}) \cdot \ldots) \cdot v_1.$$

If $\mathbf{m} = (m_n, \ldots, m_1) : F_0 \xrightarrow{m_1} F_1 \xrightarrow{m_2} \ldots \xrightarrow{m_n} F_n$ is a path of modules, then

$$\mathbf{mv} = \mathbf{m}(\mathbf{v}) = F_0 A_0 \xrightarrow{m_1 v_1} F_1 A_1 \xrightarrow{m_2 v_2} \ldots \xrightarrow{m_n v_n} F_n A_n$$

is the path obtained by evaluating $\mathbf{m}$ at the corresponding $\mathbf{v}$. When the paths are not of the same length we evaluate at the corresponding ones that make sense, so

$$\mathbf{my} = F_i A_i \xrightarrow{m_{i+1} y_{i+1}} F_{i+1} A_{i+1} \xrightarrow{m_{i+2} y_{i+2}} \ldots \xrightarrow{m_n y_n} F_n A_n.$$

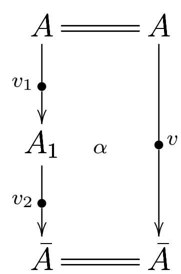
The following is now easily proved.

1.2.5. THEOREM. *Equipped with multimodulations,* $\mathbb{L}\mathrm{ax}(\mathbb{A}, \mathbb{X})$ *is a virtual double category.*

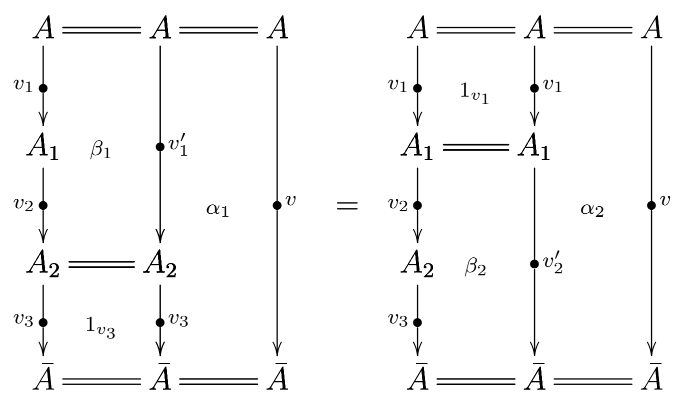
With this result, the question of whether the composite of modules exists is well-posed. Just by considering $\mathbf{V}$-profunctors, we see that certain cocompleteness conditions on $\mathbb{X}$ will be required. Under such hypotheses, [5] give a simple formula for the composite of two modules in the bicategory case. For $m_1 : F \longrightarrow\!\!\!\!\!\bullet\; G$ and $m_2 : G \longrightarrow\!\!\!\!\!\bullet\; H$ modules, and $v : A \longrightarrow\!\!\!\!\!\bullet\; \bar{A}$ they define $m(v)$ as the coequalizer

$$\sum_J m_2(v_3) \cdot G(v_2) \cdot m_1(v_1) \rightrightarrows \sum_I m_2(v_2) \cdot m_1(v_1) \longrightarrow m(v)$$

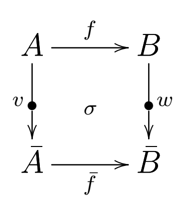
in the hom category $\mathbb{X}[FA, H\bar{A}]$ whose objects are vertical arrows $x : FA \longrightarrow\!\!\!\!\!\bullet\; H\bar{A}$ and whose morphisms are (special) cells. I is the set of "lax factorizations" of v



and J is the set of quadruples $(\alpha_1, \alpha_2, \beta_1, \beta_2)$ such that



This is a nice intuitive formula but it has several drawbacks. The main one for us is that it doesn't work for double categories. The m, thus defined, is not horizontally functorial. If



is an arbitrary cell, there is no way of defining $m(\sigma)$. To solve this, it is tempting to enlarge I to all diagrams

$$\begin{array}{ccc} A_0 & \xrightarrow{f} & A \\ {\scriptstyle v_1}\big\downarrow & & \\ A_1 & \alpha & \big\downarrow{\scriptstyle v} \\ {\scriptstyle v_2}\big\downarrow & & \\ A_2 & \xrightarrow[\bar{f}]{} & \bar{A} \end{array}$$

But now the arrows $m_2(v_2) \cdot m_1(v_1)$ do not all lie in the same hom category. One could require some stronger cocompleteness properties of $\mathbb{X}$ or, more or less equivalently, that all horizontal arrows of $\mathbb{X}$ have companions and conjoints. But this doesn't work either. There is no obvious way of defining left and right actions for m. These problems will be solved in the next two sections.

2. The Construction

2.1. Local Colimits It is clear from the examples, **V**-profunctors e.g., that we will need some cocompleteness for $\mathbb{X}$. For objects X, Y in $\mathbb{X}$ we let $\mathbb{X}[X, Y]$ denote the "vertical hom category". Its objects are vertical arrows $x : X \longrightarrow\!\!\!\!\!\bullet\;\; Y$, and its morphisms $\xi : x \Rightarrow x'$, are special cells

$$\begin{array}{ccc} X & = & X \\ {\scriptstyle x}\big\downarrow & \xi & \big\downarrow{\scriptstyle x'} \\ Y & = & Y \end{array}$$

2.1.1. Definition. We say that $\mathbb{X}$ has *local* **I** *colimits* if

(LC1) Each category $\mathbb{X}[X, Y]$ has **I** colimits;

(LC2) For every $w : W \longrightarrow\!\!\!\!\!\bullet\;\; X$ and $y : Y \longrightarrow\!\!\!\!\!\bullet\;\; Z$ the functor $\mathbb{X}[w, y] : \mathbb{X}[X, Y] \longrightarrow \mathbb{X}[W, Z]$ which takes x to $y \cdot x \cdot w$ preserves **I** colimits;

(LC3) The **I** colimits have a *further universal property.* Let $\Gamma : \mathbf{I} \longrightarrow \mathbb{X}[X, Y]$ be an **I** diagram. Given $f : X \longrightarrow X'$ and $g : Y \longrightarrow Y'$ horizontal morphisms, $x' : X' \longrightarrow\!\!\!\!\!\bullet\;\; Y'$ a vertical morphism of $\mathbb{X}$ and a cocone of cells

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle \Gamma I}\big\downarrow & \xi(I) & \big\downarrow{\scriptstyle x'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

then there exists a unique cell

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle \varinjlim \Gamma}\downarrow\bullet & \xi & \bullet\downarrow{\scriptstyle x'} \\ Y & \xrightarrow[s]{} & Y' \end{array}$$

such that $\xi\gamma(I) = \xi(I)$ ($\gamma(I)$ is the Ith colimit injection).

2.1.2. Remarks. (1) (LC3) is not saying that $\varinjlim \Gamma$ is a colimit in the category $\mathbf{X}_1$, of all vertical arrows and all cells, because the f and g are fixed. If $\mathbf{I}$ is connected however, the two concepts agree.

(2) If in $\mathbb{X}$, horizontal arrows have companions and conjoints, then (LC3) is automatic because a cell

$$\begin{array}{ccc} X & \xrightarrow{f} & X' \\ {\scriptstyle x}\downarrow & \xi & \downarrow{\scriptstyle x'} \\ Y & \xrightarrow[g]{} & Y' \end{array}$$

is equivalent to a special cell

$$\begin{array}{ccc} X & = & X \\ & & \bullet\downarrow{\scriptstyle f_*} \\ & & X' \\ {\scriptstyle x}\downarrow\bullet & \bar{\xi} & \bullet\downarrow{\scriptstyle x'} \\ & & Y' \\ & & \bullet\downarrow{\scriptstyle g^*} \\ Y & = & Y \end{array}$$

(3) (LC3) is less mysterious in the context of category theory over a base $\mathbf{B}$. The 2-category $\mathcal{CAT}/\mathbf{B}$ is cotensored over $\mathcal{C}at$. Given an object $P : \mathbf{X} \longrightarrow \mathbf{B}$ and a small category $\mathbf{I}$ then

$$\mathbf{X}^{(\mathbf{I})} \xrightarrow{P^{(\mathbf{I})}} \mathbf{B}$$

is the pullback

$$\begin{array}{ccc} \mathbf{X}^{(\mathbf{I})} & \longrightarrow & \mathbf{X}^{(\mathbf{I})} \\ {\scriptstyle P^{(\mathbf{I})}}\downarrow & & \downarrow{\scriptstyle P^{\mathbf{I}}} \\ \mathbf{B} & \xrightarrow[\Delta]{} & \mathbf{B}^{\mathbf{I}} \end{array}$$

So an object of $\mathbf{X}^{(\mathbf{I})}$ is a pair $(B, \Gamma : \mathbf{I} \longrightarrow \mathbf{X}_B)$ and a morphism $(b, t) : (B, \Gamma) \longrightarrow (B', \Gamma')$ is a morphism $b : B \longrightarrow B'$ and a natural transformation over b, $t : \Gamma \longrightarrow \Gamma'$, i.e. a natural family

$$tI : \Gamma I \longrightarrow \Gamma' I \quad \text{s.t.} \quad P(tI) = b.$$

There is an obvious diagonal functor

$$\begin{array}{ccc} \mathbf{X} & \xrightarrow{\Delta} & \mathbf{X}^{(\mathbf{I})} \\ & \searrow \quad \swarrow & \\ & \mathbf{B} & \end{array}$$

If Δ has a left adjoint over $\mathbf{B}$ (i.e. in $\mathcal{CAT}/\mathbf{B}$) it gives $\mathbf{I}$ colimits in the fibres $\mathbf{X}_B$ but with a further universal property with respect to cocones over $b : B \longrightarrow B'$.

Then $\mathbb{X}$ has local $\mathbf{I}$ colimits if

$$\begin{array}{c} \mathbf{X}_1 \\ \downarrow{\scriptstyle (\delta_0,\delta_1)} \\ \mathbf{X}_0 \times \mathbf{X}_0 \end{array}$$

has $\mathbf{I}$ colimits (in $\mathcal{CAT}/\mathbf{X}_0 \times \mathbf{X}_0$) and composition $\mathbf{X}_1 \times_{\mathbf{X}_0} \mathbf{X}_1 \xrightarrow{\bullet} \mathbf{X}_1$ distributes over them.

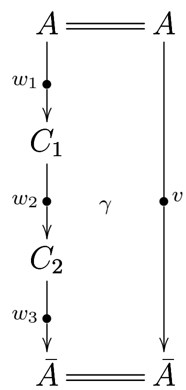
Many large double categories are locally cocomplete. For example $\mathbb{S}\text{et}$ and $\mathbb{C}\text{at}$ are, and $\mathbf{V}$-$\mathbb{S}\text{et}$ is, provided $\mathbf{V}$ has colimits and $\otimes$ distributes over them (e.g. if $\mathbf{V}$ is closed).

2.2. The Construction Let $F, G, H : \mathbb{A} \longrightarrow \mathbb{X}$ be lax functors with $\mathbb{A}$ small and $\mathbb{X}$ locally cocomplete. Let $m : F \longrightarrow\!\!\!\!\bullet\; G$ and $n : G \longrightarrow\!\!\!\!\bullet\; H$ be modules which we wish to compose. Our colimit formula is a reworking of the coequalizer formula of [5], adapted to the double category context. We will discuss how the two are related in §5.2.

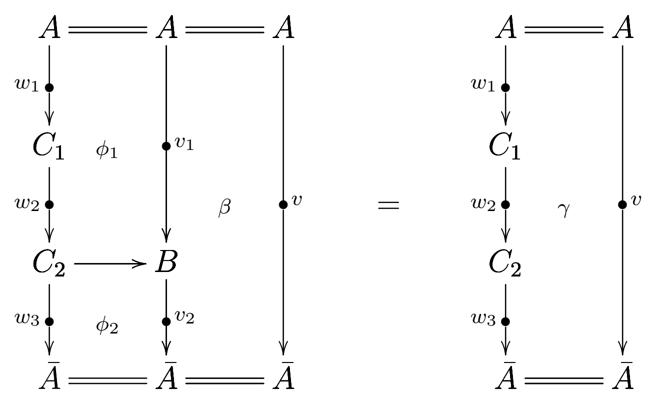
Given $v : A \longrightarrow\!\!\!\!\bullet\; \bar{A}$ we define a diagram $\Gamma_v : \mathbf{I}_v \longrightarrow \mathbb{X}[FA, H\bar{A}]$ as follows. $\mathbf{I}_v$ is a bipartite category with two kinds of objects:
lax factorizations of v

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle v_1}\downarrow\bullet & & \\ B & \beta & \bullet\downarrow{\scriptstyle v} \\ {\scriptstyle v_2}\downarrow\bullet & & \\ \bar{A} & = & \bar{A} \end{array}$$

and triple lax factorizations of v



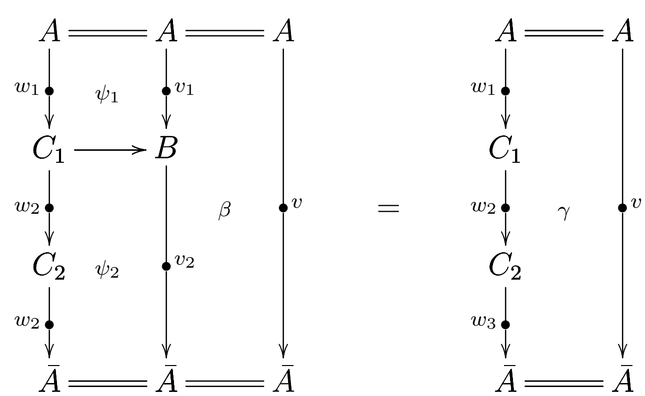
The (non-identity) morphisms go from triple lax factorizations, γ, to double ones β, and there are two kinds of these:
(i) for each horizontal morphism $C_2 \longrightarrow B$ and cells ϕ_1, ϕ_2 for which $\beta(\phi_2 \cdot \phi_1) = \gamma$ as in



we give a morphism

$$l_{\phi_2 \phi_1} : (\gamma) \longrightarrow (\beta)$$

and
(ii) for each horizontal arrow $C_1 \longrightarrow B$ and cells ψ_1, ψ_2 for which $\beta(\psi_2 \cdot \psi_1) = \gamma$ as in



we give a morphism

$$r_{\psi_2 \psi_1} : (\gamma) \longrightarrow (\beta).$$

There are no non-trivial composites.

$\Gamma_v : \mathbf{I} \longrightarrow \mathbb{X}[FA, H\bar{A}]$ is defined by

$$\Gamma_v(\beta) = nv_2 \cdot mv_1$$

$$\Gamma_v(\gamma) = nw_3 \cdot Gw_2 \cdot mw_1$$

$$\Gamma_v(l_{\phi_2\phi_1}) = nw_3 \cdot Gw_2 \cdot mw_1 \xrightarrow{nw_3\cdot\lambda} nw_3 \cdot m(w_2 \cdot w_1) \xrightarrow{n\phi_2\cdot m\phi_1} nv_2 \cdot mv_1$$

$$\Gamma_v(r_{\psi_2\psi_1}) = nw_3 \cdot Gw_2 \cdot mw_1 \xrightarrow{\rho\cdot mw_1} n(w_3 \cdot w_2) \cdot mw_1 \xrightarrow{n\psi_2\cdot m\psi_1} nv_2 \cdot mv_1.$$

In this, λ is the left action of G on m and ρ the right action of G on n.

