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KAN EXTENSIONS AND LAX IDEMPOTENT PSEUDOMONADS

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ABSTRACT. We show that colax idempotent pseudomonads and their algebras can be presented in terms of right Kan extensions. Dually, lax idempotent pseudomonads and their algebras can be presented in terms of left Kan extensions. We also show that a distributive law of a colax idempotent pseudomonad over a lax idempotent pseudomonad has a presentation in terms of Kan extensions.

1. Introduction

This paper follows [Marmolejo and Wood, 2010] and builds on the idea in [Manes, 1976], which was actually preceded by [Walters, 1970], that a monad can be presented without iterating the underlying endofunctor. [Marmolejo and Wood, 2010] extended Manes' notion of an extension operator to handle algebras but we note now that algebras were treated in a somewhat similar manner in [Walters, 1970] too. Our treatment of algebras also enabled "no iteration" descriptions of distributive laws and wreaths. Because the values of the endofunctor of a monad are term objects, the no iteration description in effect removes the need to mention terms of terms and (terms of terms of terms). This is particularly helpful in the descriptions of distributive laws and wreaths where the intent is to rewrite M-terms of A-terms as A-terms of M-terms.

When we turn to higher dimensional monads the no iteration idea is even more helpful. For then the terms tend to be n-sorted, with $n \geq 2$. For example, in completion monads with respect to classes of limits, the terms are categorical diagrams comprised of both objects and arrows. It is in fact completion monads, precisely colax idempotent pseudomonads, about which we have most to say. Such a pseudomonad $(D, d, m, \cdots)$ is what is also called a "coKZ doctrine", and characterized by adjunctions $dD \dashv m \dashv Dd$. We caution the reader that in [Marmolejo, 1997], our main reference for these pseudomonads, the subject matter is presented in terms of lax idempotent pseudomonads "KZ doctrines", for which the adjunctions are reversed to give $Dd \dashv m \dashv dD$.

The extension operator in [Manes, 1976] and those in [Marmolejo and Wood, 2010] satisfy equations. It will come as no surprise that if pseudomonads (on 2-categories say) are described in similar terms then the equalities of those papers must be replaced with invertible 2-cells — which must themselves satisfy equations. However, *colax idempotent*

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pseudomonads have all but one of their 2-cell equations given by adjunction equations. Thus it might be hoped that if colax (or lax) idempotent pseudomonads are described by extension operators then their 2-cell equations might also mediate universal properties. This is the case. The extensions which appear in describing colax [lax] idempotent pseudomonads are right [left] Kan extensions! The precise definition (Definition 3.1) in terms of Kan extensions is somewhat similar to the conditions given in [Bunge, 1974] in what is called a *coherently closed family of U-extensions* (U is a 2-functor), furthermore, the way we extend the function of objects to a pseudofunctor from the data given in Definition 3.1 is similar to the construction of a lax adjoint to U given in [Bunge, 1974].

The algebras for a colax (or lax) idempotent pseudomonad are also defined in terms of Kan extensions and proven to be essentially the same as the usual algebras.

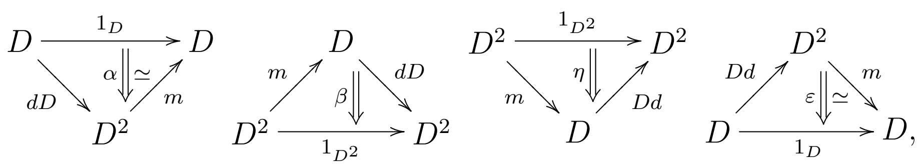
In Section 2 we begin by recalling the characterization of a colax idempotent pseudomonad $\mathbb{D} = (D, d, \cdots)$ and its algebras, in terms of adjunctions, as given in [Marmolejo,1997]. Important equations involving the derived modification $\delta : dD \to Dd$ are also recalled. In Section 3 we define right Kan pseudomonads and algebras for these. Section 4 provides a construction of a right Kan pseudomonad $\mathbb{D}'$ from a colax idempotent pseudomonad $\mathbb{D}$ and a construction of a colax idempotent pseudomonad $\mathbb{D}'$ from a right Kan pseudomonad $\mathbb{D}$. In Section 5 we show that starting with either notion as $\mathbb{D}$, the 2-category of algebras for $\mathbb{D}$ is 2-equivalent to the 2-category of algebras for $\mathbb{D}'$.

We recall in Section 6 that morphisms between pseudomonads on 2-categories can be described in terms of 2-functors between their underlying 2-categories, together with liftings to their 2-categories of algebras. Moreover, these can also be described, see [Marmolejo and Wood, 2008] in terms of transitions which are a pseudo version of Street's morphisms of monads [Street, 1972]. In Section 6 we use the work of the previous sections and these observations to give a description of transitions between colax idempotent pseudomonads in terms of extensions. Since distributive laws can be elegantly described in several ways in terms of extensions and one of their duals we are able in Section 7 to give a description of distributive laws between certain pseudomonads in terms of extensions. We note that the distributive law described in [Marmolejo, Rosebrugh, Wood, 2002], whose algebras are constructively completely distributive lattices, was produced this way, as a Kan extension. Another example is the distributive law of the small limit completion pseudomonad over the small colimit completion, whose algebras are the completely distributive categories [Marmolejo, Rosebrugh, Wood, to appear]; we also have the lextensive categories as algebras for the pseudomonad obtained from a distributive law of the finite completion pseudomonad over the finite sum completion pseudomonad; or regular categories as algebras for the finite limit completion pseudomonad over the regular factorizations pseudomonad with base $\mathbf{cat}_{\mathbf{ker}}$ as defined in [Centazzo and Wood, 2002], and many more. To illustrate how these distributive laws work in the setting of Kan extensions we examine, in Section 8, the distributive law of coFam over Fam.

2. Preliminaries

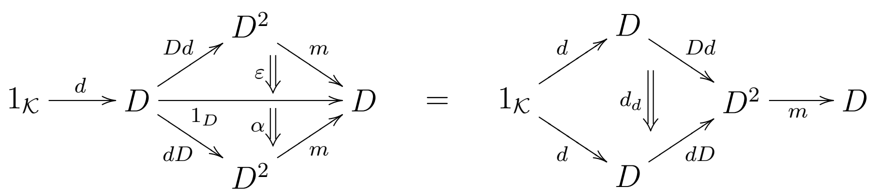
For the convenience of the reader, we recall in this section the definition of co-lax idempotent pseudomonad (also known as co-KZ pseudomonad). They first appeared in the papers of Kock [Kock, 1973] and Zöberlein [Zöberlein,1976]. In this section we largely follow (the dual of) the development given in [Marmolejo, 1997].

Let $\mathcal{K}$ be a 2-category. A *co-lax idempotent pseudomonad* $\mathbb{D} = (D, d, m, \alpha, \beta, \eta, \varepsilon)$ on $\mathcal{K}$ consists of a pseudofunctor $D : \mathcal{K} \to \mathcal{K}$, together with strong transformations $d : 1_{\mathcal{K}} \to D$ and $m : D^2 \to D$, and modifications



(1)

with α and ε invertible, that render $dD \dashv m \dashv Dd$, and such that the coherence condition



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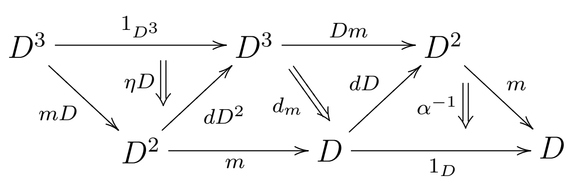
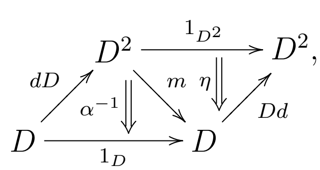
is satisfied. It is shown in [Marmolejo, 1997] that any such structure induces a pseudomonad, whose structure is given by $(D, d, m, \alpha^{-1}, \varepsilon^{-1}, \mu)$, where μ is the pasting



(3)

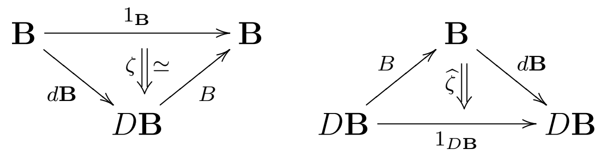
and furthermore, that for a pseudomonad $(D, d, m, \alpha^{-1}, \varepsilon^{-1}, \mu)$ to be co-lax idempotent it suffices that there exists a modification β such that $\alpha, \beta : dD \dashv m$ is an adjunction; equivalently, that there exists a modification η such that $\eta, \varepsilon : m \dashv Dd$ is an adjunction.

Recall as well that we can then produce a 2-cell $\delta : dD \to Dd$ as the pasting



that this pasting is equally the pasting of ε^{-1} and β at m, that $\delta \cdot d = d_d^{-1}$, that $m \cdot \delta = \varepsilon^{-1}\alpha^{-1}$, and that $\delta \cdot m$ is the pasting of β and η at 1_{D^2}.

The 2-category $\mathbb{D}$-Alg of $\mathbb{D}$-algebras is defined as follows. Its objects are adjunctions $\zeta, \widehat{\zeta} : d\mathbf{B} \dashv B$,



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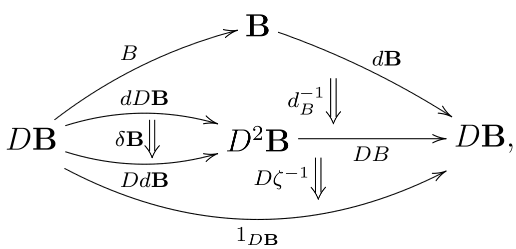
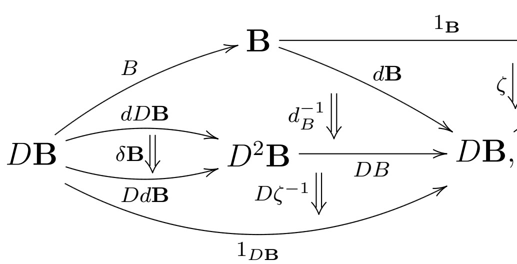
with invertible unit. The invertibility of ζ is automatic if d is fully faithful. Recall as well that $\widehat{\zeta}$ is completely determined by ζ as the pasting



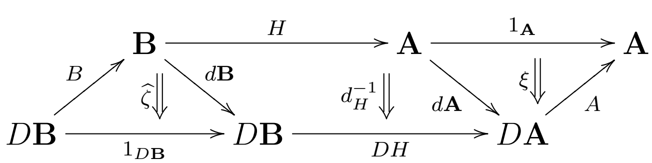
and that all we have to do to verify that a ζ as above determines an object in $\mathbb{D}$-Alg is to show that the equation



$= 1_B$ (5)

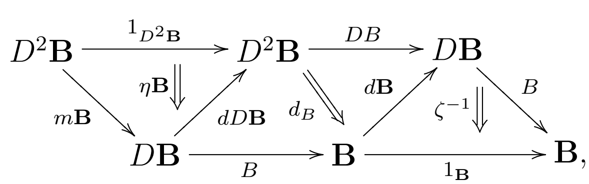
is satisfied. (Note that replacing $\mathbf{B}$ by D, B by m, and ζ by α in the definition of $\widehat{\zeta}$ gives us $\beta = \widehat{\alpha}$.)

A 1-cell from $(\mathbf{B}, B, \zeta)$ to $(\mathbf{A}, A, \xi)$ is a 1-cell $H : \mathbf{B} \to \mathbf{A}$ such that the pasting



(6)

is invertible. Given $H, K : (\mathbf{B}, B, \zeta) \to (\mathbf{A}, A, \xi)$, a 2-cell in $\mathbb{D}$-Alg is simply a 2-cell $\tau : H \to K$ in $\mathcal{K}$. Provisionally write $\mathbb{D}'$ for the pseudomonad $(D, d, m, \alpha^{-1}, \varepsilon^{-1}, \mu)$ described above. It is shown in [Marmolejo, 1997] that $\mathbb{D}$-Alg is 2-isomorphic to $\mathbb{D}'$-Alg, the usual category of algebras for a pseudomonad, since the associativity constraint needed to complete a $\mathbb{D}$-algebra $(\mathbf{B}, B, \zeta)$ to a $\mathbb{D}'$-algebra is given uniquely by the pasting



(7)

while for a 1-cell $H:(\mathbf{B}, B, \zeta) \to (\mathbf{A}, A, \xi)$, the pasting (6) uniquely completes H to a 1-cell of $\mathbb{D}'$-algebras.

3. Right Kan pseudomonads and their algebras

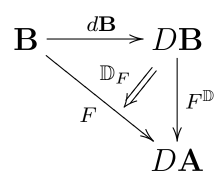
We define co-lax pseudomonads in terms of right Kan extensions. Later on we shall show that they are the usual co-lax pseudomonads as in the previous section, but for the moment (and just to be able to distinguish one from the other in this paper) we will call them right Kan pseudomonads.