2.2.1. Proposition. *Giving a cocone κ for Γ_v is equivalent to giving a family of cells*

$$\kappa(\beta) : nv_2 \cdot mv_1 \longrightarrow x$$

indexed by lax factorizations β of v, satisfying
(a) (transfer of scalars) for every triple lax factorization γ as above,

$$\begin{array}{ccccc} & & n(w_3 \cdot w_2) \cdot mw_1 & & \\ & \nearrow^{\rho\cdot mw_1} & & \searrow^{\kappa(\gamma)} & \\ nw_3 \cdot Gw_2 \cdot mw_1 & & & & x \\ & \searrow_{nw_3\cdot\lambda} & & \nearrow_{\kappa(\gamma)} & \\ & & nw_3 \cdot m(w_2 \cdot w_1) & & \end{array}$$

commutes,
(b) (naturality) for any two lax factorizations β and β' related by

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle v_1}\downarrow & \theta_1 & \downarrow{\scriptstyle v'_1} \\ B & \longrightarrow & B' \\ {\scriptstyle v_2}\downarrow & \theta_2 & \downarrow{\scriptstyle v'_2} \\ \bar{A} & = & \bar{A} \end{array}$$

$\beta'(\theta_2 \cdot \theta_1) = \beta$,

$$\begin{array}{ccc} nv_2 \cdot mv_1 & \xrightarrow{\kappa(\beta)} & \\ {\scriptstyle n\theta_2\cdot m\theta_1}\downarrow & & x \\ nv'_2 \cdot mv'_1 & \xrightarrow{\kappa(\beta')} & \end{array}$$

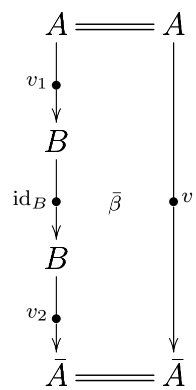
commutes.

Proof. First, let $\kappa : \Gamma_v \longrightarrow x$ be a cocone. Note that both arrows denoted by $\kappa(\gamma)$ in (a) are different as is clear from their domains. A more precise notation would be $\kappa(w_3 \cdot w_2, w_1, \gamma)$ and $\kappa(w_3, w_2 \cdot w_1, \gamma)$. Then each of the composites in the transfer of scalars diagram is equal to

$$\kappa(w_3, w_2, w_1, \gamma) : nw_3 \cdot Gw_2 \cdot mw_1 \longrightarrow x$$

by the cocone property for the morphisms $r_{1_{w_3 \cdot w_2}, 1_{w_1}}$ and $l_{1_{w_3}, 1_{w_2 \cdot w_1}}$.

For naturality, let β, β', θ_1, θ_2 be as in (b). Consider the triple lax factorization of v

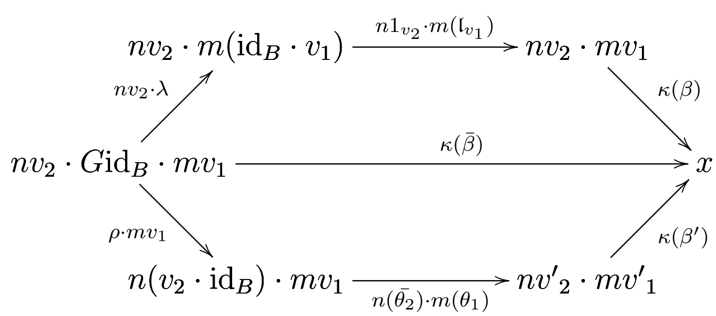


where $\bar{\beta}$ is β composed with the canonical isomorphism $v_2 \cdot \mathrm{id}_B \cdot v_1 \xrightarrow{\cong} v_2 \cdot v_1$. We have morphisms in $\mathbf{I}_v$

$$l_{1_{v_2}, 1_{\bar{v}_1}} : (\bar{\beta}) \longrightarrow (\beta)$$

$$r_{\bar{\theta}_2, \theta_1} : (\bar{\beta}) \longrightarrow (\beta')$$

which give a commutative diagram



Precede this by the canonical

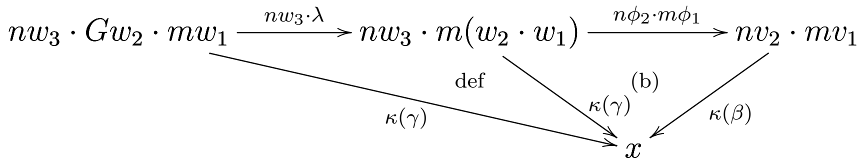
$$nv_2 \cdot mv_1 \xrightarrow{\cong} nv_2 \cdot \mathrm{id}_{GB} \cdot mv_1 \xrightarrow{nv_2 \cdot \gamma_B \cdot mv_1} nv_2 \cdot G\mathrm{id}_B \cdot mv_1$$

and the top path will be equal to $\kappa(\beta)$ whereas the bottom will be $\kappa(\beta')(n\theta_2 \cdot m\theta_1)$.

Conversely, let $\langle \kappa(\beta) : nv_2 \cdot mv_1 \longrightarrow x \rangle$ be a family of cells satisfying (a) and (b). Given a triple lax factorization (γ) as above, define

$$\kappa(\gamma) : nw_3 \cdot Gw_2 \cdot mw_1 \longrightarrow x$$

to be the common composite given by (a). Then for a morphism $l_{\phi_2,\phi_1} : (\gamma) \longrightarrow (\beta)$ in $\mathbf{I}_v$ we have $\kappa(\beta)\Gamma_v(l_{\phi_2,\phi_1})$ as the top composite in



which is the cocone property for l_{ϕ_2,ϕ_1}. The one for r_{ψ_2,ψ_1} is similar.

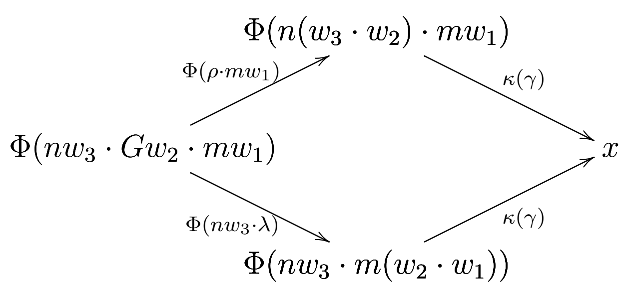
■

Actually we need a stronger form of this proposition; we need it not just for cocones on Γ_v but on $\Phi\Gamma_v$, where $\Phi : \mathbb{X}[FA, H\bar{A}] \longrightarrow \mathbf{X}$ is an arbitrary functor. In the applications that follow, $\mathbf{X}$ will be one of the vertical homs of $\mathbb{X}$, $\mathbb{X}[X_1, X_2]$, and Φ will come from vertical composition.

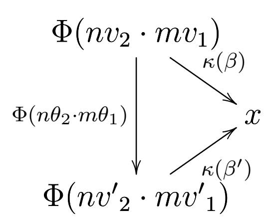
2.2.2. PROPOSITION. *Giving a cocone $\kappa : \Phi\Gamma_v \longrightarrow x$ is equivalent to giving a family*

$$\kappa(\beta) : \Phi(nv_1 \cdot mv_2) \longrightarrow x$$

indexed by lax factorizations β of v, satisfying:
(a) (transfer of scalars)



commutes,
(b) (naturality)



commutes.

PROOF. Put a Φ before all objects and arrows in $\mathbb{X}[FA, H\bar{A}]$ in the previous proof, except x and the κ's. ■

2.2.3. Definition. $(n \cdot m)(v) = \varinjlim \Gamma_v$.

Let $j_\beta : n(v_2) \cdot m(v_1) \longrightarrow (n \cdot m)(v)$ denote the colimit injection corresponding to the lax factorization (β) of v. Note that the family $\langle j_\beta \rangle$ is jointly epic. This is also true for the families $\langle y \cdot j_\beta \rangle$, $\langle j_\beta \cdot z \rangle$, $\langle y \cdot j_\beta \cdot z \rangle$ as composition with vertical arrows preserves colimits. This fact will be used repeatedly below.

2.3. The Actions Given $A' \xrightarrow{v'} A \xrightarrow{v} \bar{A} \xrightarrow{\bar{v}} \tilde{A}$ we wish to define the actions $\lambda_{n \cdot m}(\bar{v}, v)$ and $\rho_{n \cdot m}(v, v')$.

2.3.1. Proposition. *There are unique cells* $\lambda_{n \cdot m}(\bar{v}, v)$ *and* $\rho_{n \cdot m}(v, v')$ *satisfying:*

$$\begin{array}{ccc} H\bar{v} \cdot (n \cdot m)(v) & \xrightarrow{\lambda_{n \cdot m}(\bar{v}, v)} & (n \cdot m)(\bar{v} \cdot v) \\ {\scriptstyle H\bar{v} \cdot j_\beta}\uparrow & & \uparrow{\scriptstyle j_{\bar{v} \cdot \beta}} \\ H\bar{v} \cdot nv_2 \cdot mv_1 & \xrightarrow[\lambda_n(\bar{v}, v_2) \cdot mv_1]{} & n(\bar{v} \cdot v_2) \cdot mv_1 \end{array}$$

and

$$\begin{array}{ccc} (n \cdot m)(v) \cdot Fv' & \xrightarrow{\rho_{n \cdot m}(v, v')} & (n \cdot m)(v \cdot v') \\ {\scriptstyle j_\beta \cdot Fv'}\uparrow & & \uparrow{\scriptstyle j_{\beta \cdot v'}} \\ nv_2 \cdot mv_1 \cdot Fv' & \xrightarrow[nv_2 \cdot \rho_m(v_1 \cdot v')]{} & nv_2 \cdot m(v_1 \cdot v') \end{array}$$

commute for every lax factorization of v

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle v_1}\downarrow & & \\ A_1 & \beta & \downarrow{\scriptstyle v} \\ {\scriptstyle v_2}\downarrow & & \\ \bar{A} & = & \bar{A} \end{array}$$

Proof. We will only discuss the left action, the right one being similar. $H\bar{v} \cdot (\ )$ preserves colimits so $H\bar{v} \cdot (n \cdot m)(v)$ is $\varinjlim H\bar{v} \cdot \Gamma_v$ and $H\bar{v} \cdot j_\beta$ are colimit injections. So we must show that $\langle j_{\bar{v} \cdot \beta}(\lambda_n(\bar{v}, v_2) \cdot mv_1) \rangle_\beta$ determines a cocone. By proposition 2.2.2 it is sufficient to check conditions (a) and (b).

Let $(\gamma) = (w_3, w_2, w_1, \gamma)$ be a triple lax factorization of v. Condition (a) is commutativity of the outside of the diagram

$$\begin{array}{ccccc}
 & H\bar{v}\cdot n(w_3\cdot w_2)\cdot mw_1 & \xrightarrow{\lambda_n\cdot mw_1} & n(\bar{v}\cdot w_3\cdot w_2)\cdot mw_1 & \\
{\scriptstyle H\bar{v}\cdot\rho_n\cdot mw_1}\nearrow & & (1) & \nearrow{\scriptstyle \rho_n\cdot mw_1} \qquad (3) & \searrow{\scriptstyle j_{\bar{v}\cdot\gamma}} \\
H\bar{v}\cdot nw_3\cdot Gw_2\cdot mw_1 & & \xrightarrow{\lambda_n\cdot Gw_2\cdot mw_1} & n(\bar{v}\cdot w_3)\cdot Gw_2\cdot mw_1 & (n\cdot m)(\bar{v}\cdot v) \\
{\scriptstyle H(\bar{v})\cdot nw_3\cdot\lambda_m}\searrow & & (2) & \searrow{\scriptstyle n(\bar{v}\cdot w_3)\cdot\lambda_m} & \nearrow{\scriptstyle j_{\bar{v}\cdot\gamma}} \\
 & H\bar{v}\cdot nw_3\cdot m(w_2\cdot w_1) & \xrightarrow[\lambda_n\cdot m(w_2\cdot w_1)]{} & n(\bar{v}\cdot w_3)\cdot m(w_2\cdot w_1) &
\end{array}$$

Mixed associativity for n gives commutativity of (1), functoriality of vertical composition gives (2), and (3) is condition (a) for the lax factorization $\bar{v}\cdot\gamma = (\bar{v}\cdot w_3, w_2, w_1, \bar{v}\cdot\gamma)$ of $\bar{v}\cdot v$.

Let $\beta = \beta'(\theta_2\cdot\theta_1)$ as in condition (b). We have to show that the outside of the following diagram commutes.

$$\begin{array}{ccccc}
H\bar{v}\cdot nv_2\cdot mv_1 & \xrightarrow{\lambda_n\cdot mv_1} & n(\bar{v}\cdot v_2)\cdot mv_1 & \searrow{\scriptstyle j_{\bar{v}\cdot\beta}} & \\
{\scriptstyle H\bar{v}\cdot n\theta_2\cdot m\theta_1}\Big\downarrow & (1) & \Big\downarrow{\scriptstyle n(\bar{v}\cdot\theta_2)\cdot m\theta_1} & (2) & (n\cdot m)(\bar{v}\cdot v) \\
H\bar{v}\cdot nv'_2\cdot mv'_1 & \xrightarrow[\lambda_n\cdot mv'_1]{} & n(\bar{v}\cdot v'_2)\cdot mv'_1 & \nearrow{\scriptstyle j_{\bar{v}\cdot\beta'}} &
\end{array}$$

Naturality of λ_n gives (1), and (2) is condition (b) for $\bar{v}\cdot v$.

∎

3. A Factorization Property

It remains to define $n\cdot m$ on cells.

In order to do this, we need an extra condition on $\mathbb{A}$, a factorization of cells which holds vacuously for bicategories. This is where the theory for double categories diverges from that of [5]. Double cells or multicells, being two dimensional, can be factored in a variety of ways of which the one given below is but one. We envision many other such factorizations. It is the isolation of this property that we consider the main contribution of this work.

3.1. The Condition AFP

3.1.1. Definition. We say that $\mathbb{A}$ satisfies AFP if every cell of the form

$$\begin{array}{ccc} A_0 & \xrightarrow{f} & B_0 \\ {\scriptstyle v_1}\bullet\downarrow & & \\ A_1 & \alpha & \bullet\downarrow{\scriptstyle w} \\ {\scriptstyle v_2}\bullet\downarrow & & \\ A_2 & \xrightarrow[g]{} & B_2 \end{array}$$

factors as

$$\begin{array}{ccccc} A_0 & \xrightarrow{f} & B_0 & = & B_0 \\ {\scriptstyle v_1}\bullet\downarrow & \alpha_1 & \bullet\downarrow{\scriptstyle w_1} & & \\ A_1 & \longrightarrow & B_1 & \alpha_3 & \bullet\downarrow{\scriptstyle w} \\ {\scriptstyle v_2}\bullet\downarrow & \alpha_2 & \bullet\downarrow{\scriptstyle w_2} & & \\ A_2 & \xrightarrow[g]{} & B_2 & = & B_2 \end{array}$$

$\alpha = \alpha_3(\alpha_2 \cdot \alpha_1)$, and any two such factorizations are equivalent under the equivalence relation generated by identifying two factorizations

$$\begin{array}{ccccc} A_0 & \xrightarrow{f} & B_0 & = & B_0 \\ {\scriptstyle v_1}\bullet\downarrow & \gamma_1 & \bullet\downarrow{\scriptstyle u_1} & & \\ A_1 & \longrightarrow & C & \gamma_3 & \bullet\downarrow{\scriptstyle w} \\ {\scriptstyle v_2}\bullet\downarrow & \gamma_2 & \bullet\downarrow{\scriptstyle u_2} & & \\ A_2 & \xrightarrow[g]{} & B_2 & = & B_2 \end{array} \quad = \quad \begin{array}{ccccc} A_0 & \xrightarrow{f} & B_0 & = & B_0 \\ {\scriptstyle v_1}\bullet\downarrow & \delta_1 & \bullet\downarrow{\scriptstyle x_1} & & \\ A_1 & \longrightarrow & D & \delta_3 & \bullet\downarrow{\scriptstyle w} \\ {\scriptstyle v_2}\bullet\downarrow & \delta_2 & \bullet\downarrow{\scriptstyle x_2} & & \\ A_2 & \xrightarrow[g]{} & B_2 & = & B_2 \end{array}$$

if there is either

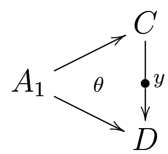
(1) a *horizontal reduction* $(\gamma_1, \gamma_2, \gamma_3) \xrightarrow{(\phi_1 \phi_2)} (\delta_1, \delta_2, \delta_3)$: there exist

$$\begin{array}{ccc} B_0 & = & B_0 \\ {\scriptstyle u_1}\bullet\downarrow & \phi_1 & \bullet\downarrow{\scriptstyle x_1} \\ C & \longrightarrow & D \\ {\scriptstyle u_2}\bullet\downarrow & \phi_2 & \bullet\downarrow{\scriptstyle x_2} \\ B_2 & = & B_2 \end{array}$$

such that

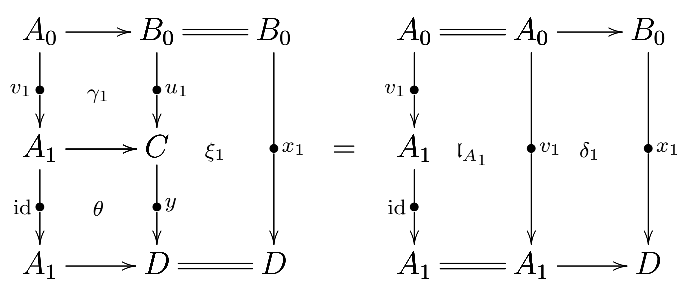
(a) $\phi_1 \gamma_1 = \delta_1$

(b) $\phi_2\gamma_2 = \delta_2$
(c) $\delta_3(\phi_2 \cdot \phi_1) = \gamma_3$
or
(2) a *vertical reduction* $(\gamma_1, \gamma_2, \gamma_3) \xrightarrow{(y,\theta)} (\delta_1, \delta_2, \delta_3)$: there exists

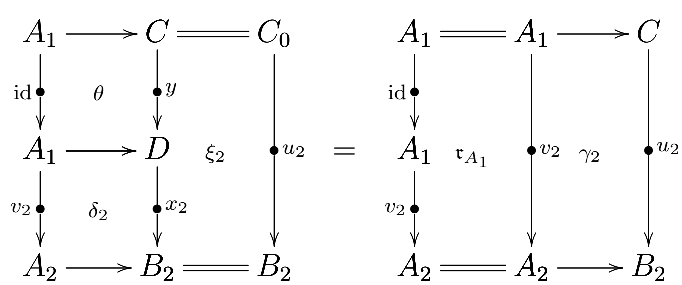


such that
(a) $\theta \cdot \gamma_1 = \delta_1$,
(b) $\delta_2 \cdot \theta = \gamma_2$,
(c) $\gamma_3 = \delta_3$.