3.1. Definition. *A right Kan pseudomonad $\mathbb{D}$ on $\mathcal{K}$ is given as follows:*

i) *A* function $D:\mathrm{Ob}(\mathcal{K}) \to \mathrm{Ob}(\mathcal{K})$*.*

ii) *For every $\mathbf{A} \in \mathcal{K}$, a 1-cell $d\mathbf{A}:\mathbf{A} \to D\mathbf{A}$.*

iii) *For every 1-cell $F:\mathbf{B} \to D\mathbf{A}$, a right Kan extension of F along $d\mathbf{B}$*



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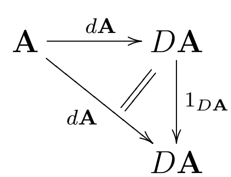
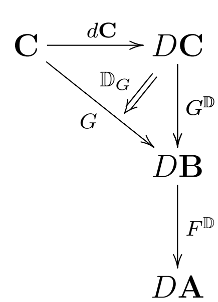
with $\mathbb{D}_F$ invertible (the latter being automatic if the 1-cell $d\mathbf{B}$ is fully faithful). Subject to the axioms

a) *For every $\mathbf{A}$ in $\mathcal{K}$,*



exhibits $1_{D\mathbf{A}}$ as a right Kan extension of $d\mathbf{A}$ along $d\mathbf{A}$.

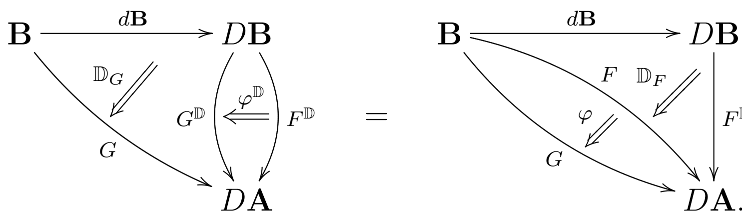
b) *For every $G:\mathbf{C} \to D\mathbf{B}$ and $F:\mathbf{B} \to D\mathbf{A}$ the 2-cell*



(9)

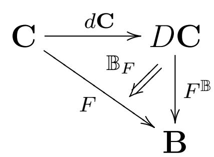
exhibits $F^{\mathbb{D}}G^{\mathbb{D}}$ as a right Kan extension of $F^{\mathbb{D}}G$ along $d\mathbf{C}$.

3.2. Remark. Observe that we can also define an effect $(\ )^{\mathbb{D}}$ on 2-cells: given $\varphi : F \to G : \mathbf{B} \to D\mathbf{A}$ in $\mathcal{K}$, we define $\varphi^{\mathbb{D}} : F^{\mathbb{D}} \to G^{\mathbb{D}}$ as the unique 2-cell such that



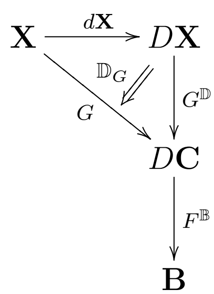
We clearly obtain a functor $(\ )^{\mathbb{D}} : \mathcal{K}(\mathbf{B}, D\mathbf{A}) \to \mathcal{K}(D\mathbf{B}, D\mathbf{A})$.

We now define the 2-category of algebras for a a right Kan pseudomonad $\mathbb{D}$ in terms of right Kan extensions. We denote it by $\mathbb{D}\text{-}\overline{\text{Alg}}$ and we define it as follows. An object $\mathbb{B}$ in $\mathbb{D}\text{-}\overline{\text{Alg}}$ consists of an object $\mathbf{B}$ in $\mathcal{K}$ together with an assignment, to every $F : \mathbf{C} \to \mathbf{B}$, of a right Kan extension $F^{\mathbb{B}} : D\mathbf{C} \to \mathbf{B}$ of F along $d\mathbf{C}$



(10)

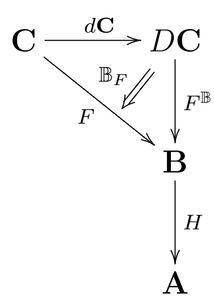
with $\mathbb{B}_F$ invertible (automatic if $d\mathbf{C}$ fully faithful), in such a way that for every $G : \mathbf{X} \to D\mathbf{C}$ in $\mathcal{K}$, the diagram



(11)

exhibits $F^{\mathbb{B}} \cdot G^{\mathbb{D}}$ as a right Kan extension of $F^{\mathbb{B}} \cdot G$ along $d\mathbf{X}$.

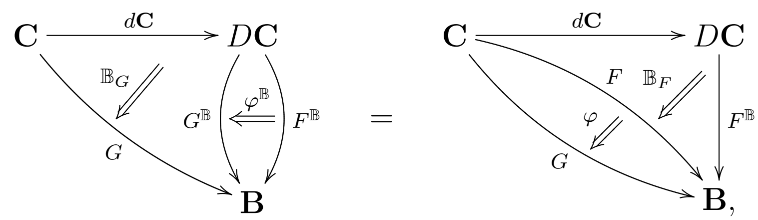
A 1-cell $H : \mathbb{B} \to \mathbb{A}$ in $\mathbb{D}\text{-}\overline{\text{Alg}}$ is a 1-cell $H : \mathbf{B} \to \mathbf{A}$ in $\mathcal{K}$ such that for every $F : \mathbf{C} \to \mathbf{B}$, the diagram



(12)

exhibits $F^{\mathbb{B}} \cdot H$ as a right Kan extension of $F \cdot H$ along $d\mathbf{C}$. A 2-cell $\tau : H \to K : \mathbb{B} \to \mathbb{A}$ is simply a 2-cell $\tau : H \to K$ in $\mathcal{K}$. Composition is as in $\mathcal{K}$. It is not hard to show that composition of 1-cells in $\mathbb{D}\text{-}\overline{\text{Alg}}$ results in a 1-cell in $\mathbb{D}\text{-}\overline{\text{Alg}}$.

3.3. Remark. As in Remark 3.2 we can, for any $\mathbb{B}$ in $\mathbb{D}$-$\overline{\mathrm{Alg}}$, induce an effect $(\ )^{\mathbb{B}}$ on 2-cells: given $\varphi : F \to G : \mathbf{C} \to \mathbf{B}$, we define $\varphi^{\mathbb{B}} : F^{\mathbb{B}} \to G^{\mathbb{B}}$ as the unique 2-cell such that



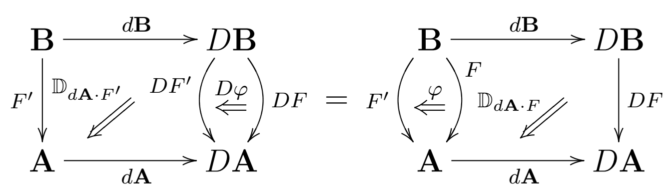
thus inducing a functor $(\ )^{\mathbb{B}} : \mathcal{K}(\mathbf{C}, \mathbf{B}) \to \mathcal{K}(D\mathbf{C}, \mathbf{B})$.

4. Right Kan pseudomonads versus co-lax idempotent pseudomonads 1

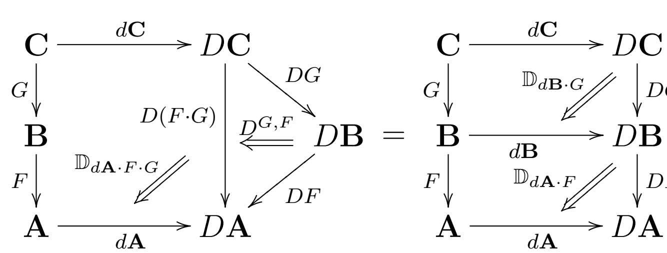
In this section we construct a colax idempotent pseudomonad from a right Kan pseudomonad, and vice versa. The constructions are given in the following two theorems.

4.1. Theorem. *Every right Kan pseudomonad on $\mathcal{K}$ induces a co-lax idempotent pseudomonad on $\mathcal{K}$.*

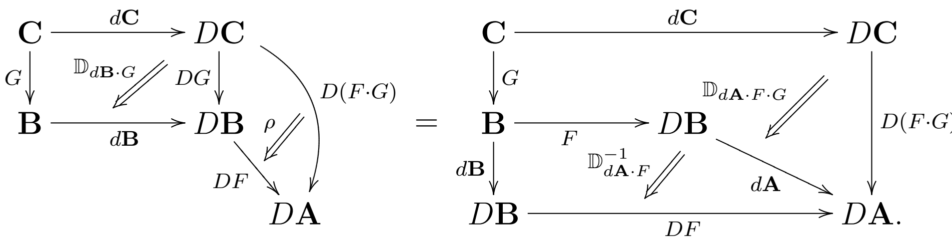
Proof. Assume we have a right Kan pseudomonad $\mathbb{D}$ on $\mathcal{K}$. We first extend D to a pseudofunctor $D : \mathcal{K} \to \mathcal{K}$. Given $\varphi : F \to F' : \mathbf{B} \to \mathbf{A}$ in $\mathcal{K}$, define $DF = (d\mathbf{A} \cdot F)^{\mathbb{D}}$, and define $D\varphi : DF \to DF'$ as $(d\mathbf{A} \cdot \varphi)^{\mathbb{D}}$, that is, $D\varphi$ is the unique 2-cell such that



(using the fact that the left most square exhibits DF' as a right Kan extension). It is then immediate that $D(1_F) = 1_{DF}$ and that for $\psi : F' \to F''$ we have $D(\psi\varphi) = (D\psi)(D\varphi)$. If $G : \mathbf{C} \to \mathbf{B}$, define $D^{G,F} : DF \cdot DG \to D(F \cdot G)$ as the unique (invertible) 2-cell such that



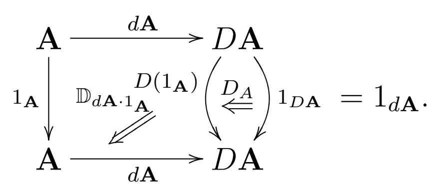
Observe that the inverse of the 2-cell $D^{G,F}$ is the unique 2-cell $\rho\colon D(F\cdot G)\to DF\cdot DG$ such that



(using (9)). It is not hard to see that for any $\gamma\colon G\to G'$ and $\varphi\colon F\to F'$

$$D(F\cdot\gamma)D^{G,F}=D^{G',F}(DF\cdot D\gamma)\text{ and }D(\varphi\cdot G)D^{G,F}=D^{G,F'}(D\varphi\cdot DG).$$

Since both 1_{DA} and $D(1_{\mathbf{A}})$ are right Kan extensions of $d\mathbf{A}$ along $d\mathbf{A}$, there is a unique isomorphism $D_{\mathbf{A}}\colon 1_{D\mathbf{A}}\to D(d\mathbf{A})$ such that



It is not hard to see that

$$D^{F,1_{\mathbf{A}}}(D_{\mathbf{A}}\cdot DF)=1_{DF}=D^{1_{\mathbf{B}},F}(DF\cdot D_{\mathbf{B}}),$$

as well as

$$D^{G\cdot H,F}(DF\cdot D^{H,G})=D^{H,F\cdot G}(D^{G,F}\cdot DH),$$

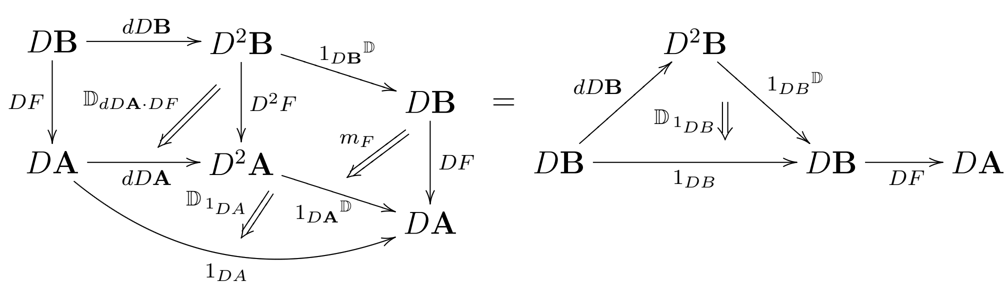
therefore $D\colon\mathcal{K}\to\mathcal{K}$ is a pseudofunctor.

Then we extend d to a strong transformation $d\colon 1_{\mathcal{K}}\to D$ by defining $d_F=\mathbb{D}_{d\mathbf{A}\cdot F}$ for $F\colon\mathbf{B}\to\mathbf{A}$ (all the relevant equations necessary to show that d is indeed a strong transformation appear above).

Next we define $m\colon D^2\to D$ such that for every $\mathbf{A}$,

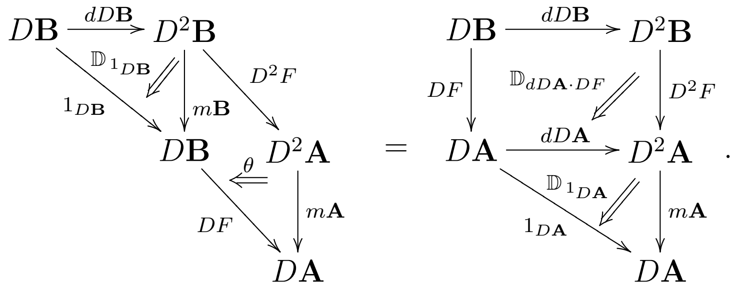
$$m\mathbf{A}=1_{D\mathbf{A}}{}^{\mathbb{D}},$$

and, using (9), define for $F\colon\mathbf{B}\to\mathbf{A}$, $m_F\colon DF\cdot m\mathbf{B}\to m\mathbf{A}\cdot D^2F$ as the unique 2-cell such that



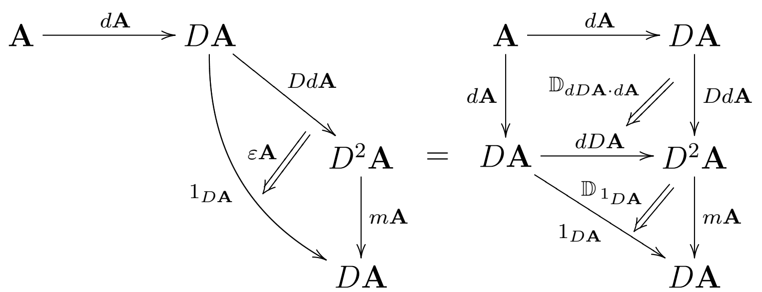
(13)

The inverse of m_F is the unique 2-cell θ such that

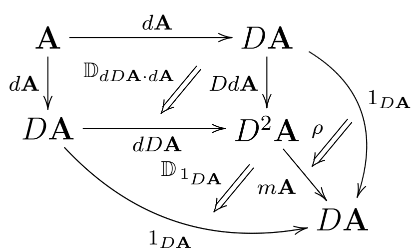


It is not hard to see that $m : D^2 \to D$ is a strong transformation.