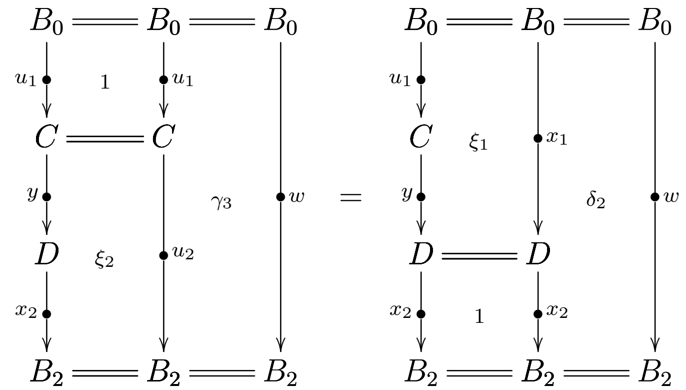
3.1.2. Note. The intent of the conditions in (2) is clear but because we are working with weak double categories we can't help certain canonical isomorphisms creeping in. More specifically, we need to require special *isomorphisms* $\xi_1 : y \cdot u_1 \Rightarrow x$ and $\xi_2 : x_2 \cdot y \Rightarrow u_2$ such that
(a)



(b)



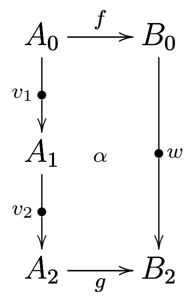
(c)



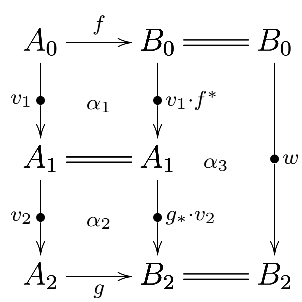
3.1.3. Examples. Bicategories, considered as vertical double categories, satisfy AFP, but more can be said.

3.1.4. Proposition. *If every horizontal arrow of $\mathbb{A}$ admits a companion and a conjoint, then $\mathbb{A}$ satisfies AFP.*

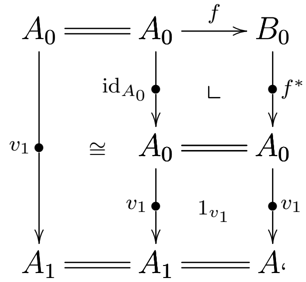
Proof. Any cell

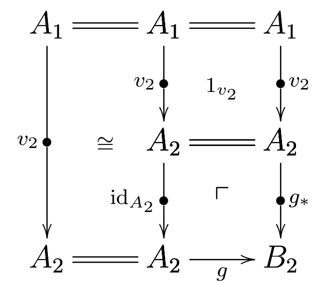


factors as

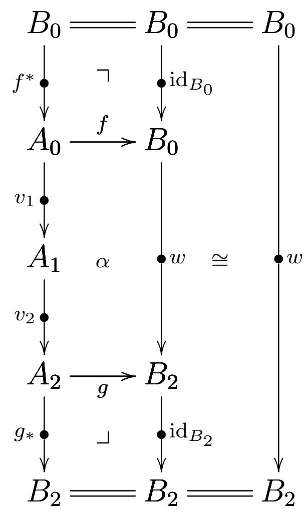


where α_1, α_2 and α_3 are respectively

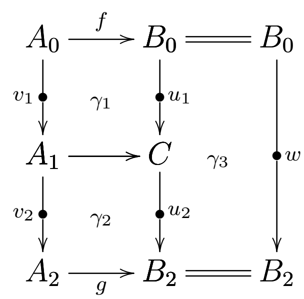




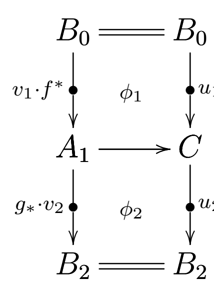
and



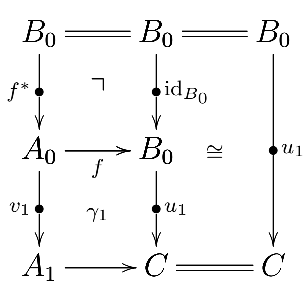
where the corner brackets represent binding cells, and $\cong$ are canonical (structural) isomorphisms. Any other factorization of α



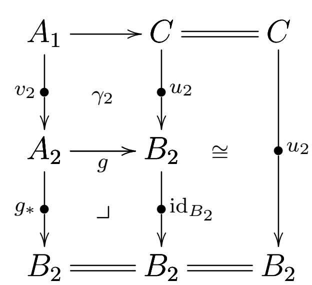
is a horizontal reduct of $(\alpha_1, \alpha_2, \alpha_3)$ via the cells



where ϕ_1 and ϕ_2 are respectively



and



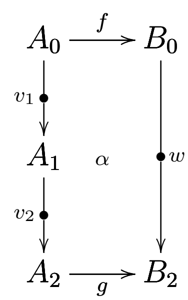
■

3.1.5. Corollary. *Any bicategory considered as a vertical double category satisfies AFP.*

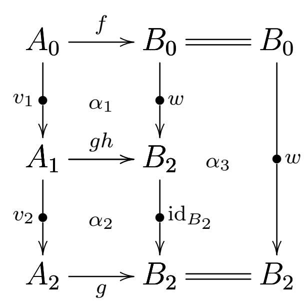
Whereas, in the proof of proposition 3.1.4, only horizontal reduction (condition (1) of AFP) was needed, the proof of the next result requires vertical reduction (condition (2)) as well.

3.1.6. Proposition. *Let $\mathbb{A}$ be a double category in which every vertical arrow is a companion of some horizontal arrow. Then $\mathbb{A}$ satisfies AFP.*

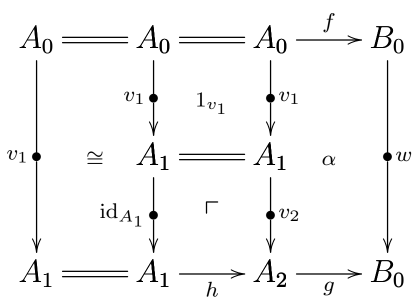
Proof. Consider a cell



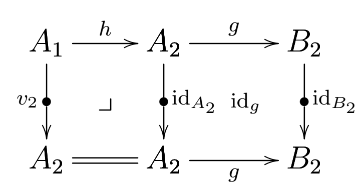
and suppose $v_2 \cong h_*$ for $h : A_1 \longrightarrow A_2$. Then α factors as



where α_1 , α_2 are respectively

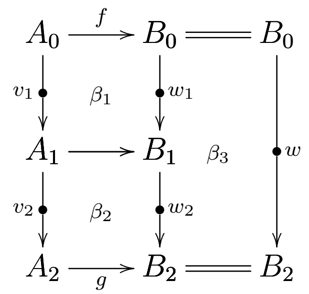


and

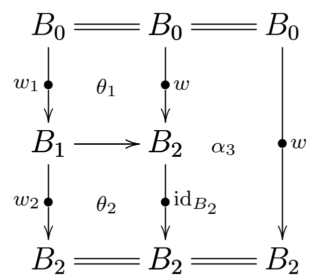


and α_3 is the canonical isomorphism.

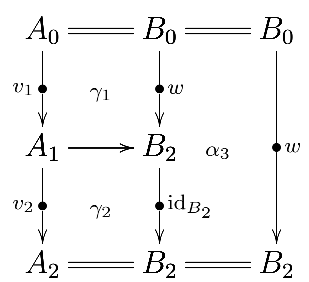
Now let



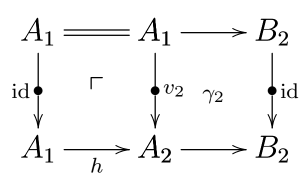
be an arbitrary factorization of α. Use the same procedure as above to factor β_3 as



which gives a horizontal reduction of the $\beta_3(\beta_2 \cdot \beta_1)$ factorization to



where $\gamma_i = \theta_i \cdot \beta_i$. Now let θ be



Then an easy calculation shows that $\theta \cdot \gamma_1 = \alpha_1$ and $\alpha_2 \cdot \theta = \gamma_2$. Thus θ gives a vertical reduction of the $\alpha_3(\gamma_2 \cdot \gamma_1)$ factorization to the $\alpha_3(\alpha_2 \cdot \alpha_1)$ one.

■

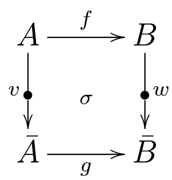
3.1.7. Corollary. *Any 2-category considered as a horizontal double category satisfies AFP.*

3.1.8. Corollary. *The double category of quintets in a 2-category satisfies AFP.*

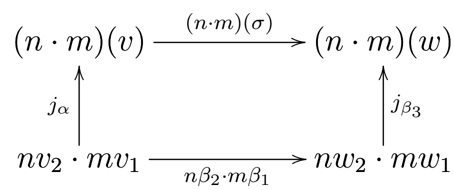
3.1.9. Corollary. *If every vertical arrow of $\mathbb{A}$ is a conjoint, then $\mathbb{A}$ satisfies AFP.*

3.2. Functoriality (defined) We are now in a position to define the action of $n \cdot m$ on cells.

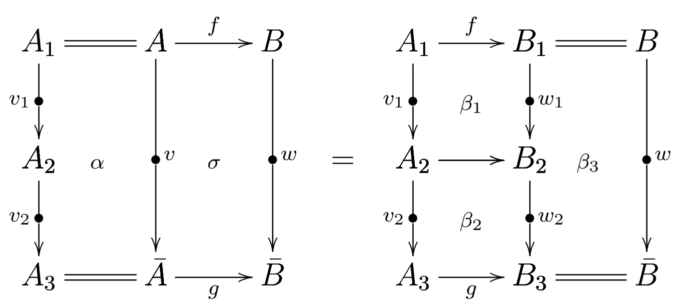
3.2.1. Proposition. *Assume that $\mathbb{A}$ satisfies AFP and let*



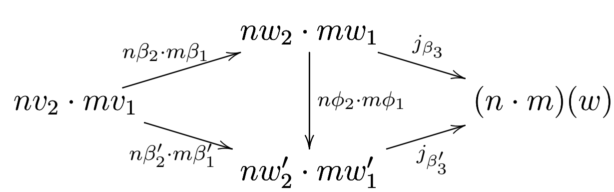
be a cell in $\mathbb{A}$. Then there exists a unique cell $(n \cdot m)(\sigma) : (n \cdot m)(v) \longrightarrow (n \cdot m)(w)$ such that



commutes for every α and factorization of $\sigma\alpha$

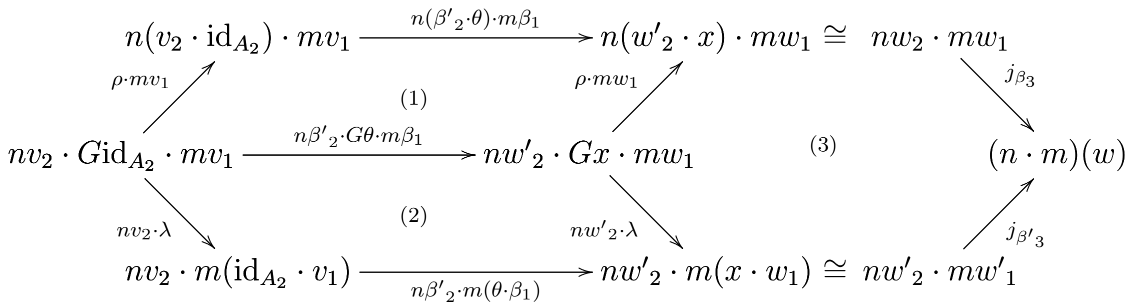


Proof. We first show that $j_{\beta_3}(n\beta_2 \cdot m\beta_1)$ is independent of the factorization.
(1) Suppose we have a horizontal reduction of $(\beta_1, \beta_2, \beta_3)$ to $(\beta_1', \beta_2', \beta_3')$ by (ϕ_1, ϕ_2). Then



commutes, the left triangle by functoriality of n and m and the right triangle by the cocone property for j (proposition 2.2.2).

(2) Suppose we have a vertical reduction of $(\beta_1, \beta_2, \beta_3)$ to $(\beta'_1, \beta'_2, \beta'_3)$ via θ. Then we have a commutative diagram



where (1) and (2) commute by naturality of ρ and λ and (3) by the cocone property for j (proposition 2.2.2). Note that the top and bottom arrows of this diagram are $n\beta_2 \cdot m\beta_1$ and $n{\beta'}_2 \cdot m{\beta'}_1$ preceded by canonical isomorphisms, so if we precompose the diagram by

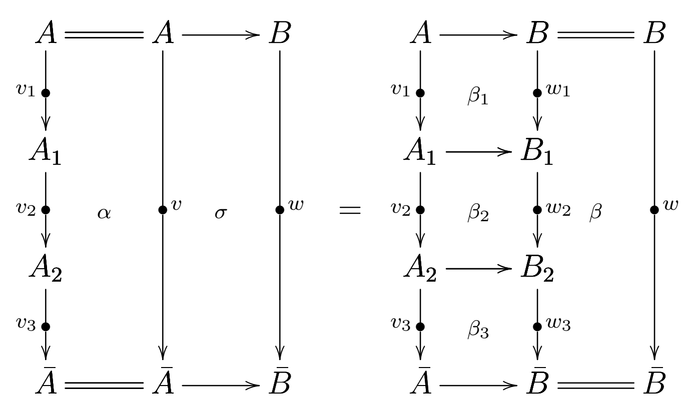
$$nv_2 \cdot mv_1 \xrightarrow{\cong} nv_2 \cdot \mathrm{id}_{GA_2} \cdot mv_1 \xrightarrow{nv_2 \cdot \gamma_G \cdot mv_1} nv_2 \cdot G(\mathrm{id}_{A_2}) \cdot mv_1$$

we get $j_{\beta_3}(n\beta_2 \cdot m\beta_1) = j_{\beta'_3}(n\beta'_2 \cdot m\beta'_1)$.

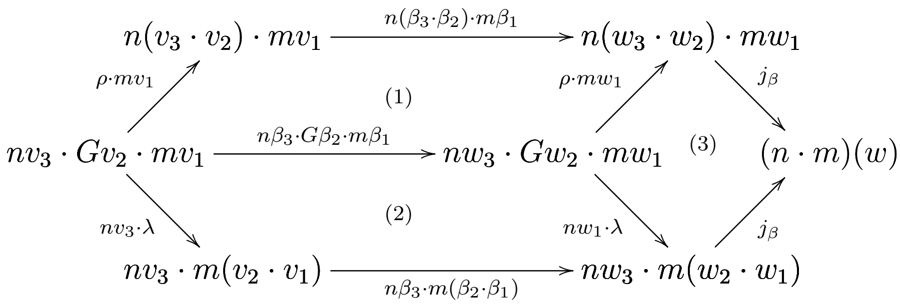
This shows that $j_{\beta_3}(n\beta_2 \cdot m\beta_1)$ is independent of the factorization of $\sigma\alpha$. We denote the common value by k_α.

Now we show that $\langle k_\alpha \rangle$ determines a cocone. We use proposition 2.2.2.

(a) (Transfer of scalars) Let $\alpha : v_3 \cdot v_2 \cdot v_2 \Rightarrow v$ be a triple lax factorization of v. Then we can factor $\sigma\alpha$ as



by first factoring $\sigma\alpha$ considered as a cell $(v_3 \cdot v_2) \cdot v_1 \longrightarrow w$ and then factoring the cell with domain $v_3 \cdot v_2$. Then the following diagram commutes



(1) and (2) by naturality of ρ and λ and (3) by the cocone property for j. The top composite is $k_\alpha(\rho \cdot mv_1)$ and the bottom is $k_\alpha(nv_3 \cdot \lambda)$. Thus $\langle k_\alpha \rangle$ determines a cocone.

Now the cocone determined by the k_α is of the type required for the "further universal property" (LC3) of definition 2.1.1. Each component has the same vertical domain, $F(f)$, and the same vertical codomain $H(g)$. Then by (LC3) there exists a unique cell $(n \cdot m)(\sigma)$: $(n \cdot m)(v) \longrightarrow (n \cdot m)(w)$ satisfying the stated conditions.

■

4. Proof of Main Theorem

In this section we prove the following theorem.

4.0.1. Theorem. *If $\mathbb{A}$ is a small double category satisfying AFT and $\mathbb{X}$ is a locally cocomplete double category, then composites of modules between lax functors $\mathbb{A} \longrightarrow \mathbb{X}$ are strongly representable.*

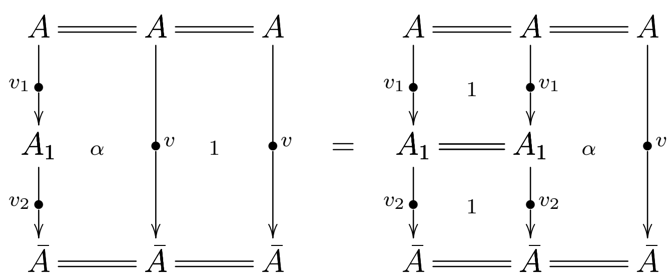
Now that the hard work has been done and everything properly set up, it is relatively straightforward to check that everything works. We set down the calculations here in the interest of completeness.

We first check that $n \cdot m$ is indeed a module $F \longrightarrow\!\!\!\!\!\bullet\; H$. The data (M1) and (M2) from definition 1.1.6 have already been specified. We use the fact that the definition of k_α in proposition 3.2.1 is independent of the factorization of $\sigma\alpha$ and make judicious choices in our proofs.

4.1. Functoriality

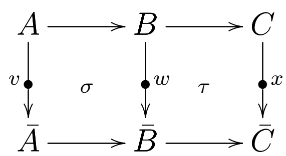
4.1.1. Proposition. *$n \cdot m$ satisfies condition (M3) "horizontal functoriality".*

Proof. For $\sigma = 1_v$ we can factor $\sigma\alpha$ as

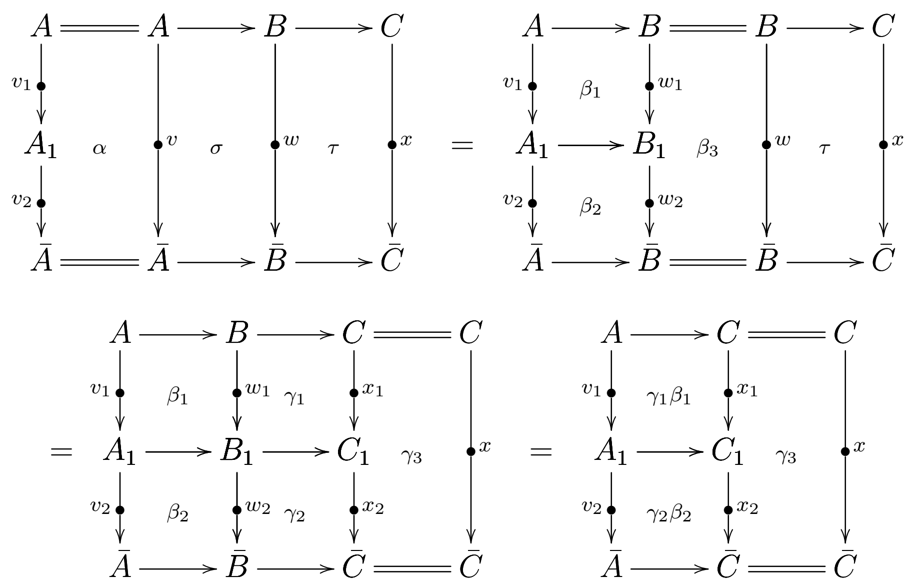


so that $k_\alpha = j_\alpha$. Thus $(n \cdot m)(1_v) = 1_{(n \cdot m)(v)}$.

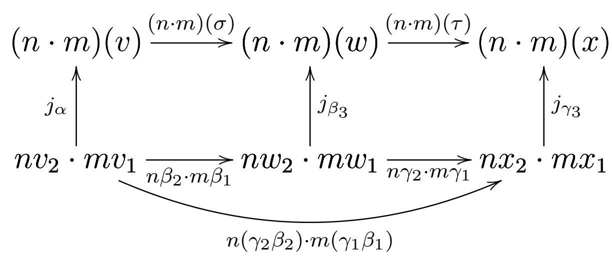
Now consider two cells



We first factor $\sigma\alpha = \beta_3(\beta_2 \cdot \beta_1)$ and then $\tau\beta_3 = \gamma(\gamma_2 \cdot \gamma_1)$



which gives the commutative diagram



so $((n \cdot m)(\tau))((n \cdot m)(\sigma)) = (n \cdot m)(\tau\sigma)$.

∎

4.2. Naturality

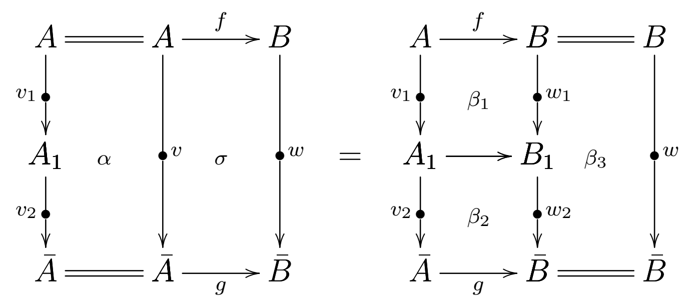
4.2.1. Proposition. *The left and right actions for $n \cdot m$ are natural (M4).*

Proof. We give the proof for the left action $\lambda_{n \cdot m}$. The right action is the same.

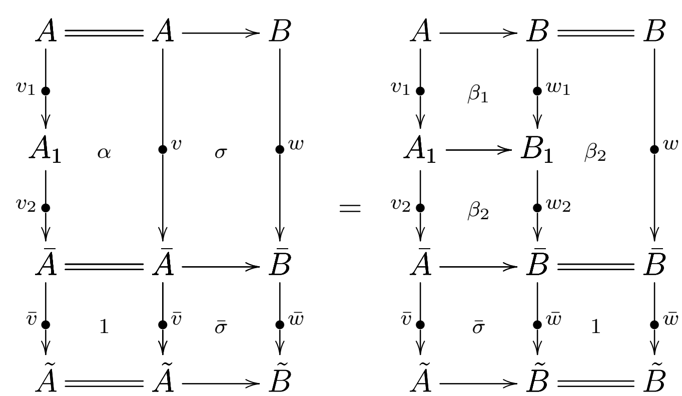
For compatible cells, $\sigma : v \longrightarrow w$ and $\bar{\sigma} : \bar{v} \longrightarrow \bar{w}$ we must establish the commutativity of

$$\begin{array}{ccc}
H\bar{v}\cdot(n\cdot m)(v) & \xrightarrow{\lambda_{n\cdot m}(\bar{v},v)} & (n\cdot m)(\bar{v}\cdot v) \\
{\scriptstyle H\bar{\sigma}\cdot(n\cdot m)(\sigma)}\downarrow & & \downarrow{\scriptstyle (n\cdot m)(\bar{\sigma}\cdot\sigma)} \\
H\bar{w}\cdot(n\cdot m)(w) & \xrightarrow[\lambda_{n\cdot m}(\bar{w},w)]{} & (n\cdot m)(\bar{w}\cdot w)
\end{array}$$

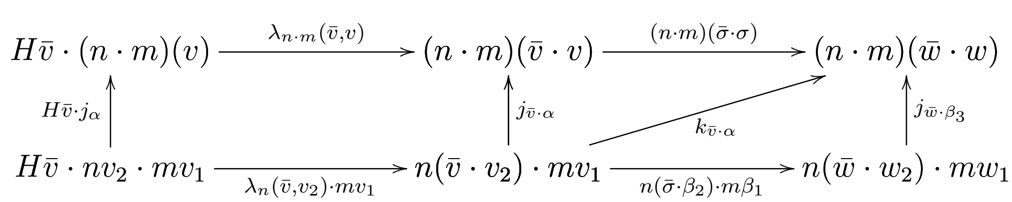
Given a lax factorization of v, $\alpha : v_2 \cdot v_1 \Rightarrow v$, we factor $\sigma\alpha$ as



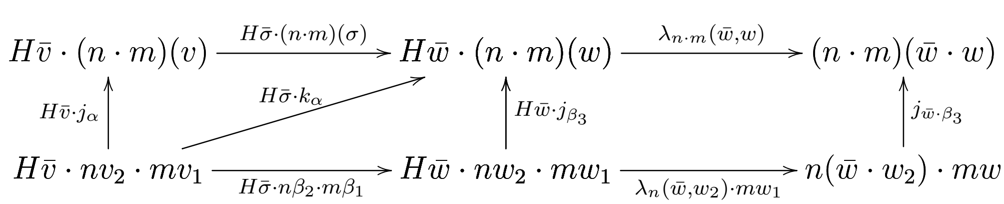
which gives us a factorization of $(\bar{\sigma}\cdot\sigma)(\bar{v}\cdot\alpha)$



Then we have commutative diagrams



and



in which the bottom rows are equal by naturality of λ_n. As the $H\bar{v}\cdot j_\alpha$ are jointly epic, we get the required commutativity.

■

4.3. Unit Laws

4.3.1. Proposition. *The left and right actions for $n\cdot m$ satisfy condition (M5), "unit laws".*

Proof. Again we just check the unit law for $\lambda_{n\cdot m}$. We must show commutativity of

$$\begin{array}{ccc} \mathrm{id}_{H\bar{A}}\cdot(n\cdot m)(v) & \xrightarrow{\eta_{\bar{A}}\cdot(n\cdot m)(v)} & H(\mathrm{id}_A)\cdot(n\cdot m)(v) \\ & \searrow^{\cong} & \downarrow{\scriptstyle \lambda_{n\cdot m}(\mathrm{id}_{\bar{A}},v)} \\ & & (n\cdot m)(\mathrm{id}_A\cdot v) \end{array}$$

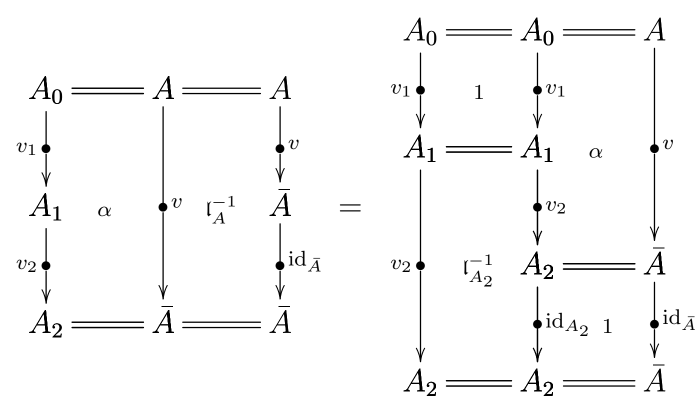
For any $\alpha : v_2\cdot v_1 \Rightarrow v$ we have a commutative diagram

$$\begin{array}{ccccc} \mathrm{id}_{H\bar{A}}\cdot(n\cdot m)(v) & \xrightarrow{\eta_{\bar{A}}\cdot(n\cdot m)(v)} & H(\mathrm{id}_{\bar{A}})\cdot(n\cdot m)(v) & \xrightarrow{\lambda_{n,m}(\mathrm{id}_{\bar{A}},v)} & (n\cdot m)(\mathrm{id}_{\bar{A}}\cdot v) \\ \uparrow{\scriptstyle \mathrm{id}_{H\bar{A}}\cdot j_\alpha} & & \uparrow{\scriptstyle H(\mathrm{id}_{\bar{A}})\cdot j_\alpha} & & \uparrow{\scriptstyle j_{\mathrm{id}_{\bar{A}}\cdot\alpha}} \\ \mathrm{id}_{H\bar{A}}\cdot nv_2\cdot mv_1 & \xrightarrow[\eta_{\bar{A}}\cdot nv_2\cdot mv_1]{} & H(\mathrm{id}_{\bar{A}})\cdot nv_2\cdot mv_1 & \xrightarrow[\lambda_n(\mathrm{id}_{\bar{A}},v_2)\cdot mv_1]{} & n(\mathrm{id}_{\bar{A}}\cdot v_2)\cdot mv_1 \end{array}$$

The canonical isomorphism represented by $\cong$ in the triangle above is in fact the top row of the commutative diagram

$$\begin{array}{ccccc} \mathrm{id}_{H\bar{A}}\cdot(n\cdot m)(v) & \xrightarrow{\iota_{H\bar{A}}} & (n\cdot m)(v) & \xrightarrow{(n\cdot m)(\iota_{\bar{A}}^{-1})} & (n\cdot m)(\mathrm{id}_{\bar{A}}\cdot v) \\ \uparrow{\scriptstyle \mathrm{id}_{H\bar{A}}\cdot j_\alpha} & & \uparrow{\scriptstyle j_\alpha} & \nearrow_{k_\alpha} & \uparrow{\scriptstyle j_{\mathrm{id}_{\bar{A}}\cdot\alpha}} \\ \mathrm{id}_{H\bar{A}}\cdot nv_2\cdot mv_1 & \xrightarrow[\iota_{H\bar{A}}\cdot mv_1]{} & nv_2\cdot mv_1 & \xrightarrow[n(\iota_{\bar{A}}^{-1})\cdot mv_1]{} & n(\mathrm{id}_{\bar{A}}\cdot v_2)\cdot mv_1 \end{array}$$

where the calculation of k_α was made using the factorization



The bottom lines of the two rectangles are equal by the unit law for n.