Now define $\alpha\mathbf{A} = {\mathbb{D}_{1_{D\mathbf{A}}}}^{-1}$, then (13) tells us that $\alpha : 1_{D\mathbf{A}} \to m \cdot dD$ is a modification. Define $\varepsilon\mathbf{A} : m\mathbf{A} \cdot Dd\mathbf{A} \to 1_{D\mathbf{A}}$ as the unique 2-cell such that

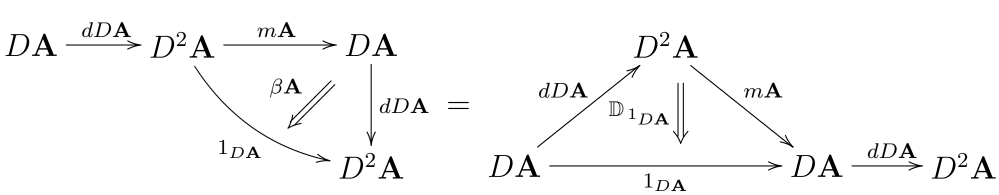


The inverse of $\varepsilon\mathbf{A}$ is the unique 2-cell ρ such that

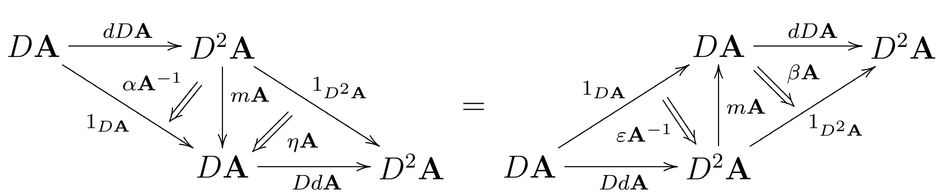


$= 1_{d\mathbf{A}}$. (14)

It is not hard to show that $\varepsilon : m \cdot Dd \to 1_D$ is a modification by pasting the relevant equation with d_F. Define $\beta\mathbf{A} : dD\mathbf{A} \cdot m\mathbf{A} \to 1_{D^2\mathbf{A}}$ as the unique 2-cell such that



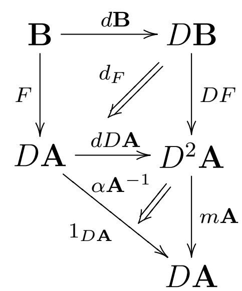
Finally define $\eta\mathbf{A} : 1_{D^2\mathbf{A}} \to Dd\mathbf{A} \cdot m\mathbf{A}$ as the unique 2-cell such that



By Section 2, the 2-cell above is $\delta\mathbf{A} : dD\mathbf{A} \to Dd\mathbf{A}$. It is not hard to see that β and η are modifications and that they determine, together with α and ε, adjunctions $dD \dashv m \dashv Dd$. Furthermore, the coherence condition (2) is given by (14). ■

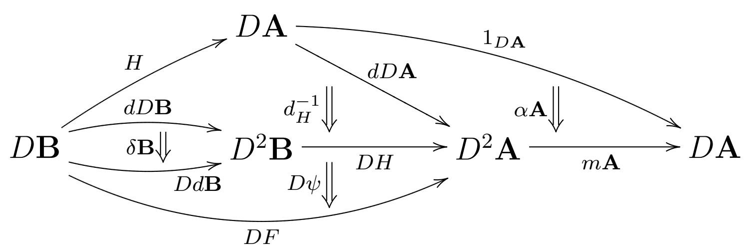
4.2. Theorem. *Every co-lax idempotent pseudomonad $\mathbb{D}$ on $\mathcal{K}$ induces a right Kan pseudomonad on $\mathcal{K}$.*

Proof. Let $\mathbb{D}$ be a co-lax idempotent pseudomonads with structure (1). We then take D and d on objects for items i) and ii) of Definition 3.1. For item iii) we define $F^{\mathbb{D}} = m\mathbf{A}\cdot DF$ and show that



(15)

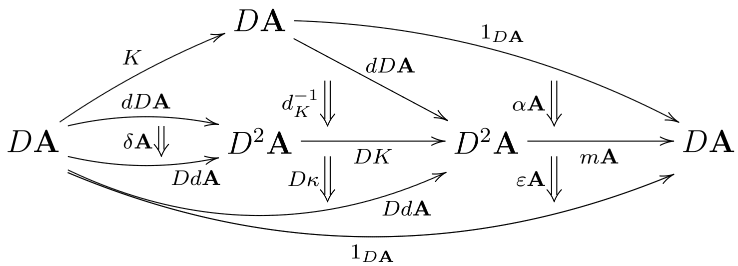
exhibits $F^{\mathbb{D}}$ as a right Kan extension of F along $d\mathbf{B}$. So take $H : D\mathbf{B} \to D\mathbf{A}$ and $\psi : H \cdot d\mathbf{B} \to F$. We show that the 2-cell



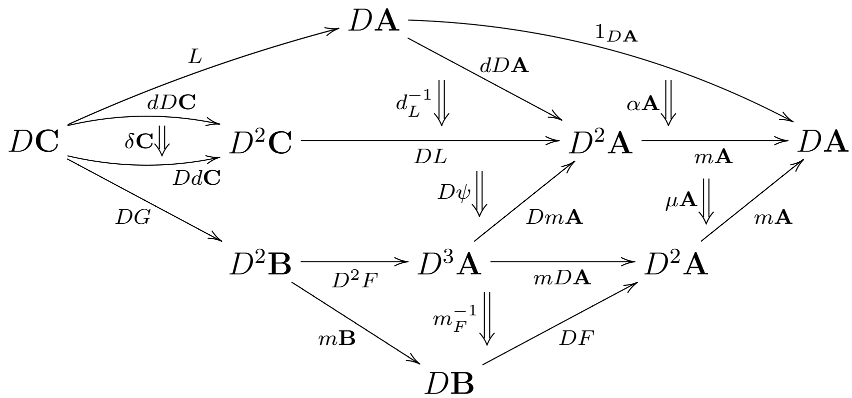
(16)

is the unique 2-cell $H \to F^{\mathbb{D}}$ that produces ψ when pasted with (15). So paste the above 2-cell with (15), substitute $\delta\mathbf{B} \cdot d\mathbf{B}$ by $d^{-1}_{d\mathbf{B}}$, then substitute the pasting of d^{-1}_H, $d^{-1}_{d\mathbf{B}}$, $D\psi$ and d_F by $dD\mathbf{A} \cdot \psi$, and cancel $\alpha\mathbf{A}$ with its inverse, thus obtaining ψ. Assume now that we have a 2-cell $\theta : H \to F^{\mathbb{D}}$ such that pasting it with (15) equals ψ. Substitute $D\psi$ in (16) by D of the pasting of θ with (15). We show that the resulting 2-cell equals θ. For this replace the pasting of $\delta\mathbf{B}$ and Dd_F by the pasting of d_{DF} and $\delta D\mathbf{A}$. Now replace the pasting of d^{-1}_H, $D\theta$ and d_{DF} by the pasting of θ and $d^{-1}_{m\mathbf{A}}$. Paste $\mu\mathbf{A}$ and its inverse at the composite $m\mathbf{A} \cdot Dm\mathbf{A}$ (where $\mu : m \cdot Dm \to m \cdot mD$ is the pasting (3)). Replace the pasting of $\alpha\mathbf{A}$, $d^{-1}_{m\mathbf{A}}$ and $\mu\mathbf{A}$ by $mD\mathbf{A} \cdot \alpha D\mathbf{A}$, and the pasting of μ^{-1} and $D\alpha\mathbf{A}^{-1}$ by $m\mathbf{A} \cdot \varepsilon D\mathbf{A}$. The pasting of $\alpha D\mathbf{A}$, $\delta D\mathbf{A}$ and $\varepsilon D\mathbf{A}$ is the identity, leaving just θ.

The proof of a) is similar, given $\kappa : K \cdot d\mathbf{A} \to d\mathbf{A}$, the relevant 2-cell to consider is



And the proof of b) is also similar, for a 2-cell $\psi : L \cdot d\mathbf{C} \to m\mathbf{A} \cdot DF \cdot G$, the relevant 2-cell is

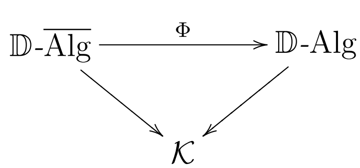


■

We compare these constructions in Section 6 below.

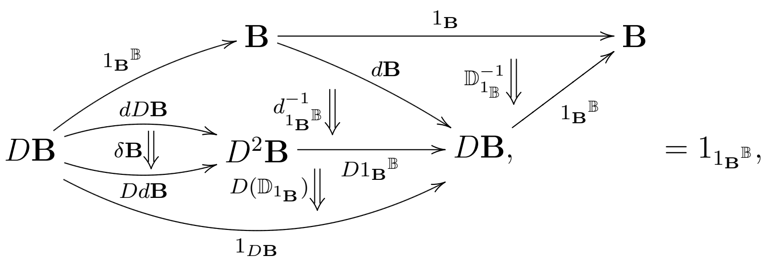
5. $\mathbb{D}$-$\overline{\mathrm{Alg}}$ versus $\mathbb{D}$-Alg

5.1. THEOREM. *Let $\mathbb{D}$ be a right Kan pseudomonad on $\mathcal{K}$, and produce the colax idempotent pseudomonad (also called $\mathbb{D}$) as in Theorem 4.1. There is a 2-equivalence $\Phi : \mathbb{D}$-$\overline{\mathrm{Alg}} \to \mathbb{D}$-Alg such that the diagram*

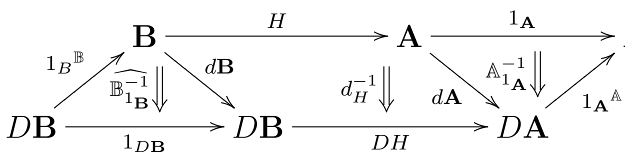


commutes, where the un-labeled arrows are forgetful 2-functors.

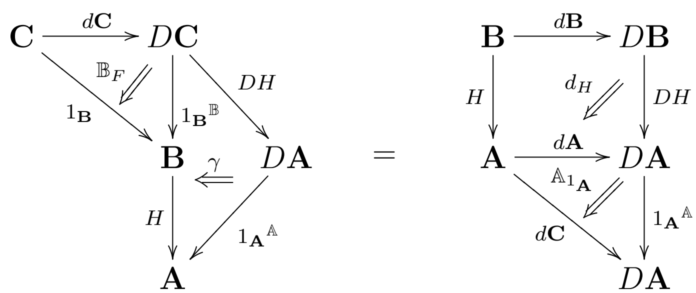
PROOF. We define $\Phi : \mathbb{D}$-$\overline{\mathrm{Alg}} \to \mathbb{D}$-Alg as follows. For $\tau : H \to K : \mathbb{B} \to \mathbb{A}$ in $\mathbb{D}$-$\overline{\mathrm{Alg}}$, define Φ of it as $\tau : H \to K : \mathbb{B}^{-1}_{1_\mathbf{B}} \to \mathbb{A}^{-1}_{1_\mathbf{A}}$. Φ will be a 2-functor if we can show that $\mathbb{B}^{-1}_{1_\mathbf{B}}$ and H are in $\mathbb{D}$-Alg. To show the first of these we must show that equation (5) is satisfied, in this case the equation is



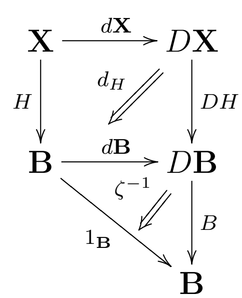
but this follows from the fact that (10), with $F = 1_{\mathbf{B}}$, is a right Kan extension. Again, since for $F = 1_{\mathbf{B}}$, (12) is a right Kan extension, we obtain the inverse of



as the unique 2-cell γ such that



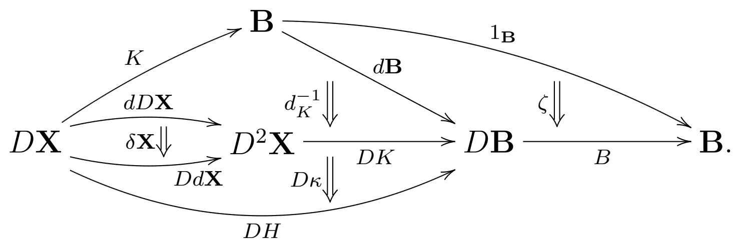
In the opposite direction define $\Psi : \mathbb{D}\text{-Alg} \to \mathbb{D}\text{-}\overline{\text{Alg}}$ as follows. For an algebra ζ as in (4), we define $\Psi(\zeta)$ such that for every $H : \mathbf{X} \to \mathbf{B}$, its extension is $B \cdot DH$, and the corresponding 2-cell is