■

4.4. Associativity

4.4.1. Proposition. *The left and right actions for $n \cdot m$ satisfy condition (M6), "associativity".*

Proof. First we check left associativity. This means that the top composites of the two following diagrams must be equal for any composable triple of vertical arrows v, $\bar{v}$, $\tilde{v}$. Any lax factorization of v, $\alpha : v_2 \cdot v_1 \Rightarrow v$ gives lax factorizations $\bar{v} \cdot \alpha : (\bar{v} \cdot v_2) \cdot v_1 \Rightarrow \bar{v} \cdot v$ and $\tilde{v} \cdot \bar{v} \cdot \alpha : (\tilde{v} \cdot \bar{v} \cdot v_2) \cdot v_1 \Rightarrow \tilde{v} \cdot \bar{v} \cdot v$ of $\bar{v} \cdot v$ and $\tilde{v} \cdot \bar{v} \cdot v$ respectively. Thus we have the following commutative diagrams.

$$\begin{array}{ccccc}
H\tilde{v} \cdot H\bar{v} \cdot (n \cdot m)(v) & \xrightarrow{H\tilde{v} \cdot \lambda(\bar{v}, v)} & H\tilde{v} \cdot (n \cdot m)(\bar{v} \cdot v) & \xrightarrow{\lambda(\tilde{v}, \bar{v} \cdot v)} & (n \cdot m)(\tilde{v} \cdot \bar{v} \cdot v) \\
{\scriptstyle H\tilde{v} \cdot H\bar{v} \cdot j_\alpha} \uparrow & & \uparrow {\scriptstyle H\tilde{v} \cdot j_{\bar{v} \cdot \alpha}} & & \uparrow {\scriptstyle j_{\tilde{v} \cdot \bar{v} \cdot \alpha}} \\
H\tilde{v} \cdot H\bar{v} \cdot nv_2 \cdot mv_1 & \xrightarrow[H\tilde{v} \cdot \lambda(\bar{v}, v_2) \cdot mv_1]{} & H\tilde{v} \cdot n(\bar{v} \cdot v_2) \cdot mv_1 & \xrightarrow[\lambda(\tilde{v}, \bar{v} \cdot v_2) \cdot mv_1]{} & n(\tilde{v} \cdot \bar{v} \cdot v_2) \cdot mv_1
\end{array}$$

and

$$\begin{array}{ccccc}
H\tilde{v} \cdot H\bar{v} \cdot (n \cdot m)(v) & \xrightarrow{\eta(\tilde{v}, \bar{v}) \cdot (n \cdot m)(v)} & H(\tilde{v} \cdot \bar{v}) \cdot (n \cdot m)(v) & \xrightarrow{\lambda(\tilde{v} \cdot \bar{v}, v)} & (n \cdot m)(\tilde{v} \cdot \bar{v} \cdot v) \\
{\scriptstyle H\tilde{v} \cdot H\bar{v} \cdot j_\alpha} \uparrow & & \uparrow {\scriptstyle H(\tilde{v} \cdot \bar{v}) \cdot j_\alpha} & & \uparrow {\scriptstyle j_{\tilde{v} \cdot \bar{v} \cdot \alpha}} \\
H\tilde{v} \cdot H\bar{v} \cdot nv_2 \cdot mv_1 & \xrightarrow[\eta(\tilde{v}, \bar{v}) \cdot nv_2 \cdot mv_1]{} & H(\tilde{v} \cdot \bar{v}) \cdot nv_2 \cdot mv_1 & \xrightarrow[\lambda(\tilde{v} \cdot \bar{v}, v_2) \cdot mv_1]{} & n(\tilde{v} \cdot \bar{v} \cdot v_2) \cdot mv_1
\end{array}$$

That the composites of the bottom rows are equal is left associativity for n, giving left associativity for $n \cdot m$.

Right associativity follows in the same way from that of m.

Middle associativity is a bit different so we set it down here. Now, we take a lax factorization of $\bar{v}$, $\alpha : v_2 \cdot v_1 \Rightarrow \bar{v}$, and the corresponding ones for $\bar{v} \cdot v$, $\tilde{v} \cdot \bar{v}$ and $\tilde{v} \cdot \bar{v} \cdot v$, $\alpha \cdot v : v_2 \cdot (v_1 \cdot v) \Rightarrow \bar{v} \cdot v$, $\tilde{v} \cdot \alpha : (\tilde{v} \cdot v_2) \cdot v_1 \Rightarrow \tilde{v} \cdot \bar{v}$, and $\tilde{v} \cdot \alpha \cdot v : (\tilde{v} \cdot v_2) \cdot (v_1 \cdot v) \Rightarrow \tilde{v} \cdot \bar{v} \cdot v$. With this we get the following commutative diagrams.

$$\begin{array}{ccccc}
H\tilde{v} \cdot (n \cdot m)(\bar{v}) \cdot Fv & \xrightarrow{\lambda(\tilde{v}, \bar{v}) \cdot Fv} & (n \cdot m)(\tilde{v} \cdot \bar{v}) \cdot Fv & \xrightarrow{\rho(\tilde{v} \cdot \bar{v}, v)} & (n \cdot n)(\tilde{v} \cdot \bar{v} \cdot v) \\
{\scriptstyle H\tilde{v} \cdot j_\alpha Fv} \uparrow & & \uparrow {\scriptstyle j_{\tilde{v} \cdot \alpha} \cdot Fv} & & \uparrow {\scriptstyle j_{\tilde{v}\alpha \cdot v}} \\
H\tilde{v} \cdot nv_2 \cdot mv_1 \cdot Fv & \xrightarrow[\lambda(\tilde{v}, v_2) \cdot mv_1 \cdot Fv]{} & n(\tilde{v} \cdot v_2) \cdot mv_1 \cdot Fv & \xrightarrow[n(\tilde{v} \cdot v_2) \cdot \rho(v_1, v)]{} & n(\tilde{v} \cdot v_2) \cdot m(v_1 \cdot v)
\end{array}$$

and

$$\begin{array}{ccccc}
H\tilde{v} \cdot (n \cdot m)(\bar{v}) \cdot Fv & \xrightarrow{H\tilde{v} \cdot \rho(\bar{v}, v)} & H\tilde{v} \cdot (n \cdot m)(\bar{v} \cdot v) & \xrightarrow{\lambda(\tilde{v}, \bar{v}, v)} & (n \cdot m)(\tilde{v} \cdot \bar{v} \cdot v) \\
{\scriptstyle H\tilde{v} \cdot j_\alpha \cdot Fv} \uparrow & & \uparrow {\scriptstyle H\tilde{v} \cdot j_{\alpha \cdot v}} & & \uparrow {\scriptstyle j_{\tilde{v} \cdot \alpha \cdot v}} \\
H\tilde{v} \cdot nv_2 \cdot mv_1 \cdot Fv & \xrightarrow[H\tilde{v} \cdot nv_2 \cdot \rho(v_1, v)]{} & H\tilde{v} \cdot nv_2 \cdot m(v_1 \cdot v) & \xrightarrow[\lambda(\tilde{v}, v_2) \cdot m(v_1 \cdot v)]{} & n(\tilde{v} \cdot v_2) \cdot m(v_1 \cdot v)
\end{array}$$

The bottom rows are equal simply by functoriality of vertical composition, so the top rows will be equal as well, which is middle associativity.

■

Thus we have proved that $n \cdot m$ is a module. It remains to see that it has the strong representability property, which we do in the next two sections.

4.5. The Canonical Bimodulation In this section we define the bimodulation

$$\begin{array}{ccc} F & = & F \\ {\scriptstyle m}\downarrow & & \\ G & \iota & \downarrow{\scriptstyle n\cdot m} \\ {\scriptstyle n}\downarrow & & \\ H & = & H \end{array}$$

Given two vertical arrows $A_0 \xrightarrow{v_1} A_1 \xrightarrow{v_2} A_2$, $v_2 \cdot v_1$ has a canonical lax factorization into v_1 and v_2

$$1_{v_2 \cdot v_1} : v_2 \cdot v_1 \Rightarrow v_2 \cdot v_1.$$

Define $\iota(v_2, v_1) = j_{1_{v_2 \cdot v_1}} : n(v_2) \cdot m(v_1) \longrightarrow (n \cdot m)(v_2 \cdot v_1)$.

4.5.1. Proposition. *ι is a bimodulation.*

Proof. First, ι satisfies (mm2), "horizontal naturality". Let

$$\begin{array}{ccc} A_0 & \longrightarrow & B_0 \\ {\scriptstyle v_1}\downarrow & \alpha_1 & \downarrow{\scriptstyle w_1} \\ A_1 & \longrightarrow & B_1 \\ {\scriptstyle v_2}\downarrow & \alpha_2 & \downarrow{\scriptstyle w_2} \\ A_2 & \longrightarrow & B_2 \end{array}$$

be cells. The composite $(\alpha_2 \cdot \alpha_1)1_{v_2 \cdot v_1}$ has the factorization $1_{w_2 \cdot w_1}(\alpha_2 \cdot \alpha_1)$

$$\begin{array}{ccccc} A_0 & = & A_0 & \longrightarrow & B_0 \\ {\scriptstyle v_1}\downarrow & & & & \\ A_1 & & \downarrow{\scriptstyle v_2\cdot v_1} & \alpha_2\cdot\alpha_1 & \downarrow{\scriptstyle w_2\cdot w_1} \\ {\scriptstyle v_2}\downarrow & & & & \\ A_2 & = & A_2 & \longrightarrow & B_2 \end{array} \quad = \quad \begin{array}{ccccc} A_0 & \longrightarrow & B_0 & = & B_0 \\ {\scriptstyle v_1}\downarrow & \alpha_1 & \downarrow{\scriptstyle w_1} & & \\ A_1 & \longrightarrow & B_1 & 1 & \downarrow{\scriptstyle w_2\cdot w_1} \\ {\scriptstyle v_2}\downarrow & \alpha_2 & \downarrow{\scriptstyle w_2} & & \\ A_2 & \longrightarrow & B_2 & = & B_2 \end{array}$$

so by proposition 3.2.1, we have that

$$\begin{array}{ccc} (n \cdot m)(v_2 \cdot v_1) & \xrightarrow{(n\cdot m)(\alpha_2\cdot\alpha_1)} & (n \cdot m)(w_2 \cdot w_1) \\ {\scriptstyle j_{1_{v_2\cdot v_1}}}\uparrow & & \uparrow{\scriptstyle j_{1_{w_2\cdot w_1}}} \\ nv_2 \cdot mv_1 & \xrightarrow[n\alpha_2\cdot m\alpha_1]{} & nw_2 \cdot mw_1 \end{array}$$

commutes, which is naturality of $n \cdot m$.

Next we check equivariance, (mm3). This says that for any three composable vertical arrows v_1, v_2, v_3 we must have commutativity of

$$\begin{array}{ccc} nv_3 \cdot Gv_2 \cdot mv_1 & \xrightarrow{nv_3 \cdot \lambda(v_2,v_1)} & nv_3 \cdot m(v_2 \cdot v_1) \\ {\scriptstyle \rho(v_3,v_2)\cdot mv_1}\downarrow & & \downarrow{\scriptstyle j_1} \\ n(v_3 \cdot v_2) \cdot mv_1 & \xrightarrow[j_1]{} & (n \cdot m)(v_3 \cdot v_2 \cdot v_1) \end{array}$$

This follows from proposition 2.2.1, part (a), because j is a cocone.

Left equivariance (mm3_l) says that

$$\begin{array}{ccc} Hv_3 \cdot nv_2 \cdot mv_1 & \xrightarrow{\lambda(v_3,v_3)\cdot mv_1} & n(v_3 \cdot v_2) \cdot mv_1 \\ {\scriptstyle Hv_3 \cdot j_{1_{v_2 \cdot v_1}}}\downarrow & & \downarrow{\scriptstyle j_{1_{v_3 \cdot v_2 \cdot v_1}}} \\ Hv_3 \cdot (n \cdot m)(v_2 \cdot v_1) & \xrightarrow[\lambda(v_3,v_2\cdot v_1)]{} & (n \cdot m)(v_3 \cdot v_2 \cdot v_1) \end{array}$$

commutes, which it does by the definition of the left action for $n \cdot m$ (proposition 2.3.1).

Right equivariance, (mm3_r), is similar.

■

4.6. The Universal Property

4.6.1. Proposition. *The bimodulation $\iota : n, m \longrightarrow n \cdot m$ satisfies the strong representability property.*

Proof. Let

$$\mathbf{p} = (F_k \xrightarrow{p_k} F_{k-1} \xrightarrow{p_{k-1}} \dots \xrightarrow{p_1} F_0 = F)$$
$$\mathbf{q} = (H = H_0 \xrightarrow{q_1} H_1 \xrightarrow{q_2} \dots \xrightarrow{q_l} H_l)$$

be paths of modules and let

$$\mu : \mathbf{q}, n, m, \mathbf{p} \longrightarrow r$$

be a multimodulation. We will show that there is a unique multimodulation $\bar{\mu}$ such that

$$\begin{array}{ccc} \mathbf{q}, n, m, \mathbf{p} & \xrightarrow{\mathbf{q},\iota,\mathbf{p}} & \mathbf{q}, n \cdot m, \mathbf{p} \\ & {\scriptstyle \mu}\searrow \quad \swarrow{\scriptstyle \bar{\mu}} & \\ & r & \end{array}$$

commutes. Thus for vertical arrows

$$\mathbf{x} = (A_k \xrightarrow{x_k} A_{k-1} \xrightarrow{x_{k-1}} \dots \xrightarrow{x_1} A_0 = A)$$

$$\mathbf{w} = (C = C_0 \xrightarrow{w_1} C_1 \xrightarrow{w_2} \dots \xrightarrow{w_l} C_l)$$

$$A \xrightarrow{v_1} B \xrightarrow{v_2} C$$

we must have commutativity of

$$\begin{array}{ccc} [\mathbf{qw}] \cdot nv_2 \cdot mv_1 \cdot [\mathbf{px}] & \xrightarrow{[\mathbf{qw}]\cdot j_{1_{v_2 \cdot v_1}} \cdot [\mathbf{px}]} & [\mathbf{qw}] \cdot (n \cdot m)(v_2 \cdot v_1) \cdot [\mathbf{px}] \\ & (1) & \\ {\scriptstyle \mu(\mathbf{w}, v_2, v_1, \mathbf{x})} \searrow & & \swarrow {\scriptstyle \bar{\mu}(\mathbf{w}, v_2 \cdot v_1, \mathbf{x})} \\ & r[\mathbf{w}, v_2, v_1, \mathbf{x}] & \end{array}$$

Let $v : A \to C$ be an arbitrary vertical arrow and

$$\begin{array}{ccc} A & = & A \\ {\scriptstyle v_1} \downarrow & & \\ B & \alpha & \downarrow {\scriptstyle v} \\ {\scriptstyle v_2} \downarrow & & \\ C & = & C \end{array}$$

a lax factorization of v. Then for $\mathbf{x}$ and $\mathbf{w}$ as above, naturality of $\bar{\mu}$ would give the commutative square

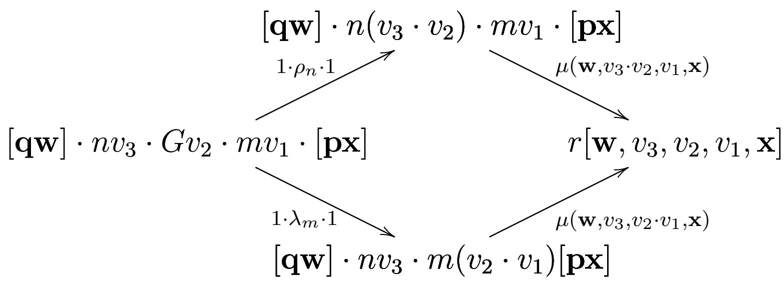
$$\begin{array}{ccc} [\mathbf{qw}] \cdot (n \cdot m)(v_2 \cdot v_1) \cdot [\mathbf{px}] & \xrightarrow{[\mathbf{qw}]\cdot (n \cdot m)\alpha \cdot [\mathbf{px}]} & [\mathbf{qw}] \cdot (n \cdot m)v \cdot [\mathbf{px}] \\ {\scriptstyle \bar{\mu}(\mathbf{w}, v_2 \cdot v_1, \mathbf{x})} \downarrow & & \downarrow {\scriptstyle \bar{\mu}(\mathbf{w}, v, \mathbf{x})} \\ r[\mathbf{w}, v_2, v_1, \mathbf{x}] & \xrightarrow[r[\mathbf{w}, \alpha, \mathbf{x}]]{} & r[\mathbf{w}, v, \mathbf{x}] \end{array}$$

which, if pasted onto the right side of (1), gives the commuting

$$\begin{array}{ccc} [\mathbf{qw}] \cdot nv_2 \cdot mv_1 \cdot [\mathbf{px}] & \xrightarrow{[\mathbf{qw}]\cdot j_\alpha \cdot [\mathbf{px}]} & [\mathbf{qw}] \cdot (n \cdot m)v \cdot [\mathbf{px}] \\ {\scriptstyle \mu(\mathbf{w}, v_2, v_1, \mathbf{x})} \downarrow & (2) & \downarrow {\scriptstyle \bar{\mu}(\mathbf{w}, v, \mathbf{x})} \\ r[\mathbf{w}, v_2, v_1, \mathbf{x}] & \xrightarrow[r[\mathbf{w}, \alpha, \mathbf{x}]]{} & r[\mathbf{w}, v, \mathbf{x}] \end{array}$$

because $((n \cdot m)\alpha)(j_1) = j_\alpha$. Now the family $\langle [\mathbf{qw}] \cdot j_\alpha \cdot [\mathbf{px}] \rangle_\alpha$ is jointly epic so, if $\bar{\mu}$ exists, it is unique.

The next step is to show $r[\mathbf{w}, \alpha, \mathbf{x}]\mu(\mathbf{w}, v_2, v_1, \mathbf{x})$ determines a cocone as in proposition 2.2.1, which will show that $\bar{\mu}(\mathbf{w}, v, \mathbf{x})$ does exist. Condition (a) says that for any triple lax factorization $\gamma : v_3 \cdot v_2 \cdot v_1 \longrightarrow v$ the diagram

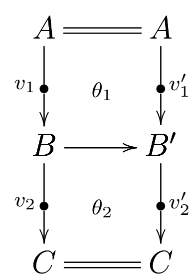


must commute when followed by

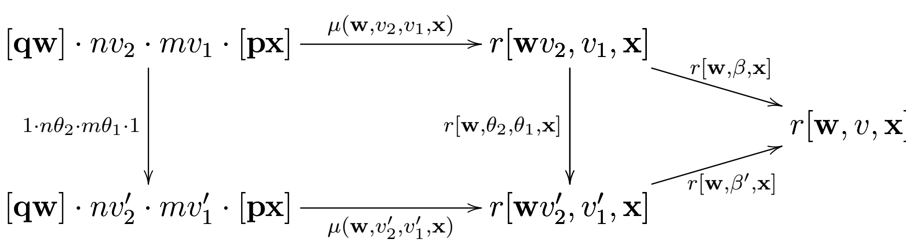
$$r[\mathbf{w}, \gamma, \mathbf{x}] : r[\mathbf{w}, v_3, v_2, v_1, \mathbf{x}] \longrightarrow r[\mathbf{w}, v, \mathbf{x}].$$

But it commutes even before that by associativity of μ.

For condition (b) consider two lax factorizations of v, $\beta : v_2 \cdot v_1 \Rightarrow v$, $\beta' : v_2' \cdot v_1' \Rightarrow v$ related by $\beta'(\theta_2 \cdot \theta_1) = \beta$ for



We have to show that



commutes, which it does, the square by naturality of μ and the triangle by functoriality of r. This shows that the $\bar{\mu}(\mathbf{w}, v, \mathbf{x})$ exist (and are unique) satisfying (2).

It remains to show that this definition makes $\bar{\mu}$ into a multimodulation. There are a number of things to check but the method of proof is the same for all of them. The condition is expressed as the commutativity of a diagram involving $\bar{\mu}$ which we reduce to a similar diagram for μ by precomposing with an arbitrary $1 \cdot j_\alpha \cdot 1$ (for suitable identities).

To check naturality of $\bar{\mu}$ we take a path of cells

$$
\begin{array}{ccc}
A_l & \longrightarrow & A'_l \\
{\scriptstyle \mathbf{x}}\big\downarrow\!\bullet & \xi & \bullet\!\big\downarrow{\scriptstyle \mathbf{x}'} \\
A & \longrightarrow & A' \\
{\scriptstyle v}\big\downarrow\!\bullet & \sigma & \bullet\!\big\downarrow{\scriptstyle v'} \\
C & \longrightarrow & C' \\
{\scriptstyle \mathbf{w}}\big\downarrow\!\bullet & \omega & \bullet\!\big\downarrow{\scriptstyle \mathbf{w}'} \\
C_k & \longrightarrow & C_k
\end{array}
$$

and must show that

$$
\begin{array}{ccc}
[\mathbf{qw}] \cdot (n \cdot m)v \cdot [\mathbf{px}] & \xrightarrow{\bar{\mu}(\mathbf{w},v,\mathbf{x})} & r[\mathbf{w}, v, \mathbf{x}] \\
{\scriptstyle [\mathbf{q}\omega]\cdot(n\cdot m)\sigma\cdot[\mathbf{p}\xi]}\Big\downarrow & (*) & \Big\downarrow{\scriptstyle r[\omega,\sigma,\xi]} \\
[\mathbf{qw}'] \cdot (n \cdot m)v' \cdot [\mathbf{px}'] & \xrightarrow[\bar{\mu}(\mathbf{w}',v',\mathbf{x}')]{} & r[\mathbf{w}', v', \mathbf{x}']
\end{array}
$$

commutes. For this it will be sufficient to show that it does when preceded by $[\mathbf{qw}]\cdot j_\alpha \cdot [\mathbf{px}]$ for $\alpha : v_2 \cdot v_1 \Rightarrow v$, an arbitrary lax factorization of v. Factor $\sigma\alpha$ as

$$
\begin{array}{ccccc}
A & = & A & \longrightarrow & A' \\
{\scriptstyle v_1}\big\downarrow & & \Big| & & \Big| \\
B & \alpha & {\scriptstyle v}\Big\downarrow & \sigma & \Big\downarrow{\scriptstyle v'} \\
{\scriptstyle v_2}\big\downarrow & & \Big| & & \Big| \\
C & = & C & \longrightarrow & C'
\end{array}
\quad = \quad
\begin{array}{ccccc}
A & \longrightarrow & A' & = & A' \\
{\scriptstyle v_1}\big\downarrow & \beta_1 & \big\downarrow{\scriptstyle v'_1} & & \Big| \\
B & \longrightarrow & B' & \beta & \Big\downarrow{\scriptstyle v'} \\
{\scriptstyle v_1}\big\downarrow & \beta_2 & \big\downarrow{\scriptstyle v'_2} & & \Big| \\
C & \longrightarrow & C' & = & C'
\end{array}
$$

Then by definition of $(n \cdot m)\sigma$ and functoriality of vertical composition, we get the commutative square

$$
\begin{array}{ccc}
[\mathbf{qw}] \cdot nv_2 \cdot mv_1 \cdot [\mathbf{px}] & \xrightarrow{1\cdot j_\alpha \cdot 1} & [\mathbf{qw}] \cdot (n \cdot m)v \cdot [\mathbf{px}] \\
{\scriptstyle [\mathbf{q}\omega]\cdot n\beta_2\cdot m\beta_1[\mathbf{p}\xi]}\Big\downarrow & & \Big\downarrow{\scriptstyle [\mathbf{q}\omega]\cdot(n\cdot m)\sigma\cdot[\mathbf{p}\xi]} \\
[\mathbf{qw}'] \cdot nv'_2 \cdot mv'_1 \cdot [\mathbf{px}'] & \xrightarrow[1\cdot j_\beta \cdot 1]{} & [\mathbf{qw}'] \cdot (n \cdot m)v' \cdot [\mathbf{px}']
\end{array}
$$

If we paste this onto $(*)$ and use the definition of $\bar{\mu}$ we get

$$\begin{array}{ccccc}
[\mathbf{qw}]\cdot nv_2\cdot mv_1\cdot[\mathbf{px}] & \xrightarrow{\mu(\mathbf{w},v_2,v_1,\mathbf{x})} & r[\mathbf{w},v_2,v_1,\mathbf{x}] & \xrightarrow{r[\mathbf{w},\alpha,\mathbf{x}]} & r[\mathbf{w},v,\mathbf{x}] \\
{\scriptstyle [\mathbf{q}\omega]\cdot n\beta_2\cdot m\beta_1\cdot[\mathbf{p}\xi]}\downarrow & & \downarrow{\scriptstyle r[\omega,\beta_2,\beta_1,\xi]} & & \downarrow{\scriptstyle r[\omega,\sigma,\xi]} \\
[\mathbf{qw}']\cdot nv_2'\cdot mv_1'\cdot[\mathbf{px}'] & \xrightarrow[\mu(\mathbf{w}',v_2',v_1',\mathbf{x}')]{} & r[\mathbf{w}',v_2',v_1',\mathbf{x}] & \xrightarrow[r[\mathbf{w}',\beta,\mathbf{x}']]{} & r[\mathbf{w}',v',\mathbf{x}']
\end{array}$$

which commutes by naturality of μ and functoriality of r. This establishes naturality of $\bar{\mu}$.

Checking the equivariance of $\bar{\mu}$ involves evaluating one of the F's or one of the H's at a vertical arrow. There are eight cases to consider:
(1) F_k, $k>0$ (right equivariance),
(2) F_i, $0<i<k$ (doesn't involve $n\cdot m$),
(3) F_0, $k>0$ (involves $n\cdot m$),
(4) F_0, $k=0$ (involves $n\cdot m$),
and four dual ones for the H's.

For (1), let $\mathbf{x}$, $\mathbf{w}$, α be as before, and $x': A_k' \longrightarrow\!\!\!\!\!\!\bullet\;\; A_k$. We must show that

$$\begin{array}{ccc}
[\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{px}]\cdot F_k x' & \xrightarrow{\bar{\mu}(\mathbf{w},v,\mathbf{x})\cdot F_k x'} & r[\mathbf{w},v,\mathbf{x}]\cdot F_k x' \\
{\scriptstyle 1\cdot\rho_{p_k}}\downarrow & & \downarrow{\scriptstyle \rho_r} \\
[\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{p}(\mathbf{x}\cdot x')] & \xrightarrow[\bar{\mu}(\mathbf{w},v,\mathbf{x}\cdot x')]{} & r[\mathbf{w},v,\mathbf{x}\cdot x']
\end{array}$$

commutes. Paste onto this the commutative diagram

$$\begin{array}{ccc}
[\mathbf{qw}]\cdot nv_2\cdot mv_1\cdot[\mathbf{px}]\cdot F_k x' & \xrightarrow{1\cdot j_\alpha\cdot 1} & [\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{px}]\cdot F_k x' \\
{\scriptstyle 1\cdot\rho_{p_k}}\downarrow & & \downarrow{\scriptstyle 1\cdot\rho_{p_k}} \\
[\mathbf{qw}]\cdot nv_2\cdot mv_1'[\mathbf{p}(\mathbf{x}\cdot x')] & \xrightarrow[1\cdot j_\alpha\cdot 1]{} & [\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{p}(\mathbf{x}\cdot x')]
\end{array}$$

and use the definition of $\bar{\mu}$ to get

$$\begin{array}{ccccc}
[\mathbf{qw}]\cdot nv_2\cdot mv_1\cdot[\mathbf{px}]\cdot F_k x' & \xrightarrow{\mu(\mathbf{w},v_2,v_1,\mathbf{x})\cdot F_k x'} & r[\mathbf{w},v_2,v_1,\mathbf{x}]\cdot F_k x' & \xrightarrow{r[\mathbf{w},\alpha,\mathbf{x}]\cdot F_k x'} & r[\mathbf{w},v,\mathbf{x}]\cdot F_k x' \\
{\scriptstyle 1\cdot\rho_{p_k}}\downarrow & & \downarrow{\scriptstyle \rho_r} & & \downarrow{\scriptstyle \rho_r} \\
[\mathbf{qw}]\cdot nv_2\cdot mv_1\cdot[\mathbf{p}(\mathbf{x}\cdot x')] & \xrightarrow[\mu(\mathbf{w},v_2,v_1,\mathbf{x}\cdot x')]{} & r[\mathbf{w},v_2,v_1,\mathbf{x}\cdot x'] & \xrightarrow[r[\mathbf{w},\alpha,\mathbf{x}\cdot x']]{} & r[\mathbf{w},v,\mathbf{x}\cdot x']
\end{array}$$

which commutes, the left square by right equivariance of μ and the right square by naturality of ρ_r.

To prove equivariance for case (2) we replace the path $\mathbf{x}$ by a path $(\mathbf{x}, y, \mathbf{z})$ where $\mathbf{x}$ and $\mathbf{z}$ are paths of lengths i and $k-i$ and y a single arrow. We must show that the following diagram commutes

$$\begin{array}{ccccc}
 & & [\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{px}]\cdot F_i y\cdot[\mathbf{pz}] & & \\
 & {\scriptstyle 1\cdot\lambda_{p_{i+1}}\cdot 1}\swarrow & & \searrow{\scriptstyle 1\cdot\rho_{p_i}\cdot 1} & \\
[\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{px}]\cdot[\mathbf{p}(y\cdot\mathbf{z})] & & & & [\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{p}(\mathbf{x}\cdot y)]\cdot[\mathbf{pz}] \\
 & {\scriptstyle \bar{\mu}(\mathbf{w},v,\mathbf{x},y\cdot\mathbf{z})}\searrow & & \swarrow{\scriptstyle \bar{\mu}(\mathbf{w},v,\mathbf{x}\cdot y,\mathbf{z})} & \\
 & & r[\mathbf{w}, v, \mathbf{x}, y, \mathbf{z}] & &
\end{array}$$

If we precede this by $[\mathbf{qw}]\cdot j_\alpha\cdot[\mathbf{px}]\cdot F_i y\cdot[\mathbf{pz}]$ we get the corresponding diagram for μ followed by $r[\mathbf{w}, \alpha, \mathbf{x}, y, \mathbf{z}]$. This uses the definition of $\bar{\mu}$ and the fact that j_α and ρ_{p_i} (resp. $\lambda_{p_{i+1}}$) act on different factors and so can be permuted.

For case (3) we must show that

$$\begin{array}{ccc}
[\mathbf{qw}]\cdot(n\cdot m)v\cdot Fx'\cdot[\mathbf{px}] & \xrightarrow{1\cdot\rho_{n\cdot m}\cdot 1} & [\mathbf{qw}]\cdot(n\cdot m)(v\cdot x')\cdot Fx'\cdot[\mathbf{px}] \\
{\scriptstyle 1\cdot\lambda_{p_1}\cdot 1}\downarrow & & \downarrow{\scriptstyle \bar{\mu}(\mathbf{w},v\cdot x',\mathbf{x})} \\
[\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{p}(x'\cdot\mathbf{x})] & \xrightarrow[\bar{\mu}(\mathbf{w},v,x'\cdot\mathbf{x})]{} & r[\mathbf{w}, v, x', \mathbf{x}]
\end{array}$$

commutes. Now we must use the definition of $\rho_{n\cdot m}$ (proposition 2.3.1), which gives the commutative

$$\begin{array}{ccc}
[\mathbf{qw}]\cdot nv_2\cdot mv_1\cdot Fx'\cdot[\mathbf{px}] & \xrightarrow{1\cdot\rho_m\cdot 1} & [\mathbf{qw}]\cdot nv_2\cdot m(v_1\cdot x')\cdot[\mathbf{px}] \\
{\scriptstyle 1\cdot j_\alpha\cdot 1}\downarrow & & \downarrow{\scriptstyle 1\cdot j_{\alpha\cdot x'}\cdot 1} \\
[\mathbf{qw}]\cdot(n\cdot m)v\cdot Fx'\cdot[\mathbf{px}] & \xrightarrow[1\cdot\rho_{n\cdot m}\cdot 1]{} & [\mathbf{qw}]\cdot(n\cdot m)(v\cdot x')\cdot[\mathbf{px}]
\end{array}$$

that we paste on top of the diagram we want to commute. We use the definition of $\bar{\mu}$ and the fact that $1\cdot j_\alpha\cdot 1$ and $1\cdot\lambda_{p_1}\cdot 1$ act on different factors to permute them, to get

$$\begin{array}{ccc}
[\mathbf{qw}]\cdot nv_2\cdot mv_1\cdot Fx'\cdot[\mathbf{px}] & \xrightarrow{1\cdot\rho_n\cdot 1} & [\mathbf{qw}]\cdot nv_2\cdot m(v_1\cdot x')\cdot[\mathbf{px}] \\
{\scriptstyle 1\cdot\lambda_{p_1}\cdot 1}\downarrow & & \downarrow{\scriptstyle \mu(\mathbf{w},v_2,v_1\cdot x',\mathbf{x})} \\
[\mathbf{qw}]\cdot nv_2\cdot mv_1\cdot[\mathbf{p}(x'\cdot\mathbf{x})] & \xrightarrow[\mu(\mathbf{w},v_2,v_1,x'\cdot\mathbf{x})]{} & r[\mathbf{w}, v_2, v_1, x', \mathbf{x}] \\
{\scriptstyle 1\cdot j_\alpha 1}\downarrow & & \downarrow{\scriptstyle r[\mathbf{w},\alpha,x',\mathbf{x}]} \\
[\mathbf{qw}]\cdot(n\cdot m)v\cdot[\mathbf{p}(x'\cdot\mathbf{x})] & \xrightarrow[\bar{\mu}(\mathbf{w},v,x'\cdot\mathbf{x})]{} & r[\mathbf{w}, v, x', \mathbf{x}]
\end{array}$$

The top square commutes by equivariance of μ and the bottom by definition of $\bar{\mu}$.