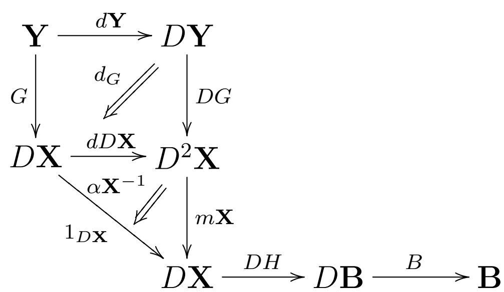
(17)

To see that $\Psi(\zeta)$ is well defined, we must show that (17) exhibits $B \cdot DH$ as a right Kan extension of H along $d\mathbf{X}$. Given $K : D\mathbf{X} \to \mathbf{B}$ and $\kappa : K \cdot d\mathbf{X} \to H$, the unique 2-cell

$K \to B \cdot DH$ that pasted with (17) equals κ is

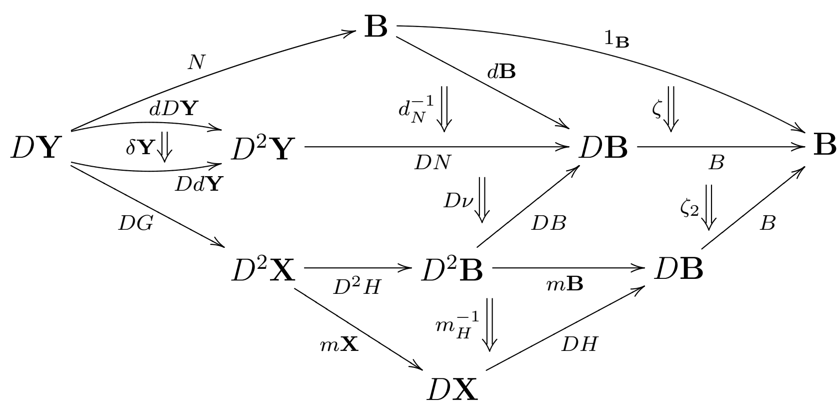


Furthermore, we must show that for any $G : \mathbf{Y} \to D\mathbf{X}$ and $H : \mathbf{X} \to \mathbf{B}$, the 2-cell



(18)

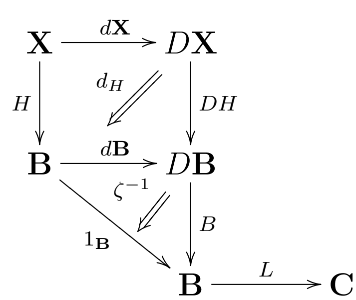
exhibits $B \cdot DH \cdot m\mathbf{X} \cdot DG$ as a right Kan extension of $B \cdot DH \cdot G$ along $d\mathbf{Y}$. Given $N : D\mathbf{Y} \to \mathbf{B}$ and $\nu : N \cdot d\mathbf{Y} \to B \cdot DH \cdot G$, the 2-cell



where ζ_2 is the 2-cell given by (7), is the unique 2-cell that pasted with (18) equals ν.

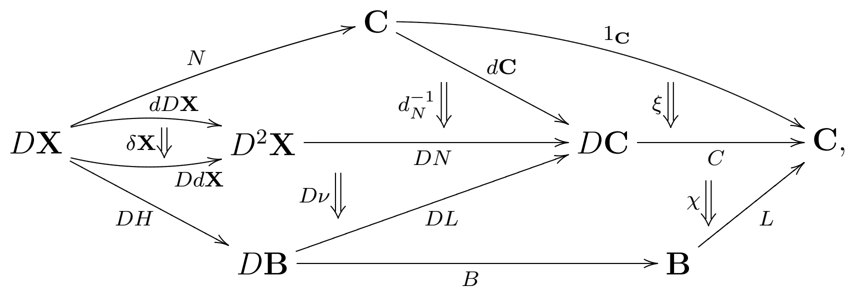
We thus conclude that $\Psi(\zeta)$ is an object of $\mathbb{D}$-$\overline{\mathrm{Alg}}$.

Given a 1-cell $L : \zeta \to \xi$ (with $\xi : id_{\mathbf{C}} \to C \cdot d\mathbf{C}$) in $\mathbb{D}$-Alg, we want to show that $L : \Psi(\zeta) \to \Psi(\xi)$ is a 1-cell in $\mathbb{D}$-$\overline{\mathrm{Alg}}$. Thus we must show that



(19)

exhibits $L \cdot B \cdot DH$ as right Kan extension of $L \cdot H$ along $d\mathbf{X}$ for any $H : \mathbf{X} \to \mathbf{B}$. Given $N : D\mathbf{X} \to \mathbf{C}$ and $\nu : N \cdot d\mathbf{X} \to L \cdot H$, the unique 2-cell $N \to L \cdot B \cdot DH$ that pasted with (19) equals ν is



where χ is the inverse of the 2-cell induced by L that corresponds to (6), given by the fact that L is a 1-cell of algebras. Thus we define $\Psi(L) = L$.

For a 2-cell $\lambda : L \to L' : \zeta \to \xi$ in $\mathbb{D}$-Alg, define $\Psi(\lambda) = \lambda$.

It is routine to verify that $\Phi \circ \Psi$ and $\Psi \circ \Phi$ are isomorphic to the corresponding identities. ∎

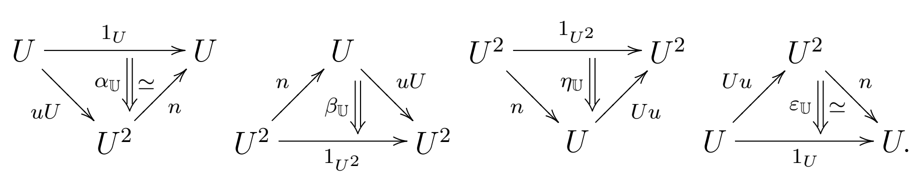
Similar arguments produce the following

5.2. Theorem. *Given colax idempotent pseudomonad $\mathbb{D}$, produce its associated right Kan pseudomonad (also called $\mathbb{D}$), as in Theorem 4.2. Then* $\mathbb{D}$-Alg *and* $\mathbb{D}$-$\overline{\text{Alg}}$ *are equivalent.* ∎

6. Right Kan pseudomonads versus co-lax idempotent pseudomonads 2

If $\mathbb{U}$ and $\mathbb{D}$ are pseudomonads on the 2-categories $\mathcal{L}$ and $\mathcal{K}$ respectively then, following [Marmolejo, 1999] we can describe morphisms from $(\mathcal{L}, \mathbb{U})$ to $(\mathcal{K}, \mathbb{D})$ in terms of liftings of 2-functors $F : \mathcal{L} \to \mathcal{K}$ to 2-functors $\widehat{F} : \mathbb{U}\text{-Alg} \to \mathbb{D}\text{-Alg}$ (that commute with the forgetful 2-functors). In Theorem 3.5 of [Marmolejo and Wood, 2008] we showed that such liftings are essentially the same as *transitions* from $\mathbb{U}$ to $\mathbb{D}$ along F. The latter are a pseudo version of the morphisms of monads found in [Street, 1972] (where they are called monad functors) and consist of strong transformations $r : DF \to FU$ together with two invertible modifications (corresponding to the two equalities of [Street, 1972]) subject to two equations. We refer the reader to [Marmolejo and Wood, 2008] for the definitions of these and of coherent isomorphisms between them.

We consider the particular case of $\mathbb{D} = (D, d, m, \alpha_{\mathbb{D}}, \beta_{\mathbb{D}}, \eta_{\mathbb{D}}, \varepsilon_{\mathbb{D}})$ and $\mathbb{U} = (U, u, n, \alpha_{\mathbb{U}}, \beta_{\mathbb{U}}, \eta_{\mathbb{U}}, \varepsilon_{\mathbb{U}})$ colax idempotent pseudomonads (with $\mathbb{D}$ as in (1), but with subindex $\mathbb{D}$ on the 2-cells that conform $\mathbb{D}$, and we use the same letters for the corresponding 2-cells that conform $\mathbb{U}$, but with subindex $\mathbb{U}$; thus the structure for $\mathbb{U}$ is



(20)

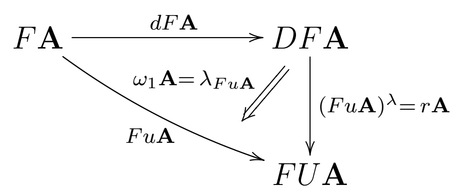
We follow the same pattern with δ, thus $\delta_{\mathbb{D}} : dD \to Dd$, and $\delta_{\mathbb{U}} : uU \to Uu$). It follows from the previous Section that morphisms of monads between them and hence also transitions, can be described in terms of algebras for the corresponding right Kan pseudomonads.

6.1. Theorem. *Let $\mathbb{U}$ and $\mathbb{D}$ be colax idempotent pseudomonads on 2-categories $\mathcal{L}$ and $\mathcal{K}$ respectively. A transition from $\mathbb{U}$ to $\mathbb{D}$ along a 2-functor $F : \mathcal{L} \to \mathcal{K}$ can be given by the following data: for every $\mathbf{A}$ in $\mathcal{L}$, a $\mathbb{D}$-algebra $(FU\mathbf{A}, (\ )^{\lambda})$, such that for every $L : \mathbf{B} \to U\mathbf{A}$ in $\mathcal{L}$,*

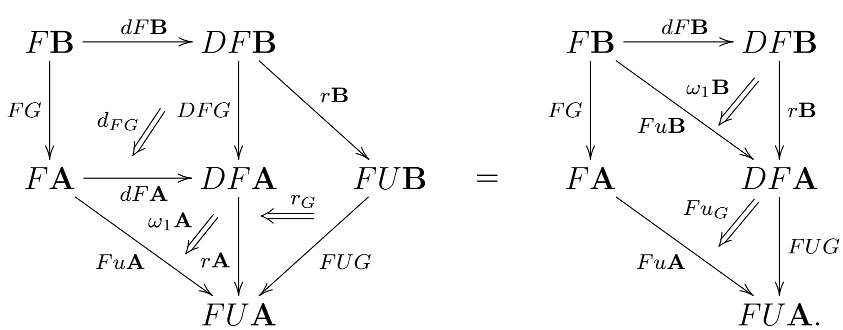
$$F(L^{\mathbb{U}}) : (FU\mathbf{B}, (\ )^{\lambda}) \to (FU\mathbf{A}, (\ )^{\lambda})$$

is a morphism of $\mathbb{D}$-algebras. Every transition from $\mathbb{U}$ to $\mathbb{D}$ along F is coherently isomorphic to one that arises in this way.

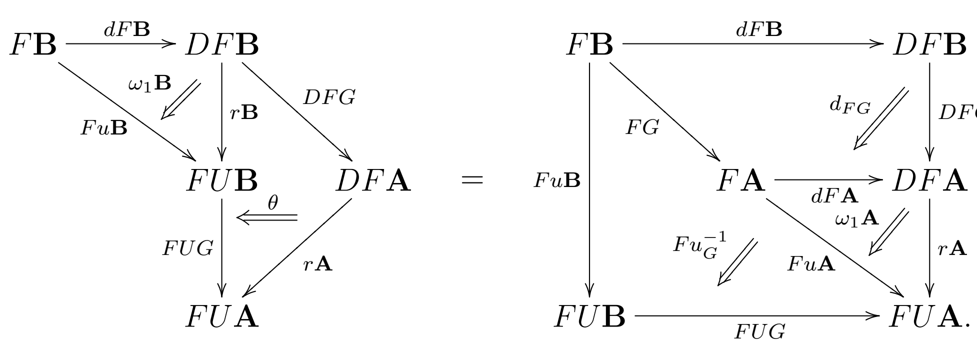
Proof. For every $\mathbf{A}$ in $\mathcal{L}$ define $r\mathbf{A} = (Fu\mathbf{A})^{\lambda}$ and $\omega_1\mathbf{A} = \lambda_{Fu\mathbf{A}}$:



To make r a strong transformation observe that, for every $G : \mathbf{B} \to \mathbf{A}$, $r\mathbf{A} \cdot d_{FG} = (Fu\mathbf{A})^{\lambda} \cdot \mathbb{D}_{dF\mathbf{A} \cdot FG}$ exhibits $r\mathbf{A} \cdot DFG$ as a right Kan extension of $r\mathbf{A} \cdot dFA \cdot FG$ along $dF\mathbf{B}$. Thus we define r_G as the unique 2-cell such that