Case (4) is somewhere between cases (1) and (3) and so is its proof. We must establish the commutativity of

$$\begin{array}{ccc} [\mathbf{qw}]\cdot(n\cdot m)v\cdot F_0x & \xrightarrow{\bar{\mu}(\mathbf{w},v)\cdot F_0x} & r[\mathbf{w},v]\cdot F_0x \\ {\scriptstyle 1\cdot\rho_{n\cdot m}}\big\downarrow & & \big\downarrow{\scriptstyle \rho_r} \\ [\mathbf{qw}]\cdot(n\cdot m)(v\cdot x) & \xrightarrow[\bar{\mu}(\mathbf{w},v\cdot x)]{} & r[\mathbf{w},v\cdot x] \end{array}$$

Paste onto this the commutative diagram (proposition 2.3.1)

$$\begin{array}{ccc} [\mathbf{qw}]\cdot nv_1\cdot mv_2\cdot F_0x & \xrightarrow{1\cdot j_\alpha 1} & [\mathbf{qw}]\cdot(n\cdot m)v\cdot F_0x \\ {\scriptstyle 1\cdot\rho_m}\big\downarrow & & \big\downarrow{\scriptstyle 1\cdot\rho_{n\cdot m}} \\ [\mathbf{qw}]\cdot nv_1\cdot m(v_2\cdot x) & \xrightarrow[1\cdot j_{\alpha\cdot x}]{} & [\mathbf{qw}]\cdot(n\cdot m)(v\cdot x) \end{array}$$

and use the definition of $\bar{\mu}$ to get

$$\begin{array}{ccccc} [\mathbf{qw}]\cdot nv_1\cdot mv_2\cdot F_0x & \xrightarrow{\mu(\mathbf{w},v_1,v_2)\cdot F_0x} & r[\mathbf{w},v_1,v_2]\cdot F_0x & \xrightarrow{r[\mathbf{w},\alpha]\cdot F_0x} & r[\mathbf{w},v]\cdot F_0x \\ {\scriptstyle 1\cdot\rho_m}\big\downarrow & & \big\downarrow{\scriptstyle \rho_r} & & \big\downarrow{\scriptstyle \rho_r} \\ [\mathbf{qw}]\cdot nv_1\cdot m(v_2\cdot x) & \xrightarrow[\mu(\mathbf{w},v_1,v_2\cdot x)]{} & r[\mathbf{w},v_1,v_2\cdot x] & \xrightarrow[r[w,\alpha\cdot x]]{} & r[\mathbf{w},v\cdot x] \end{array}$$

which commutes, the left square by right equivariance of μ and the right square by naturality of ρ_r.

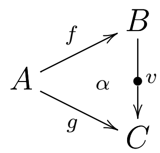
The other four cases are dual.

■

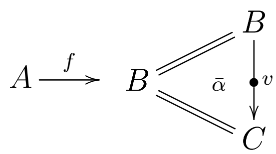
5. Complements

5.1. Identities In this section we show that for arbitrary $\mathbb{A}$ and $\mathbb{X}$, identities are strongly representable in $\mathbb{L}\mathrm{ax}(\mathbb{A},\mathbb{X})$. This justifies our claim in [15], theorem 4.3. It is also necessary in order to see that, under the hypotheses of theorem 4.0.1, $\mathbb{L}\mathrm{ax}(\mathbb{A},\mathbb{X})$ is a double category.

We have to check this separately because, for one thing, nullary modulations are not exactly the specialization of multimodulations to the nullary case, although a more global definition of multimodulation than the one given above would indeed show it in that light. Another reason is that, as Peter Lumsdaine has pointed out, our AFP is a binary condition and those usually come with a corresponding nullary one. But if we generalize our AFP to n-ary factorizations, which is easily enough done, and then specialize to $n = 0$, we get a ridiculously strong condition, which says (in part) that every cell



factors as



which implies that $B = C$ and $g = f$.

We recall the definition of the module we called id_F (in corollary 3.19 of [15]) and Id_G (in theorem 4.3 of [15]).

5.1.1. Definition. For any lax functor $F : \mathbb{A} \longrightarrow \mathbb{X}$ there is a canonical module $\mathrm{id}_F : F \longrightarrow\!\!\!\!\!\bullet\;\, F$ defined by the formulas

$$\mathrm{id}_F(v) = F(v) : FA \longrightarrow\!\!\!\!\!\bullet\;\, F\bar{A}$$

$$\mathrm{id}_F(\sigma) = F(\sigma) : Fv \longrightarrow Fw$$

$$\lambda_{\mathrm{id}_F}(\bar{v}, v) = \phi_F(\bar{v}, v) : F\bar{v} \cdot Fv \longrightarrow F(\bar{v} \cdot v)$$

$$\rho_{\mathrm{id}_F}(\bar{v}, v) = \phi_F(\bar{v}, v) : F\bar{v} \cdot Fv \longrightarrow F(\bar{v} \cdot v).$$

5.1.2. Proposition. *The above definitions make* id_F *into a module* $F \longrightarrow\!\!\!\!\!\bullet\;\, F$.

Proof. Condition (M3) follows immediately from (LF3), (M4) from (LF4), (M5) and (M6) from (LF5).

■

5.1.3. Definition. If $t : F \longrightarrow G$ is a natural transformation between the lax functors F and G, there is a canonical modulation $\mathrm{id}_t : \mathrm{id}_F \longrightarrow \mathrm{id}_G$ defined by

$$\mathrm{id}_t(v) = t(v) : Fv \longrightarrow Gv.$$

5.1.4. Proposition. *The above definition makes* id_t *into a modulation.*

Proof. Condition (m2) follows immediately from (NT2) and condition (m3) from (NT3).

■

We also recall the definition of null modulation. Conditions (nm1) and (nm2) below are just a specialization of (mm1) and (mm2) from section 1 above, but (mm3), (mm3_l) and (mm3_r) don't make sense when $n = 0$ and must be replaced by (nm3).

5.1.5. Definition. Let F, G_0, G_1 be lax functors $\mathbb{A} \longrightarrow \mathbb{X}$ and $p : G_0 \longrightarrow\!\!\!\!\!\bullet\;\; G_1$ a module, $t : F \longrightarrow G$, $s : F \longrightarrow H$ natural transformations. A *null modulation*

$$\mu : F \longrightarrow p$$

$$\begin{array}{ccc} & & G \\ & \overset{t}{\nearrow} & \\ F & \mu & \Big\downarrow \bullet\, p \\ & \underset{s}{\searrow} & \\ & & H \end{array}$$

consists of the following data:

(nm1) For every object A, a cell

$$\mu A : \text{id}_{FA} \longrightarrow p(\text{id}_A).$$

These must satisfy

(nm2) (Horizontal naturality) For any horizontal arrow $f : A \longrightarrow A'$,

$$\begin{array}{ccc} \text{id}_{FA} & \xrightarrow{\text{id}_{Ff}} & \text{id}_{FA'} \\ {\scriptstyle \mu A}\Big\downarrow & & \Big\downarrow{\scriptstyle \mu A'} \\ p(\text{id}_A) & \xrightarrow[p(\text{id}_f)]{} & p(\text{id}_{A'}) \end{array}$$

commutes.

(nm3) (Equivariance) For every vertical arrow $v : A \longrightarrow\!\!\!\!\!\bullet\;\; \bar{A}$,

$$\begin{array}{ccccc} Fv \cdot \text{id}_{FA} & \xrightarrow{\cong} & \text{id}_{F\bar{A}} \cdot Fv & \xrightarrow{\mu\bar{A}\cdot tv} & p(\text{id}_{\bar{A}}) \cdot Gv \\ {\scriptstyle sv\cdot\mu A}\Big\downarrow & & & & \Big\downarrow{\scriptstyle \rho_p} \\ Hv \cdot p(\text{id}_A) & \xrightarrow[\lambda_p]{} & p(v \cdot \text{id}_A) & \xrightarrow[\cong]{} & p(\text{id}_{\bar{A}} \cdot v) \end{array}$$

commutes.

We can now define the null modulation ι which will turn out to be the universal one.

5.1.6. Definition. For any lax functor F we let

$$\iota A = \phi A : \text{id}_{FA} \longrightarrow F(\text{id}_A).$$

5.1.7. Proposition. *ι is a null modulation from F into* id_F.

Proof. Conditions (nm2) and (nm3) follow immediately from (LF4) and (LF5) respectively.

■

There are two bimodulations associated with an arbitrary module $m : F \longrightarrow\!\!\!\!\!\bullet\ G$ which will be useful in the proof below.

5.1.8. Definition. For a module $m : F \longrightarrow\!\!\!\!\!\bullet\ G$ we have bimodulations

$$\lambda : \mathrm{id}_G, m \longrightarrow m$$

$$\rho : m, \mathrm{id}_F \longrightarrow m$$

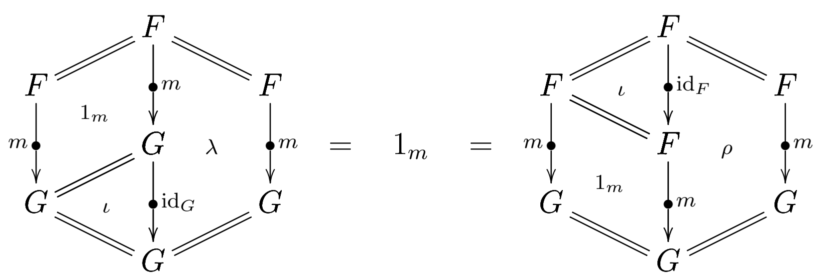
defined by

$$\lambda(v_2, v_1) = \lambda_m(v_2, v_1) : Gv_2 \cdot mv_1 \longrightarrow m(v_2 \cdot v_1)$$

$$\rho(v_2, v_1) = \rho_m(v_2, v_1) : mv_2 \cdot Fv_1 \longrightarrow m(v_2 \cdot v_1)$$

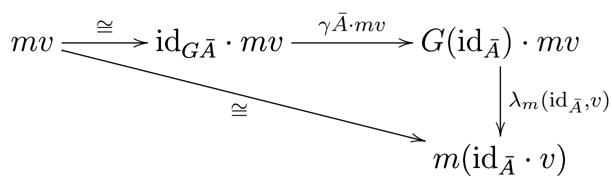
i.e. λ and ρ are the left and right actions for m.

5.1.9. Proposition. *λ and ρ are bimodulations and satisfy*



Proof. The conditions for λ and ρ to be bimodulations correspond to those for m to be a module. For λ, e.g., (mm2) is (M4), (mm3) and (mm3$_l$) are the first condition of (M6), and (mm3$_r$) is the second condition of (M6).

Evaluating $\lambda(\iota, 1_m)$ at $v : A \longrightarrow\!\!\!\!\!\bullet\ \bar{A}$ gives the commutative



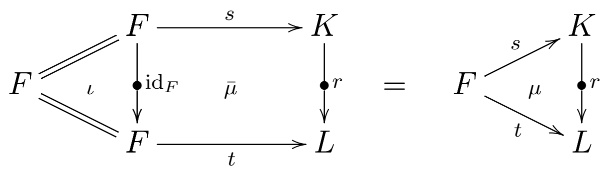
by (M5), and this is the first equation above.

The argument for ρ is dual.

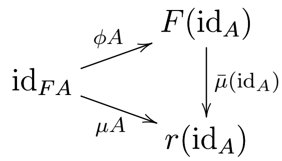
■

5.1.10. Theorem. *Identities in* $\mathbb{L}\mathrm{ax}(\mathbb{A}, \mathbb{X})$ *are strongly represented by* id_F *with universal null modulation* ι.

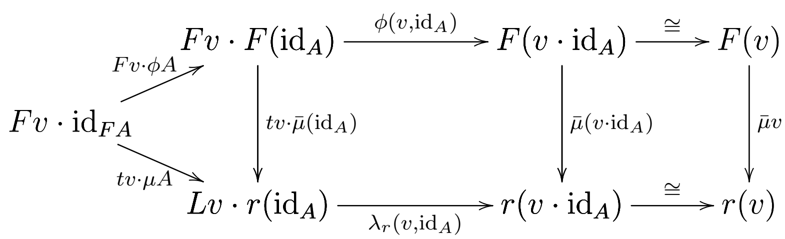
Proof. We first prove weak representability. Given a null modulation $\mu : F \longrightarrow r$ we want to show that there is a unique modulation $\bar{\mu} : \mathrm{id}_F \longrightarrow r$ such that



i.e. for every A



commutes. If $\bar{\mu}$ exists, its left associativity gives commutativity of the middle part of the diagram

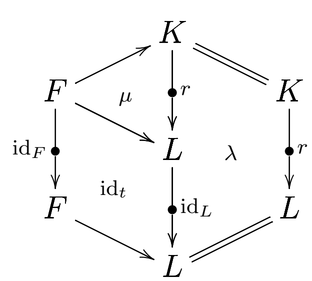


The right square commutes by naturality of $\bar{\mu}$ and the triangle by functoriality of vertical composition and the defining property of $\bar{\mu}$ above. The right unit law for F says that the top composite is a canonical isomorphism, so $\bar{\mu}v$ is uniquely determined.

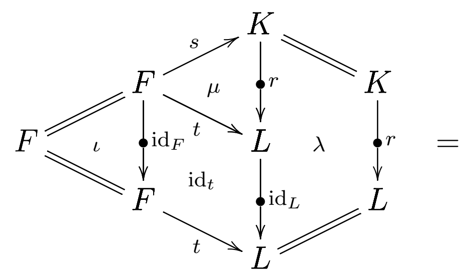
In fact $\bar{\mu}v$ is the composite

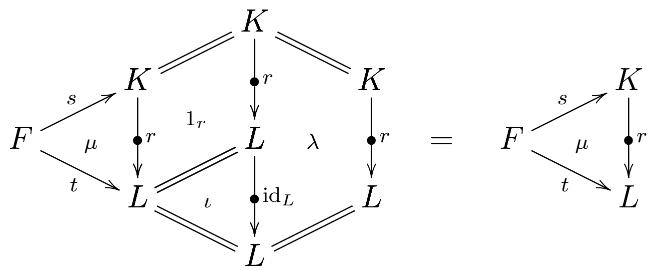
$$Fv \xrightarrow{\cong} Fv \cdot \mathrm{id}_{FA} \xrightarrow{tv\cdot\mu A} Lv \cdot r(\mathrm{id}_A) \xrightarrow{\lambda_r(v,\mathrm{id}_A)} r(v \cdot \mathrm{id}_A) \xrightarrow{\cong} rv$$

which is the composite modulation

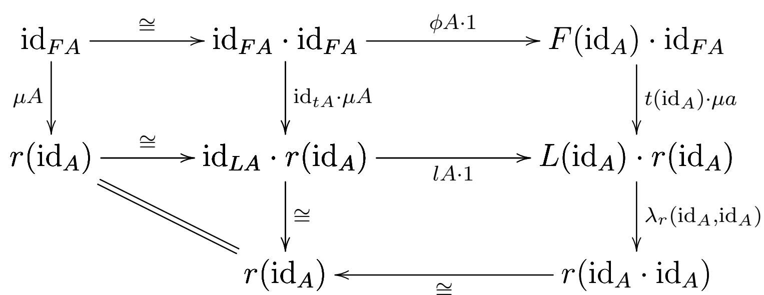


at v. So $\bar{\mu}$ is a modulation. That it satisfies the required commutativity condition follows from the equalities





The validity of these equations is checked by evaluating at A to get



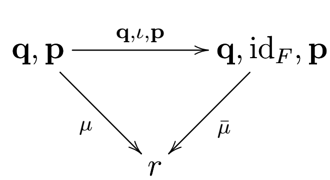
where lA is the structure morphism for L and the top right square commutes by the unit law for t (NT3). The bottom square is the unit law for ι. The top diagram above corresponds to the path top, right, bottom, and the second diagram to the path right, down, right, down, left.