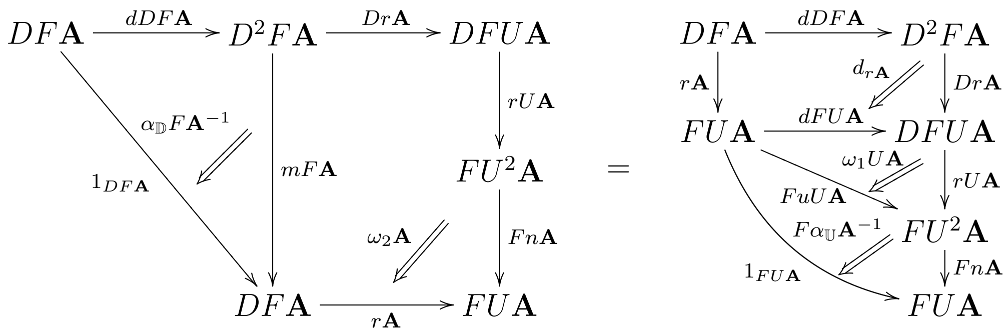


The inverse of r_G is the unique 2-cell θ (given by the fact that $FUG \cdot \omega_1\mathbf{B} = F((u\mathbf{A} \cdot G)^{\mathbb{U}}) \cdot \lambda_{Fu\mathbf{B}}$ exhibits $FUG \cdot r\mathbf{B}$ as a right Kan extension of $FUG \cdot Fu\mathbf{B}$ along $d\mathbf{B}$) such that

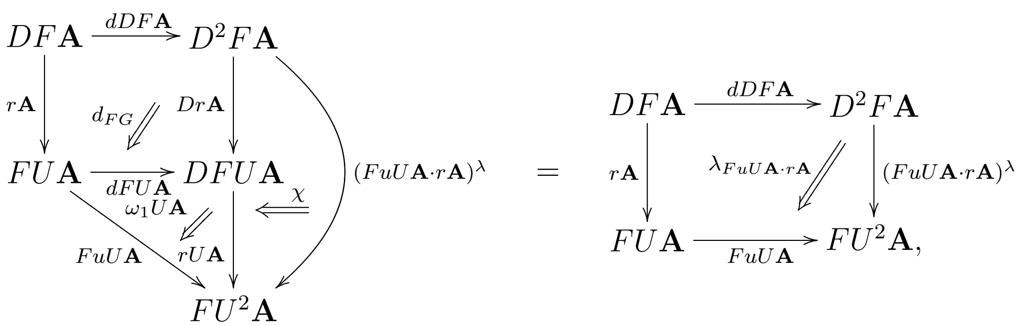


It is routine to verify that $r : DF \to FU$ is a strong transformation and the equation defining r_G above tells us that $\omega_1 : r \cdot dF \to Fu$ is a modification.

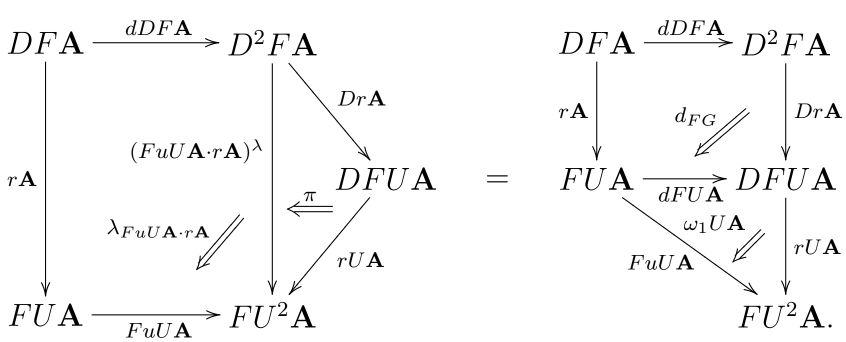
To define $\omega_2\mathbf{A}$ we observe that $r\mathbf{A} \cdot \alpha F\mathbf{A}^{-1} = (Fu\mathbf{A})^\lambda \cdot \mathbb{D}_{1_{DF\mathbf{A}}}$ exhibits $r\mathbf{A} \cdot mF\mathbf{A}$ as a right Kan extension of $r\mathbf{A} \cdot mF\mathbf{A}$ along $dDF\mathbf{A}$, thus we can define $\omega_2\mathbf{A}$ as the unique 2-cell such that



(This is the equation in Theorem 2.3 of [Marmolejo and Wood, 2008].) To induce the inverse of $\omega_2\mathbf{A}$ we observe first that $rU\mathbf{A} \cdot Dr\mathbf{A} \cong (FuU\mathbf{A} \cdot r\mathbf{A})^\lambda$: in one direction take the unique 2-cell χ such that

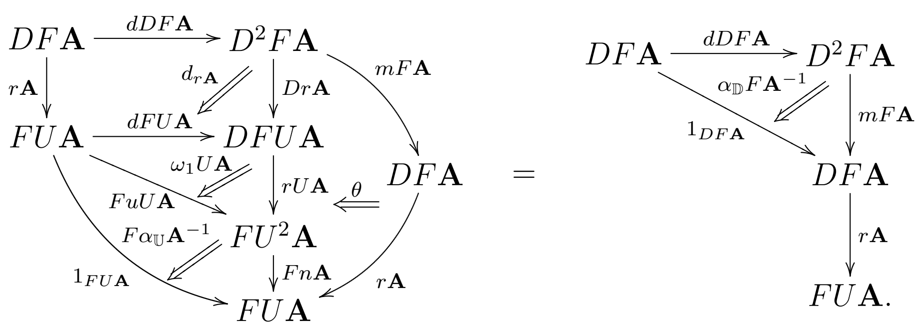


while in the other take the unique 2-cell π such that



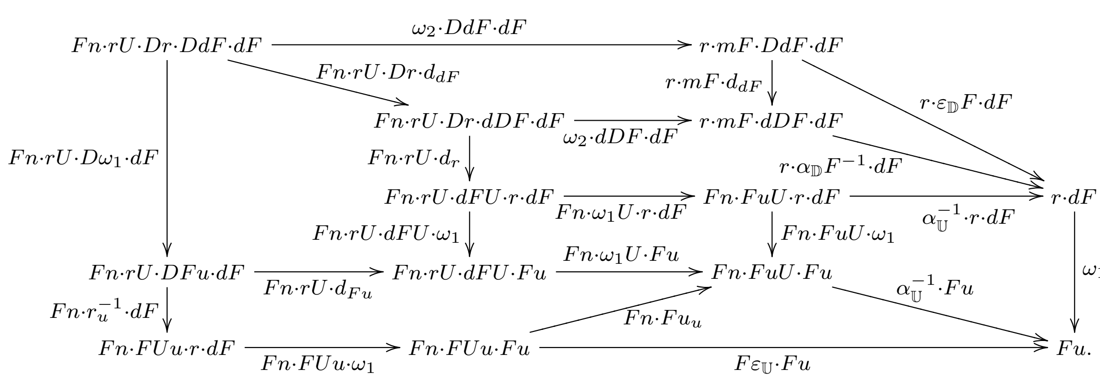
The isomorphism χ just exhibited and the equality $Fn\mathbf{A} = F((1_{U\mathbf{A}})^{\mathbb{U}})$ show that the pasting of $d_{r\mathbf{A}}$, $\omega_1 U\mathbf{A}$ and $F\alpha_{\mathbf{A}}^{-1}$ exhibits $Fn\mathbf{A} \cdot rU\mathbf{A} \cdot Dr\mathbf{A}$ as a right Kan extension of

$r\mathbf{A}$ along $dDF\mathbf{A}$. Thus, the inverse of $\omega_2\mathbf{A}$ is the unique 2-cell θ such that



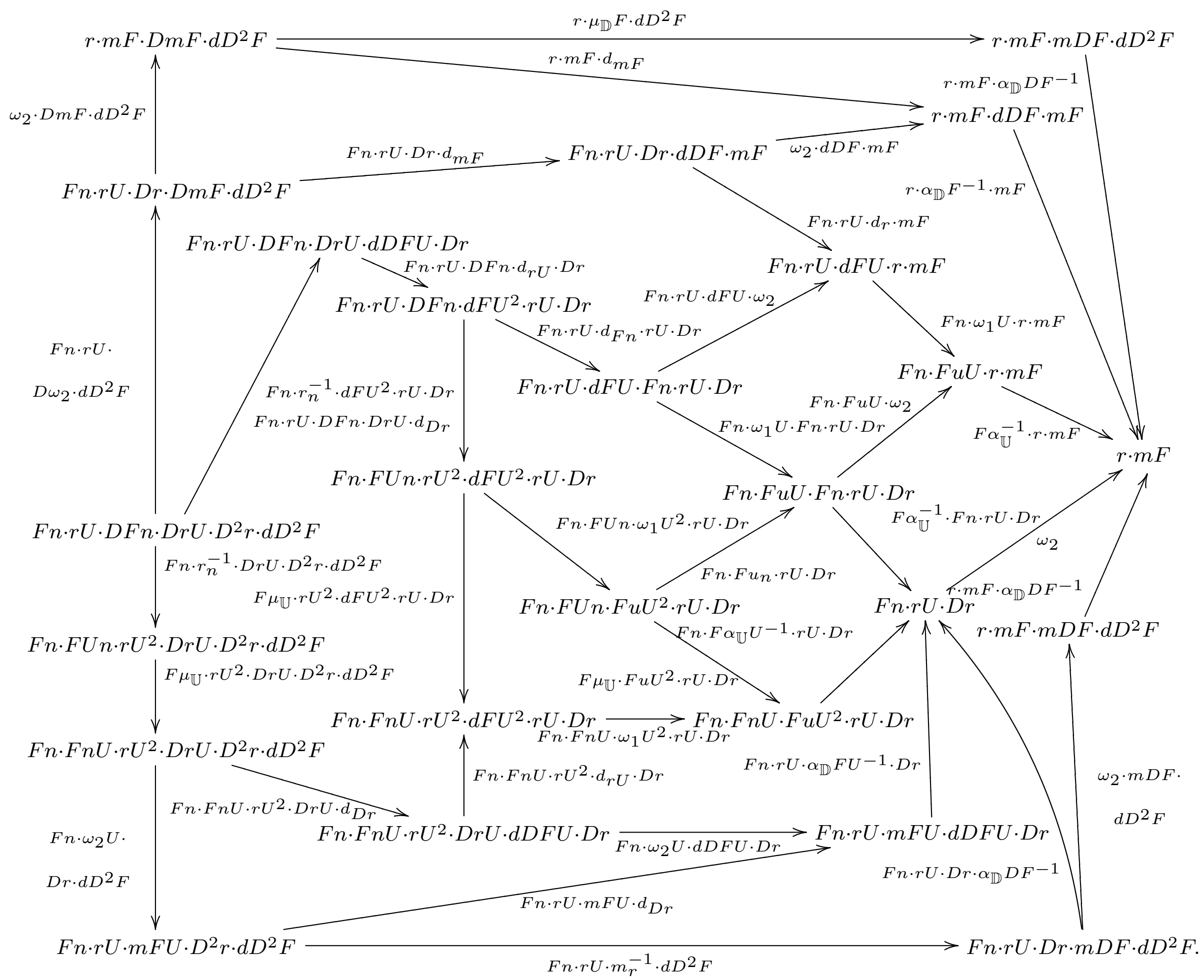
We need to verify that $\omega_2 : Fn \cdot rU \cdot Dr \to r \cdot mF$ is a modification. Given any $G : \mathbf{B} \to \mathbf{A}$, one shows that the pasting of d_{DFG} and $\alpha_{\mathbb{D}}F\mathbf{A}^{-1}$ followed by $r\mathbf{A}$ exhibits $r\mathbf{A}\cdot mF\mathbf{A}\cdot D^2FG$ as a right Kan extension of $r\mathbf{A} \cdot DFG$ along $dF\mathbf{B}$. Then one has to prove that the two pastings that need to be equal to show that ω_2 is a modification, are equal when preceded by $dDF\mathbf{B}$ and pasted with d_{DFG} and $\alpha_{\mathbb{D}}F\mathbf{A}^{-1}$.

The coherence conditions in Definition 2.1 of [Marmolejo and Wood, 2008] remain to be shown. The first ends in r, so it suffices to show that both pastings are equal when preceded by dF and pasted with ω_1. The following commutative diagram shows this:



For the other condition, observe first that, for every $\mathbf{A}$, $r\mathbf{A} \cdot mF\mathbf{A} \cdot \alpha_{\mathbb{D}}DF\mathbf{A}^{-1}$ exhibits $r\mathbf{A}\cdot mF\mathbf{A}\cdot mDF\mathbf{A}$ as a right Kan extension of $r\mathbf{A}\cdot mF\mathbf{A}$ along $dD^2F\mathbf{A}$. It then suffices to show that both pastings are the same when preceded by dD^2F and pasted with $\alpha_{\mathbb{D}}DF^{-1}$.

The following commutative diagram shows that these are equal:



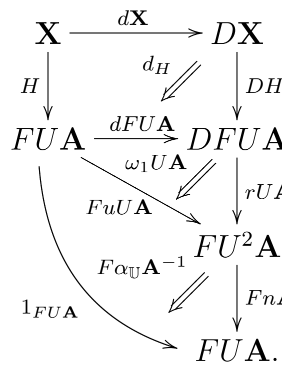
Assume now that we have a transition (r, ω_1, ω_2) from $\mathbb{U}$ to $\mathbb{D}$ along F. Consider the composite

$$\mathbb{U}\text{-Alg} \xrightarrow{\widehat{F}} \mathbb{D}\text{-Alg} \xrightarrow{\Psi} \mathbb{D}\text{-}\overline{\text{Alg}},$$

where $\widehat{F}:\mathbb{U}\text{-Alg} \to \mathbb{D}\text{-Alg}$ is the lifting of F determined by the transition (r, ω_1, ω_2) as in Proposition 2.2 of [Marmolejo and Wood, 2008], and $\Psi:\mathbb{D}\text{-Alg} \to \mathbb{D}\text{-}\overline{\text{Alg}}$ was defined in the proof of Theorem 5.1. If we apply this composite to the free $\mathbb{U}$-algebra $\alpha_{\mathbb{U}}\mathbf{A}$, $\mathbf{A}$ in $\mathcal{L}$, we obtain the following $\mathbb{U}$-algebra structure on $FU\mathbf{A}$: for $H:\mathbf{X} \to FU\mathbf{A}$, H^λ is the composite

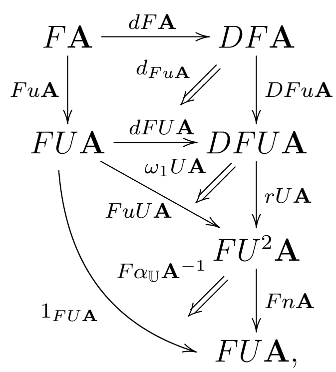
$$D\mathbf{X} \xrightarrow{DH} DFU\mathbf{A} \xrightarrow{rU\mathbf{A}} FU^2\mathbf{A} \xrightarrow{Dn\mathbf{A}} FU\mathbf{A}$$

and λ_H is the pasting

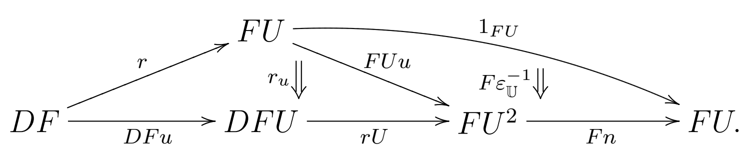


(21)

Furthermore, since for every $L\colon \mathbf{B} \to U\mathbf{A}$ we have that $L^{\mathbb{U}}\colon \alpha_{\mathbb{U}}\mathbf{B} \to \alpha_{\mathbb{U}}\mathbf{A}$ is a morphism of $\mathbb{U}$-algebras, the same composite of functors tells us that $F(L^{\mathbb{U}})\colon (FU\mathbf{B}, (\ )^{\lambda}) \to (FU\mathbf{A}, (\ )^{\lambda})$ is a $\mathbb{D}$-algebra morphism. According to the first part of this proof, the $\widehat{\omega_1}$ of the induced transition from $\mathbb{U}$ to $\mathbb{D}$ along F is



with its corresponding $\widehat{\omega_2}$, and the invertible modification that makes this and (r, ω_1, ω_2) coherently isomorphic is given by



This completes the proof. ∎

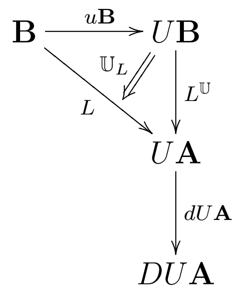
7. Distributive laws

In this section we deal with distributive laws. We treat the particular case of a distributive law of a colax idempotent pseudomonad over a lax idempotent pseudomonad, but observe that the other cases are similar. We point out that the composite pseudomonad resulting from a distributive law of a colax idempotent pseudomonad over another colax idempotent pseudomonad turns out to be colax idempotent (see Theorem 11.7 in [Marmolejo, 1999]). We begin with the following

7.1. Lemma. *Let $\mathbb{D}$ be a pseudomonad on $\mathcal{K}$ and let $\mathbb{U}$ be a colax idempotent monad (as in (20)) on $\mathcal{K}$. If there is a distributive law of $\mathbb{U}$ over $\mathbb{D}$, then*

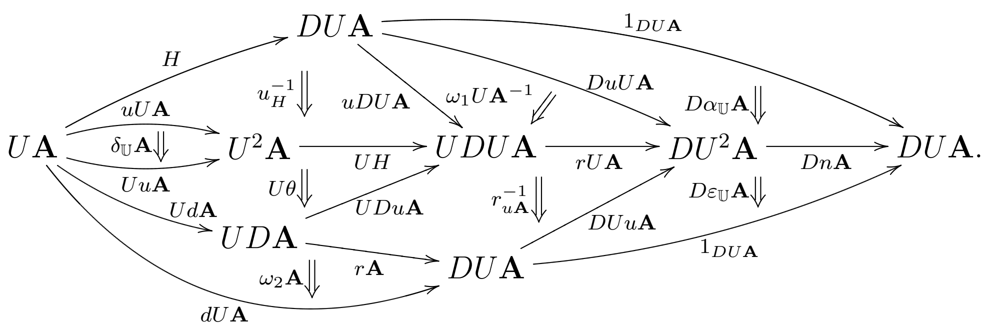
(i) *For every $\mathbf{A}$, $d^{-1}_{u\mathbf{A}}$ exhibits $dU\mathbf{A}$ as a right Kan extension of $Du\mathbf{A} \cdot d\mathbf{A}$ along $u\mathbf{A}$.*

(ii) *For every $L : \mathbf{B} \to U\mathbf{A}$ in $\mathcal{K}$,*

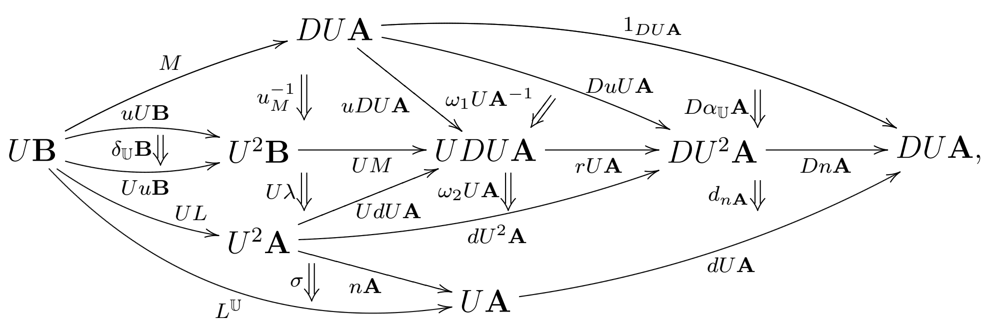


exhibits $dU\mathbf{A} \cdot L^{\mathbb{U}}$ as a right Kan extension of $dU\mathbf{A} \cdot L$ along $u\mathbf{B}$.

Proof. Let $(r, \omega_1, \omega_2, \omega_3, \omega_4)$ be a distributive law of $\mathbb{U}$ over $\mathbb{D}$ as in Proposition 5.1 in [Marmolejo and Wood, 2008]. Actually, the only part of the structure for a distributive law that we need to prove this is r, w_1, w_2 and the coherence condition (10) of that article. For (i) let $H : U\mathbf{A} \to DU\mathbf{A}$ and $\theta : H \cdot u\mathbf{A} \to Du\mathbf{A} \cdot d\mathbf{A}$, then the unique 2-cell $H \to dU\mathbf{A}$ that pasted with $d^{-1}_{u\mathbf{A}}$ is θ is given by the pasting

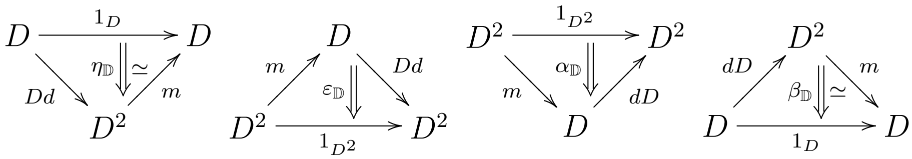


For (ii) let $M : U\mathbf{B} \to DU\mathbf{A}$ and $\lambda : M \cdot u\mathbf{B} \to dU\mathbf{A} \cdot L$, then the unique 2-cell $M \to dU\mathbf{A} \cdot L^{\mathbb{U}}$ that pasted with $dU\mathbf{A} \cdot \mathbb{U}_L$ is λ is given by the pasting



where $\sigma : n\mathbf{A} \cdot UL \to L^{\mathbb{U}}$ is the unique 2-cell that pasted with $\mathbb{U}_L$ equals the pasting of u_L and $\alpha_{\mathbb{U}}\mathbf{A}$. ■

For the next theorem we take $\mathbb{U}$ a colax idempotent pseudomonad with structure as in (20), but $\mathbb{D}$ is now a lax idempotent pseudomonad. We take the data for $\mathbb{D}$ as follows:



so that $Dd \dashv m \dashv dD$.

7.2. Theorem. *Assume that $\mathbb{U}$ is a colax idempotent monad on $\mathcal{K}$, and that $\mathbb{D}$ is a lax idempotent monad on $\mathcal{K}$ such that the conditions* (i) *and* (ii) *of Lemma 7.1 are satisfied. Then a distributive law of $\mathbb{U}$ over $\mathbb{D}$ can be given by the following data:*

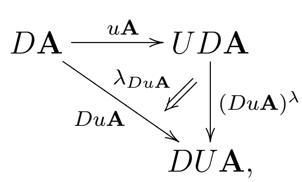
(iii) *For every $\mathbf{A}$ in $\mathcal{K}$, a $\mathbb{U}$-algebra structure $(DU\mathbf{A}, (\ )^\lambda)$,*

such that the following two conditions are satisfied:

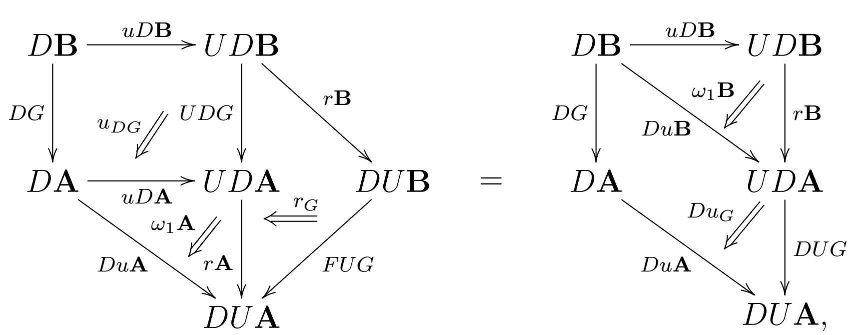
(iv) *For every $L : \mathbf{B} \to U\mathbf{A}$, $D(L^{\mathbb{U}}) : (DU\mathbf{B}, (\ )^\lambda) \to (DU\mathbf{A}, (\ )^\lambda)$ is 1-cell of $\mathbb{U}$-algebras.*

(v) *For every $H : \mathbf{C} \to DU\mathbf{A}$, $(H^\lambda)^{\mathbb{D}} : (DU\mathbf{C}, (\ )^\lambda) \to (DU\mathbf{A}, (\ )^\lambda)$ is an algebra morphism.*

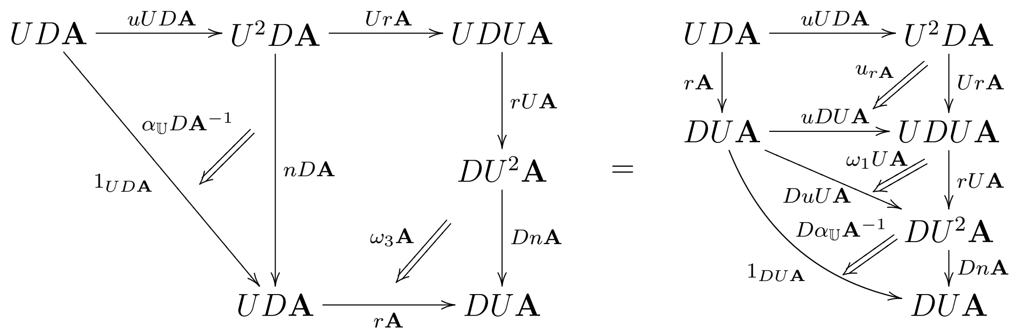
Proof. According to Theorem 6.1 we get a transition from $\mathbb{U}$ to $\mathbb{U}$ along D if we define, for every $\mathbf{A}$ in $\mathcal{K}$, $r\mathbf{A} = (Du\mathbf{A})^\lambda$, define $\omega_1\mathbf{A}$ as the 2-cell $\lambda_{Du\mathbf{A}}$:



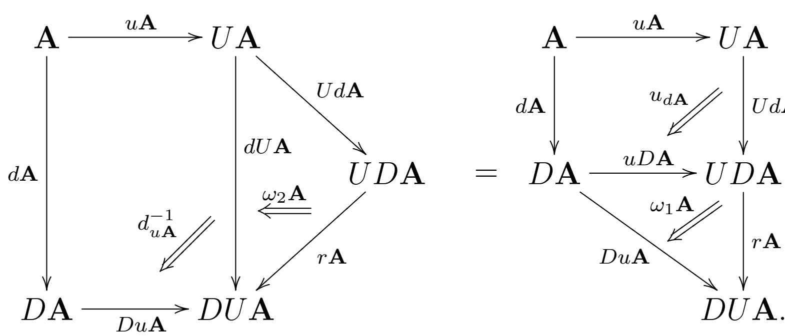
define r_G, for $G : \mathbf{B} \to \mathbf{A}$, as the unique 2-cell such that