To prove strong representability, we adopt the notation of 4.6. Thus we start with paths of modules

$$\mathbf{p} = F_k \xrightarrow{p_k} F_{k-1} \xrightarrow{p_{k-1}} \ldots \xrightarrow{p_1} F_0$$

$$\mathbf{q} = H_0 \xrightarrow{q_1} H_1 \xrightarrow{q_2} \ldots \xrightarrow{q_l} H_l$$

with $F_0 = F = H_0$. We have to show that for any multimodulation $\mu : \mathbf{q}, \mathbf{p} \longrightarrow r$, there is a unique multimodulation $\bar{\mu} : \mathbf{q}, \mathrm{id}_F, \mathbf{p} \longrightarrow r$ such that



commutes.

Thus for paths of vertical arrows

$$\mathbf{x} = A_k \xrightarrow{x_k} A_{k-1} \xrightarrow{x_{k-1}} \dots \xrightarrow{x_1} A_0$$

$$\mathbf{w} = C_0 \xrightarrow{w_1} C_1 \xrightarrow{w_2} \dots \xrightarrow{w_l} C_l$$

with $A_0 = A = C_0$, we must have commutativity of

$$\begin{array}{ccccc}
[\mathbf{qw}] \cdot [\mathbf{px}] & \xrightarrow{\cong} & [\mathbf{qw}] \cdot \mathrm{id}_{FA} \cdot [\mathbf{px}] & \xrightarrow{1\cdot\phi A\cdot 1} & [\mathbf{qw}] \cdot F(\mathrm{id}_A) \cdot [\mathbf{px}] \\
 & \searrow{\scriptstyle \mu(\mathbf{w},\mathbf{x})} & & \swarrow{\scriptstyle \bar{\mu}(\mathbf{w},\mathrm{id}_A,\mathbf{x})} & \\
 & & r[\mathbf{w}, \mathrm{id}_A, \mathbf{x}] & & \\
 & & \swarrow{\scriptstyle \cong} & & \\
 & r[\mathbf{w}, \mathbf{x}] & & &
\end{array}$$

We've already considered the case where both $\mathbf{x}$ and $\mathbf{w}$ are empty, so we can assume that one of them is not, say $\mathbf{x}$ (the other case being dual).

Now, take $\mathbf{x}$ and $\mathbf{w}$ as above but with $y : A_0 \longrightarrow C_0$. Then if $\bar{\mu}$ does exist, inner equivariance says that

$$\begin{array}{ccc}
[\mathbf{qw}] \cdot F(\mathrm{id}_{A_0}) \cdot F(y) \cdot [\mathbf{px}] & \xrightarrow{1\cdot\lambda_{p_1}\cdot 1} & [\mathbf{qw}] \cdot F(\mathrm{id}_{A_0}) \cdot [\mathbf{p}(y \cdot \mathbf{x}] \\
{\scriptstyle 1\cdot\phi(\mathrm{id}_{A_0},y)\cdot 1}\downarrow & & \downarrow{\scriptstyle \bar{\mu}(\mathbf{w},\mathrm{id}_{A_0},y\cdot\mathbf{x})} \\
[\mathbf{qw}] \cdot F(\mathrm{id}_{A_0} \cdot y) \cdot [\mathbf{px}] & \xrightarrow[\bar{\mu}(\mathbf{w},\mathrm{id}_{A_0}\cdot y,\mathbf{x})]{} & r[\mathbf{w}, \mathrm{id}_{A_0}, y, \mathbf{x}]
\end{array}$$

commutes. If we precede this square by $1 \cdot \phi A_0 \cdot 1$, the bottom composite gives the commutative

$$\begin{array}{ccc}
[\mathbf{qw}] \cdot \mathrm{id}_{A_0} \cdot Fy \cdot [\mathbf{px}] & \xrightarrow{1\cdot\phi A_0\cdot 1} & [\mathbf{qw}] \cdot F(\mathrm{id}_{A_0}) \cdot Fy \cdot [\mathbf{px}] \\
{\scriptstyle \cong}\downarrow & & \downarrow{\scriptstyle 1\cdot\phi(\mathrm{id}_A,y)\cdot 1} \\
[\mathbf{qw}] \cdot Fy \cdot [\mathbf{px}] & \xrightarrow{\cong} & [\mathbf{qw}] \cdot F(\mathrm{id}_{A_0} \cdot y) \cdot [\mathbf{px}] \\
{\scriptstyle \bar{\mu}(\mathbf{w},y,\mathbf{x})}\downarrow & & \downarrow{\scriptstyle \bar{\mu}(\mathbf{w},\mathrm{id}_{A_0}\cdot y,\mathbf{x})} \\
r[\mathbf{w}, y, \mathbf{x}] & \xrightarrow[\cong]{} & r[\mathbf{w}, \mathrm{id}_{A_0}, y, \mathbf{x}]
\end{array}$$

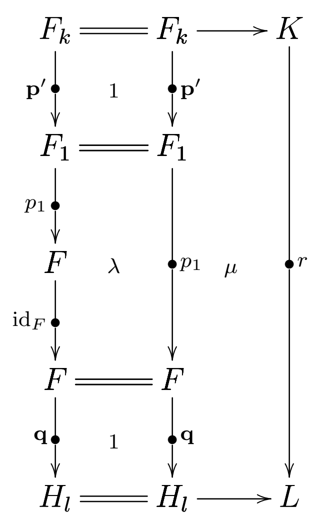
The top composite gives the commutative

$$\begin{array}{ccc}
[\mathbf{qw}]\cdot Fy\cdot[\mathbf{px}] & \xrightarrow{1\cdot\phi A_0\cdot 1} & [\mathbf{qw}]\cdot F(\mathrm{id}_{A_0}\cdot y)\cdot[\mathbf{px}] \\
{\scriptstyle 1\cdot\lambda_{p_1}\cdot 1}\downarrow & & \downarrow{\scriptstyle 1\cdot\lambda_{p_1}\cdot 1} \\
[\mathbf{qw}]\cdot \mathrm{id}_{FA_0}\cdot[\mathbf{p}(y\cdot\mathbf{x})] & \xrightarrow{1\cdot\phi A_0\cdot 1} & [\mathbf{qw}]\cdot F(\mathrm{id}_{A_0})\cdot[\mathbf{p}(y\cdot\mathbf{x})] \\
{\scriptstyle\cong}\downarrow & & \\
[\mathbf{qw}]\cdot[\mathbf{p}(y\cdot\mathbf{x})] & & \downarrow{\scriptstyle \bar{\mu}(\mathbf{w},\mathrm{id}_{A_0},y\cdot\mathbf{x})} \\
{\scriptstyle \mu(\mathbf{w},y\cdot\mathbf{x})}\downarrow & & \\
r[\mathbf{w},y,\mathbf{x}] & \xrightarrow[\cong]{} & r[\mathbf{w},\mathrm{id}_{A_0},y,\mathbf{x}]
\end{array}$$

where the unlabeled isomorphisms are canonical. This shows that $\bar{\mu}$, if it exists, is unique. In fact we can solve for $\bar{\mu}(\mathbf{w},y,\mathbf{x})$. It is the composite

$$[\mathbf{qw}]\cdot Fy\cdot[\mathbf{px}] \xrightarrow{1\cdot\lambda_{p_1}\cdot 1} [\mathbf{qw}]\cdot[\mathbf{p}(y\cdot\mathbf{x})] \xrightarrow{\mu(\mathbf{w},y\cdot\mathbf{x})} r[\mathbf{w},y,\mathbf{x}].$$

If we write $\mathbf{p}$ as p_1, $\mathbf{p}'$ we see that this means that $\bar{\mu}$ is the composite of multimodulations $\mu(1,\lambda,1)$



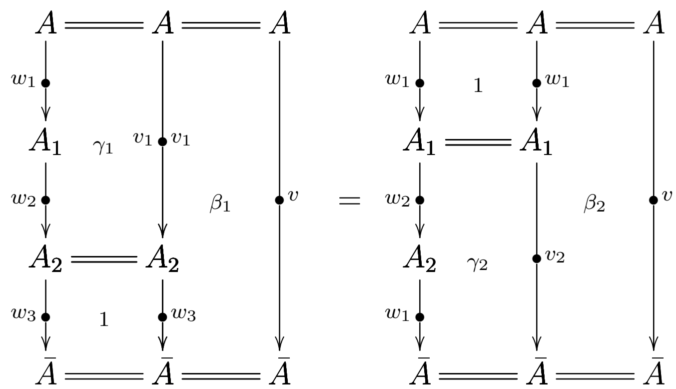
where λ is the bimodulation of proposition 5.1.9. It follows immediately that $\bar{\mu}$ is a multimodulation. By the same proposition it also follows that $\bar{\mu}(1,\iota,1)=\mu$. This completes the proof.

■

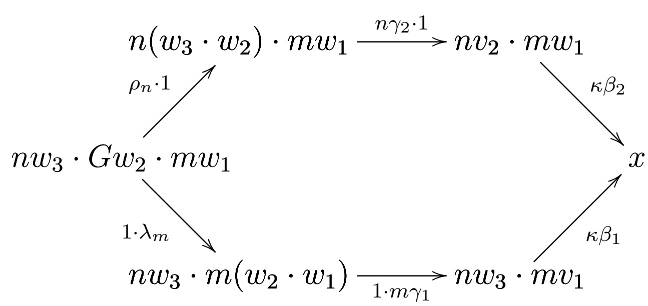
5.2. Reduction to [5] In this section we show that if $\mathbb{A}$ and $\mathbb{X}$ are bicategories, i.e. all horizontal arrows are identities, our colimit formula for $m\cdot n$ (definition 2.2.3) is equivalent to the one given in [5], §2.4. By proposition 2.2.1, a cocone κ on Γ_v is given by a family of cells

$$\kappa(\beta): nv_2\cdot mv_1 \longrightarrow x$$

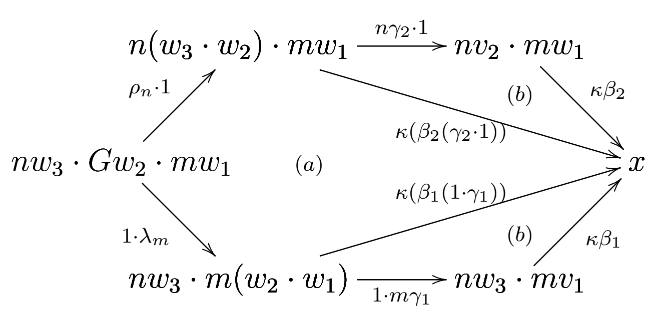
indexed by lax factorizations of v satisfying two conditions: (a) transfer of scalars and (b) naturality. A cocone on their diagram is also a family $\kappa(\beta) : nv_2 \cdot m_1 \longrightarrow x$ as above satisfying just one condition (c):
For all β, γ_1, β_2, γ_2, $\beta_1(1 \cdot \gamma_1) = \beta_2(\gamma_2 \cdot 1)$ as in



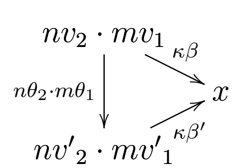
we have the commutative hexagon



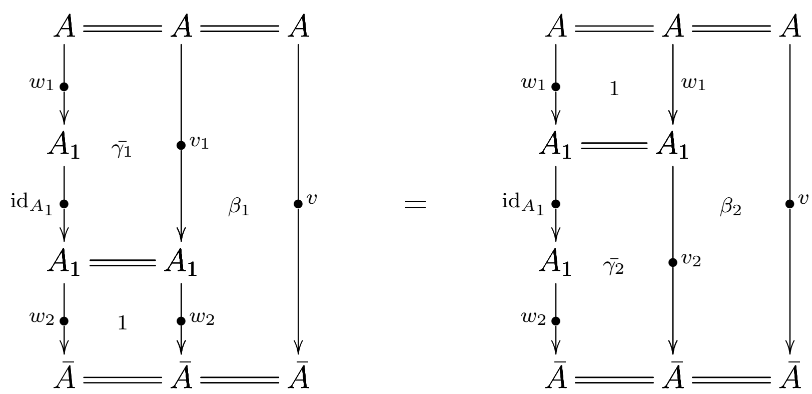
If κ satisfies (a) and (b) then it satisfies (c) as can be seen from



On the other hand, (a) is a special case of (c) where γ_1 and γ_2 are identities. Showing that naturality (condition (b)) follows from (c) is a bit trickier. Let $\beta : v_2 \cdot v_1 \longrightarrow v$ and $\beta' : v'_2 \cdot v'_1 \longrightarrow v$ be lax factorizations of v such that $\beta'(\theta_2 \cdot \theta_1) = \beta$ for $\theta_i : v_i \longrightarrow v'_i$. We have to show that



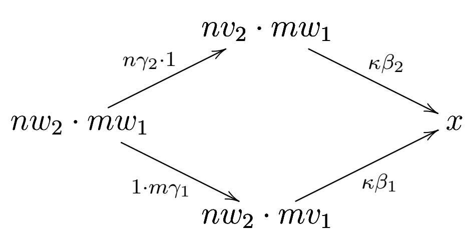
commutes. A consequence of (c) is the following. Let $\gamma_1 : w_1 \longrightarrow v_1$ and $\gamma_2 : w_2 \longrightarrow v_2$, $\beta_1 : w_2 \cdot v_1 \longrightarrow v$, $\beta_2 : v_2 \cdot w_1 \longrightarrow v$ be such that $\beta_1(1 \cdot \gamma_1) = \beta_2(\gamma_2 \cdot 1)$. Then the pentagon of condition (c) when applied to the situation



and preceded by

$$nw_2 \cdot mw_1 \xrightarrow{\cong} nw_2 \cdot \mathrm{id}_{GA_1} \cdot mw_1 \xrightarrow{1 \cdot \gamma A_1 \cdot 1} nw_2 \cdot G(\mathrm{id}_{A_1}) \cdot mw_1$$

reduces to



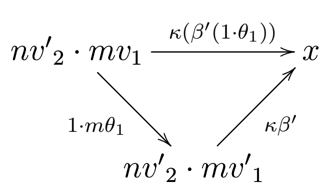
being commutative. Now specialize this to

$$\gamma_1 = \theta_1, \quad \beta_1 = \beta'$$

and

$$\gamma_2 = 1_{v'_2}, \quad \beta_2 = \beta'(1 \cdot \theta_1)$$

to get commutativity of



Then let

$$\gamma_1 = 1_{v_1}, \quad \beta_1 = \beta$$

and

$$\gamma_2 = \theta_2, \quad \beta_2 = \beta'(1 \cdot \theta_1)$$

to get commutativity of

$$\begin{array}{ccc} & nv'_2 \cdot mv_1 & \\ {\scriptstyle n\theta_2 \cdot 1} \nearrow & & \searrow {\scriptstyle \kappa(\beta'(1\cdot\theta_1))} \\ nv_2 \cdot mv_1 & \xrightarrow[\kappa\beta]{} & x \end{array}$$

Pasting these two diagrams along $\kappa(\beta'(1 \cdot \theta_1)$ gives the required commutativity (b).

Thus we see that, when $\mathbb{A}$ and $\mathbb{X}$ are bicategories (i.e., double categories whose horizontal arrows are all identities) our construction is equivalent to that of [5]. It follows that composition of modules is associative up to coherent isomorphism as claimed in *loc. cit.*

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Department of Mathematics and Statistics, Dalhousie University
Halifax, NS, Canada, B3H 3J5

Email: `pare@mathstat.dal.ca`

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