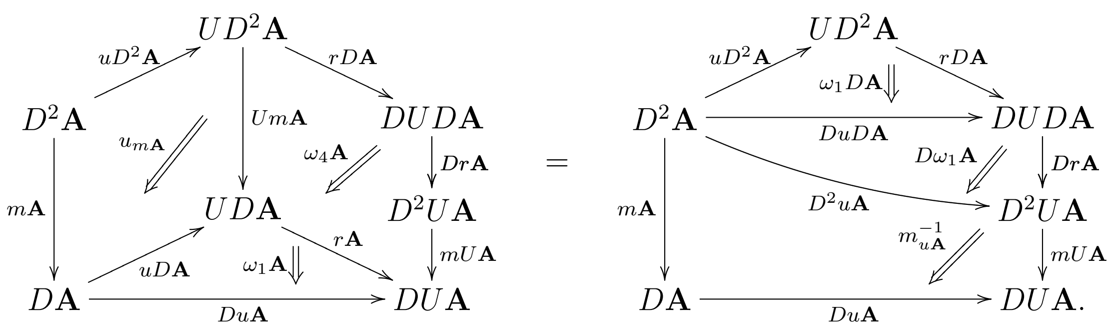
and define $\omega_3\mathbf{A}$ as the unique 2-cell such that



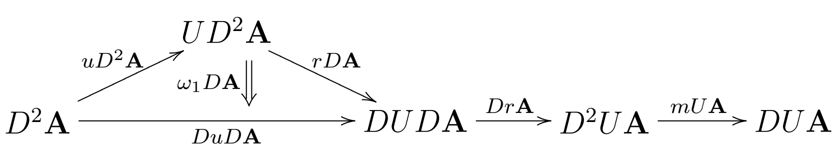
We define $\omega_2\mathbf{A}$ as the unique 2-cell such that such that



Then ω_2 is an invertible modification. Now we define $\omega_4\mathbf{A}$ as the unique 2-cell such that

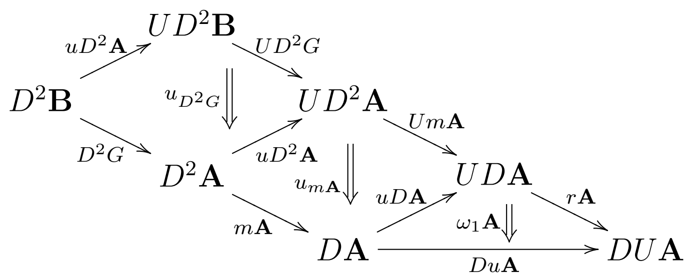


One induces the inverse of $\omega_4\mathbf{A}$ using the fact that



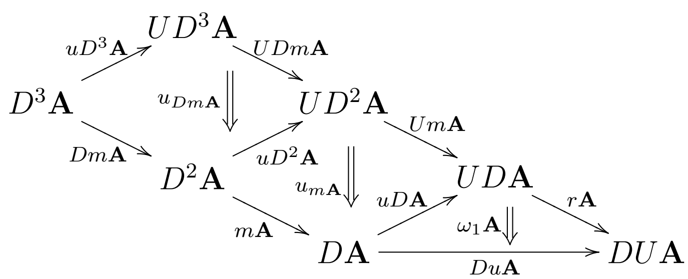
exhibits $mU\mathbf{A} \cdot Dr\mathbf{A} \cdot rD\mathbf{A}$ as a right Kan extension of $mU\mathbf{A} \cdot Dr\mathbf{A} \cdot DuD\mathbf{A}$ along $uD^2\mathbf{A}$; the proof that it is indeed a right Kan extension follows from the fact that $mU\mathbf{A} \cdot Dr\mathbf{A} \simeq$

$(r\mathbf{A})^{\mathbb{D}} = ((Du\mathbf{A})^{\lambda})^{\mathbb{D}} : (DUD\mathbf{A}, (\ )^{\lambda}) \to (DU\mathbf{A}, (\ )^{\lambda})$ is a 1-cell of $\mathbb{U}$-algebras. To show that ω_4 is a modification, one shows that for every $G : \mathbf{B} \to \mathbf{A}$,



exhibits $r\mathbf{A} \cdot Um\mathbf{A} \cdot UD^2G$ as a right Kan extension of $Du\mathbf{A} \cdot m\mathbf{A} \cdot D^2G$, since $r\mathbf{A} \cdot Um\mathbf{A} \cdot UD^2G \simeq (Du\mathbf{A} \cdot m\mathbf{A} \cdot D^2G)^{\lambda}$.

Next we show that (r, ω_2, ω_4) is an op-transition from $\mathbb{D}$ to $\mathbb{D}$ along U. Coherence condition (7) of [Marmolejo and Wood, 2008] follows from the fact that $\omega_1\mathbf{A}$ exhibits $r\mathbf{A}$ as a right Kan extension of $Du\mathbf{A}$ along $uD\mathbf{A}$, using the defining equation of $\omega_2\mathbf{A}$. And coherence condition (8) of [Marmolejo and Wood, 2008] follows from the fact that

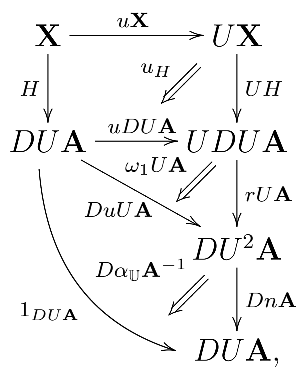


exhibits $r\mathbf{A} \cdot Um\mathbf{A} \cdot UDm\mathbf{A}$ as a right Kan extension of $Du\mathbf{A} \cdot m\mathbf{A} \cdot Dm\mathbf{A}$ along $uD^3\mathbf{A}$, this because $r\mathbf{A} \cdot Um\mathbf{A} \cdot UDm\mathbf{A} \simeq (Du\mathbf{A} \cdot m\mathbf{A} \cdot Dm\mathbf{A})^{\lambda}$.

Thus we have a transition (r, ω_1, ω_3) from $\mathbb{U}$ to $\mathbb{U}$ along D and an op-transition (r, ω_2, ω_4) from $\mathbb{D}$ to $\mathbb{D}$ along U. We are left with the verification that the coherence conditions of Proposition 5.1 of [Marmolejo and Wood, 2008] are satisfied. Condition (10) of that paper is the defining equation of ω_2, while (11) of the same paper follows from the fact that $dU\mathbf{A} \cdot \alpha_{\mathbb{U}}^{-1}$ exhibits $dU\mathbf{A} \cdot n\mathbf{A}$ as a right Kan extension of $dU\mathbf{A}$ along $uU\mathbf{A}$. And (12) of that paper is the defining equation for ω_4, leaving us only with coherence condition (13) of the same paper. This coherence condition follows from the fact that $r\mathbf{A} \cdot Um\mathbf{A} \cdot \alpha_{\mathbb{U}}D\mathbf{A}$ exhibits $r\mathbf{A} \cdot Um\mathbf{A} \cdot nD^2\mathbf{A}$ as a right Kan extension of $r\mathbf{A} \cdot Um\mathbf{A}$ along $uUD^2\mathbf{A}$ (since $r\mathbf{A} \cdot Um\mathbf{A} \cdot nD^2\mathbf{A} \simeq (r\mathbf{A} \cdot Um\mathbf{A})^{\lambda}$). ∎

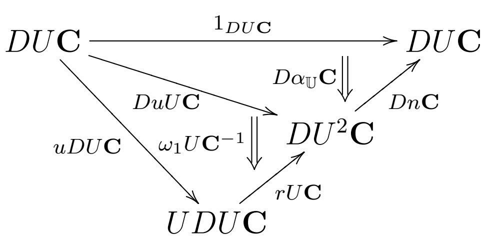
We must now show that every distributive law of $\mathbb{U}$ over $\mathbb{D}$, with $\mathbb{U}$ colax idempotent and $\mathbb{D}$ lax idempotent, arises essentially in this way. Let $(r, \omega_1, \omega_2, \omega_3, \omega_4)$ be a distributive law of $\mathbb{U}$ over $\mathbb{D}$. Then we have that conditions (i) and (ii) of Lema 7.1 are satisfied, and we must obtain conditions (iii), (iv) and (v) of Theorem 7.2, and show that the distributive law obtained from Theorem 7.2 is essentially the distributive law $(r, \omega_1, \omega_2, \omega_3, \omega_4)$.

Observe that (D, ω_1, ω_3) is a transition from $\mathbb{U}$ to $\mathbb{U}$ along D. Then Theorem 6.1 gives us the $\mathbb{U}$-algebra structure on $DU\mathbf{A}$ corresponding to (21), which in this case assigns to an $H : \mathbf{X} \to DU\mathbf{A}$ the right Kan extension

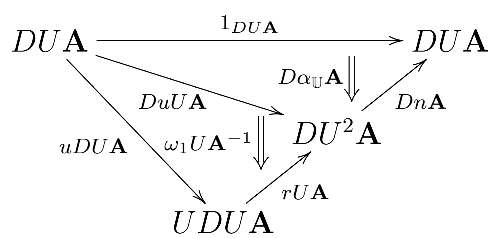


and for every $L : \mathbf{B} \to U\mathbf{A}$, $D(L^{\mathbb{U}})$ is a 1-cell of $\mathbb{U}$-algebras. This gives us conditions (iii) and (iv) of Theorem 7.2.

We are left with showing that, for any $H : \mathbf{C} \to DU\mathbf{A}$, $(H^{\lambda})^{\mathbb{D}} : (DU\mathbf{C}, (-)^{\lambda}) \to (DU\mathbf{A}, (-)^{\lambda})$ is a $\mathbb{U}$-algebra morphism. To do this we observe that $(DU\mathbf{C}, (-)^{\lambda})$ and $(DU\mathbf{A}, (-)^{\lambda})$ are the images under the 2-functor $\Psi : \mathbb{U}\text{-Alg} \to \mathbb{U}\text{-}\overline{\text{Alg}}$ of the $\mathbb{U}$-algebras given by



and



respectively (these in turn are the images of the free algebras $\alpha_{\mathbb{U}}\mathbf{C}$ and $\alpha_{\mathbb{U}}\mathbf{A}$ under the lifting $\mathbb{U}$-Alg $\to$ $\mathbb{U}$-Alg induced by the transition (D, ω_1, ω_3)), thus it suffices to show that

$$(H^{\lambda})^{\mathbb{D}} = \; DU\mathbf{C} \xrightarrow{DUH} DUDU\mathbf{A} \xrightarrow{DrU\mathbf{A}} D^2U^2\mathbf{A} \xrightarrow{D^2n\mathbf{A}} D^2U\mathbf{A} \xrightarrow{mU\mathbf{A}} DU\mathbf{A}$$

is a 1-cell between these latter $\mathbb{U}$-algebras. According to (6), we must show that

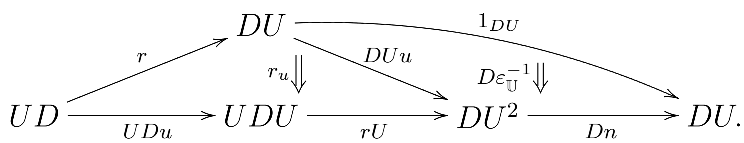
$$Dn\mathbf{A} \cdot rU\mathbf{A} \cdot UmU\mathbf{A} \cdot UD^2n\mathbf{A} \cdot UDrU\mathbf{A} \cdot UDUH \cdot UDn\mathbf{C} \cdot UrU\mathbf{C} \cdot \delta_{\mathbb{U}}DU\mathbf{C}$$

is invertible; and one uses the available isomorphisms to produce $nDU\mathbf{C}$ just after $\delta_{\mathbb{U}}DU\mathbf{C}$ to conclude that the above 2-cell is indeed invertible. One then applies the construction given in Theorem 7.2 to produce a new distributive law $(s, \pi_1, \pi_2, \pi_3, \pi_4)$. The claim is that the original distributive law $(r, \omega_1, \omega_2, \omega_3, \omega_4)$ is coherently isomorphic to this new one in the following sense:

7.3. Definition. *Let $\mathbb{U}$ and $\mathbb{D}$ be pseudomonads on $\mathcal{K}$, and let $(r, \omega_1, \omega_2, \omega_3, \omega_4)$ and $(s, \pi_1, \pi_2, \pi_3, \pi_4)$ be distributive laws of $\mathbb{U}$ over $\mathbb{D}$. We say that the distributive laws are* coherently isomorphic *if there is an invertible $\alpha : r \to s$ that makes the transitions (r, ω_1, ω_3) and (s, π_1, π_3) coherently isomorphic, and makes the op-transitions (r, ω_2, ω_4) and (s, π_2, π_4) coherently isomorphic.*

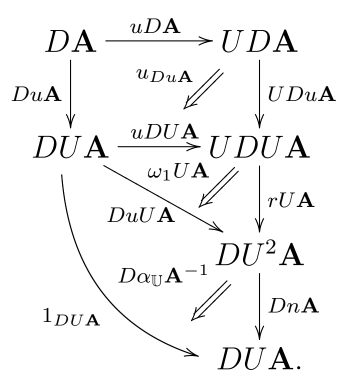
7.4. Theorem. *Let $\mathbb{U}$ be a colax idempotent monad, $\mathbb{D}$ a lax idempotent monad on $\mathcal{K}$, and $(r, \omega_1, \omega_2, \omega_3, \omega_4)$ a distributive law of $\mathbb{U}$ over $\mathbb{D}$. If $(s, \pi_1, \pi_2, \pi_3, \pi_4)$ is the distributive law produced just before Definition 7.3, then $(r, \omega_1, \omega_2, \omega_3, \omega_4)$ and $(s, \pi_1, \pi_2, \pi_3, \pi_4)$ are coherently isomorphic distributive laws.*

Proof. Theorem 6.1 already gives us (r, ω_1, ω_3) and (s, π_1, π_3) coherently isomorphic by the 2-cell

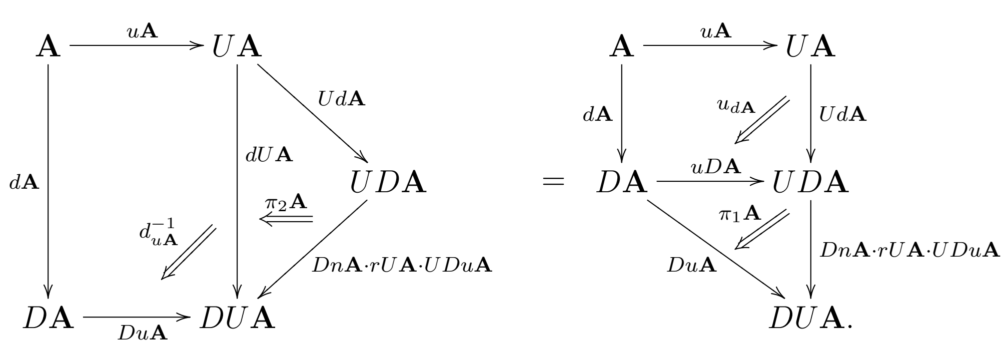


(22)

We must show that it also makes (r, ω_2, ω_4) and (s, π_2, π_4) coherently isomorphic. We have that $s\mathbf{A} = Dn\mathbf{A} \cdot rU\mathbf{A} \cdot UDu\mathbf{A}$ and, π_1 is the pasting



To show that (22) at $\mathbf{A}$ pasted with π_2 equals ω_2 we use the fact that $d_{u\mathbf{A}}^{-1}$ exhibits $dU\mathbf{A}$ as a right Kan extension of $Du\mathbf{A} \cdot d\mathbf{A}$ along $u\mathbf{A}$ and the defining equation of π_2, namely



The case for ω_3 and π_3 is similar to the one just shown. ∎

7.5. Remark. Of course we still have not shown that Definition 7.3 is good, in the sense that the structures induced (liftings, composite pseudomonads, coherent structures) are essentially the same for two coherently isomorphic distributive laws. However, this would take us too far from the objectives of the present paper. We defer the treatment of this issue to a paper that will deal with the "no-iteration" version of the algebras for a general pseudomonad, and the corresponding version of a distributive law.

8. Example

Let $\mathbb{U}$ be coFam on **Cat**. That is, $U\colon \mathrm{Ob}(\mathbf{Cat}) \to \mathrm{Ob}(\mathbf{Cat})$ is given as follows. For a category $\mathbf{A}$, the objects of $U\mathbf{A}$ are finite families $\langle A_i\rangle_{i\in I}$ of objects of $\mathbf{A}$. A morphism $\langle A_i\rangle_{i\in I} \to \langle B_j\rangle_{j\in J}$ in $U\mathbf{A}$ consists of a function $\varphi\colon J \to I$ together with a family of morphisms $\langle f_j\colon A_{\varphi(j)} \to B_j\rangle_{j\in J}$ in $\mathbf{A}$. The identity on $\langle A_i\rangle_{i\in I}$ is $(1_I, \langle 1_{A_i}\rangle_{i\in I})$, whereas composition of $(\varphi, \langle f_j\rangle_{j\in J})\colon \langle A_i\rangle_{i\in I} \to \langle B_j\rangle_{j\in J}$ and $(\psi, \langle g_k\rangle_{k\in K})\colon \langle B_j\rangle_{j\in J} \to \langle C_k\rangle_{k\in K}$ is $(\varphi\psi, \langle g_k \cdot f_{\psi(k)}\rangle_{k\in K})$.

The functor $u\mathbf{A}\colon \mathbf{A} \to U\mathbf{A}$ sends an object A to the family with exactly one element $\langle A\rangle_{\{*\}}$, and $f\colon A \to B$ to $(1_{\{*\}}, \langle f\rangle_{\{*\}})$.

We observe that $U\mathbf{A}$ has finite products. Given a finite set I, and for every $i \in I$ an element $\langle A_{ij}\rangle_{j\in J_i}$ in $U\mathbf{A}$, then

$$\prod_{i\in I}\langle A_{ij}\rangle_{j\in J_i} = \langle A_{ij}\rangle_{(i,j)\in\coprod_{i\in I} J_i},$$

with the i-th projection given by

$$(\sigma_i\colon J_i \to \textstyle\coprod_{i\in I} J_i, \langle 1_{A_{ij}}\rangle_{j\in J_i})\colon \langle A_{ij}\rangle_{(i,j)\in\coprod_{i\in I} J_i} \to \langle A_{ij}\rangle_{j\in J_i}.$$

Given a functor $F\colon \mathbf{B} \to U\mathbf{A}$, $F^{\mathbb{U}}\colon U\mathbf{B} \to U\mathbf{A}$ is such that $F(\langle B_j\rangle_{j\in J}) = \prod_{j\in J} FB_j$, and given a morphism $(\gamma, \langle g_j\rangle_{j\in J})\colon \langle C_k\rangle_{k\in K} \to \langle B_j\rangle_{j\in J}$, define $F^{\mathbb{U}}$ on it such that the diagram

$$\begin{array}{ccc} F^{\mathbb{U}}(\langle C_k\rangle_{k\in K}) & \xrightarrow{\pi_{\gamma(j)}} & FC_{\gamma(j)} \\ {\scriptstyle F^{\mathbb{U}}((\gamma,\langle g_j\rangle_{j\in J}))}\downarrow & & \downarrow{\scriptstyle g_j} \\ F^{\mathbb{U}}(\langle B_j\rangle_{j\in J}) & \xrightarrow{\pi_j} & FB_j \end{array}$$

commutes for every $j \in J$. Then the diagram

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{u\mathbf{B}} & U\mathbf{B} \\ & {\scriptstyle F}\searrow & \downarrow{\scriptstyle F^{\mathbb{U}}} \\ & & U\mathbf{A} \end{array}$$

commutes (provided we make the convention that a unary product is simply the object involved). And it exhibits $F^{\mathbb{U}}$ as a right Kan extension of F along $u\mathbf{B}$. Indeed, given

$$\begin{array}{ccc} \mathbf{B} & \xrightarrow{u\mathbf{B}} & U\mathbf{B} \\ & {\scriptstyle F}\searrow \ \overset{\theta}{\Swarrow} & \downarrow{\scriptstyle H} \\ & & U\mathbf{A}, \end{array}$$

then the unique natural transformation $\widehat{\theta} : H \to F^{\mathbb{U}}$ that preceded by $u\mathbf{A}$ is θ, is given, at $\langle B_j \rangle_{j \in J}$, by the morphism that makes the diagram

$$\begin{array}{ccc} H(\langle B_j \rangle_{j \in J}) & \xrightarrow{H((\ulcorner j \urcorner, \langle 1_{B_j} \rangle_{\{*\}}))} & H(\langle B_j \rangle_{\{*\}}) \\ {\scriptstyle \widehat{\theta}\langle B_j \rangle_{j \in J}}\downarrow & & \downarrow{\scriptstyle \theta B_j} \\ F^{\mathbb{U}}(\langle B_j \rangle_{j \in J}) & \xrightarrow[F^{\mathbb{U}}((\ulcorner j \urcorner, \langle 1_{B_j} \rangle_{\{*\}}))]{} & F^{\mathbb{U}}(\langle B_j \rangle_{j \in J}) \end{array}$$

commute for all $j \in J$.

It is a routine exercise to verify that $F^{\mathbb{U}} : U\mathbf{B} \to U\mathbf{A}$ preserves finite products, and that $\langle B_j \rangle_{j \in J} = \prod_{j \in J} \langle B_j \rangle_{\{*\}}$ in $U\mathbf{B}$. Then we can verify condition b) of Theorem 3.1. Indeed, given $G : \mathbf{C} \to U\mathbf{B}$, $H : U\mathbf{C} \to U\mathbf{A}$ and $\theta : H \cdot u\mathbf{C} \to F^{\mathbb{U}} \cdot G$, then the unique natural transformation $\widehat{\theta} : H \to F^{\mathbb{U}} \cdot G^{\mathbb{U}}$ that preceded by $u\mathbf{C}$ is θ is given, at $\langle C_i \rangle_{i \in I}$ in $U\mathbf{C}$, by the unique arrow that makes the diagram

$$\begin{array}{ccc} H(\langle C_i \rangle_{i \in I}) & \xrightarrow{H(\pi_i)} & H(\langle C_i \rangle_{\{*\}}) \\ {\scriptstyle \widehat{\theta}\langle C_i \rangle_{i \in I}}\downarrow & & \downarrow{\scriptstyle \theta C_i} \\ F^{\mathbb{U}}G^{\mathbb{U}}(\langle C_i \rangle_{i \in I}) & \xrightarrow[F^{\mathbb{U}}G^{\mathbb{U}}(\pi_i)]{} & F^{\mathbb{U}}G^{\mathbb{U}}(\langle C_i \rangle_{\{*\}}) \end{array}$$

commute for all $i \in I$.

We have shown, using the techniques of this paper, that $\mathbb{U}$ is a colax idempotent monad. It is well known that the algebras for $\mathbb{U}$ are categories with finite products and functors that preserve finite products.

Dually, as $\mathbb{D}$ we take Fam. Thus $D\mathbf{A} = (U(\mathbf{A}^{\mathrm{op}}))^{\mathrm{op}}$, and the rest of the structure can be read from this from the description of $\mathbb{U}$. Of course, $\mathbb{D}$ is a lax idempotent monad.

It is well known that there is a distributive law of $\mathbb{U}$ over $\mathbb{D}$; the main ingredient being the fact that if $\mathbf{A}$ has finite products then Fam$\mathbf{A}$ also has finite products. Here we verify the conditions of this paper.

We observe first that $DU\mathbf{A}$ has finite products. Indeed, given a finite set I, and for every $i \in I$ an element $\langle \langle A_{ijk} \rangle_{k \in K_{ij}} \rangle_{j \in J_i}$ in $DU\mathbf{A}$, then the product of the family is given by the object

$$\langle \langle A_{it(i)k} \rangle_{k \in \coprod_{i \in I} K_{i,t(i)}} \rangle_{t \in \prod_{i \in I} J_i},$$

with the i-th projection given by the projection $\pi_i : \prod_{i\in I} J_i \to J_i$ together with, for every $t \in \prod_{i\in I} J_i$, the morphism

$$(\sigma_i : K_{it(i)} \to \coprod_{i\in I} K_{it(i)}, \langle 1_{A_{it(i)k}} \rangle_{k\in K_{it(i)}}) : \langle A_{it(i)k} \rangle_{k\in \coprod_{i\in I} K_{i,t(i)}} \to \langle A_{ijk} \rangle_{k\in K_{ij}}.$$

It is not hard to verify that the conditions of Lemma 7.1 are satisfied. Indeed, to see that $d_{u\mathbf{A}}^{-1}$ exhibits $dU\mathbf{A}$ as a right Kan extension of $Du\mathbf{A} \cdot d\mathbf{A}$ along $u\mathbf{A}$, take $\theta : H \cdot u\mathbf{A} \to Du\mathbf{A} \cdot d\mathbf{A}$, then the unique 2-cell $\widehat{\theta} : H \to dU\mathbf{A}$ that pasted with $d_{u\mathbf{A}}^{-1}$ produces θ is constructed as follows. Given $\langle A_i \rangle_{i\in I}$ in $U\mathbf{A}$, we observe that

$$\prod_{i\in I} \langle \langle A_i \rangle_{\{*\}} \rangle_{\{*\}} = \langle \langle A_i \rangle_{i\in I} \rangle_{\{*\}}$$

in $DU\mathbf{A}$. Thus $\widehat{\theta}\langle A_i \rangle_{i\in I} : H(\langle A_i \rangle_{i\in I}) \to dU\mathbf{A}(\langle A_i \rangle_{i\in I}) = \langle \langle A_i \rangle_{i\in I} \rangle_{\{*\}}$ is the unique arrow such that the diagram

$$\begin{array}{ccc}
H(\langle A_i \rangle_{i\in I}) & \xrightarrow{H((\ulcorner i \urcorner, \langle A_i \rangle_{\{*\}}))} & H(\langle A_i \rangle_{\{*\}}) \\
{\scriptstyle \widehat{\theta}\langle A_i \rangle_{i\in I}} \downarrow & & \downarrow {\scriptstyle \theta A_i} \\
\langle \langle A_i \rangle_{i\in I} \rangle_{\{*\}} & \xrightarrow[\pi_i]{} & \langle \langle A_i \rangle_{\{*\}} \rangle_{\{*\}}
\end{array}$$

commutes.

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