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CANONICAL AND OP-CANONICAL LAX ALGEBRAS

GAVIN J. SEAL

ABSTRACT. The definition of a category of $(\mathsf{T}, \mathbf{V})$-algebras, where $\mathbf{V}$ is a unital commutative quantale and T is a **Set**-monad, requires the existence of a certain lax extension of T. In this article, we present a general construction of such an extension. This leads to the formation of two categories of $(\mathsf{T}, \mathbf{V})$-algebras: the category $\mathbf{Alg}(\mathsf{T}, \mathbf{V})$ of *canonical* $(\mathsf{T}, \mathbf{V})$*-algebras*, and the category $\mathbf{Alg}(\mathsf{T}', \mathbf{V})$ of *op-canonical* $(\mathsf{T}, \mathbf{V})$*-algebras*. The usual topological-like examples of categories of $(\mathsf{T}, \mathbf{V})$-algebras (preordered sets, topological, metric and approach spaces) are obtained in this way, and the category of closure spaces appears as a category of canonical $(\mathsf{P}, \mathbf{V})$-algebras, where P is the powerset monad. This unified presentation allows us to study how these categories are related, and it is shown that under suitable hypotheses both $\mathbf{Alg}(\mathsf{T}, \mathbf{V})$ and $\mathbf{Alg}(\mathsf{T}', \mathbf{V})$ embed coreflectively into $\mathbf{Alg}(\mathsf{P}, \mathbf{V})$.

1. Introduction

Following the description by Manes [11] of the category of compact Hausdorff spaces as the Eilenberg-Moore category of the ultrafilter monad U, Barr [1] showed that by weakening the axioms used to define a monad and its algebras, the resulting Eilenberg-Moore category could be seen to be isomorphic to the category **Top** of topological spaces. The category **Met** of premetric spaces benefitted from a similar treatment in Lawvere's fundamental paper [9], via the identity monad I this time. In recent years, Clementino, Hofmann, and Tholen [2, 6, 5] extended these results and provided a unified setting that presented each of these categories as a particular instance of the category $\mathbf{Alg}(\mathsf{T}, \mathbf{V})$ of so-called $(\mathsf{T}, \mathbf{V})$*-algebras*, where T is a **Set**-monad and $\mathbf{V}$ a unital commutative quantale. For example, if $\mathbf{V}$ is the two-element lattice $\mathbf{2}$, the category $\mathbf{Alg}(\mathsf{T}, \mathbf{2})$ is isomorphic to either the category **Ord** of preordered sets or to **Top**, by taking T to be the identity or the ultrafilter monad, respectively. In the same way, if $\mathbf{V}$ is the extended real half-line $\overline{\mathbf{R}}_+$, then $\mathbf{Alg}(\mathsf{T}, \overline{\mathbf{R}}_+)$ is isomorphic to either **Met** or to the category **App** of approach spaces depending on whether $\mathsf{T} = \mathsf{I}$ or U.

Although the scope of this unified setting is striking, closure spaces do not seem to appear as such $(\mathsf{T}, \mathbf{V})$-algebras. This gap comes as a surprise, since all the mentioned structures are intimately linked to certain "closure-like" operators. Also, the powerset monad—which is a natural candidate for T—does not appear to provide any meaningful

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example. A similar situation seems to occur for the filter monad F, whose corresponding $(\mathsf{F},\mathbf{V})$-algebras lack a certain monotonicity condition to describe topological spaces conveniently. Note however that by lifting the theory to a larger setting (see [13] and [8]), it is possible to include the monotonicity condition in the definition of the algebras.

The original intent of the present work was to close the "closure space/powerset" gap by showing that closure spaces *could* be described as $(\mathsf{T},\mathbf{V})$-algebras via the powerset monad, modulo a slight modification in the axioms used to define $(\mathsf{T},\mathbf{V})$-algebras. In the process, a crucial property of lax algebras appeared, namely that *the monotonicity condition is a consequence of the reflexivity and transitivity of the structure matrices.* This led to reconsider the case of the filter monad, and it resulted that **Top** could be shown to be isomorphic to a category of $(\mathsf{F},\mathbf{2})$-algebras without the use of any additional construction. By investigating further the algebras related to this monad, it also followed that **App** could be described as a category of $(\mathsf{F},\overline{\mathbf{R}}_+)$-algebras.

In fact, an important aspect of the theory was beginning to emerge. Indeed, before considering the category $\mathbf{Alg}(\mathsf{T},\mathbf{V})$ itself, a certain extension of the monad T is required. One approach to the existence of such an extension is discussed in [3], and unicity is obtained for $\mathbf{V}=\mathbf{2}$. With the weaker axioms introduced here however, it is possible to put forth two other extensions of T by assuming similar conditions on T and $\mathbf{V}$, but with very different techniques. Because all the significant examples may be obtained in this manner, the extensions described here are called the *canonical* and *op-canonical* extensions of T, depending on whether the structures of the resulting $(\mathsf{T},\mathbf{V})$-algebras are monotone increasing or decreasing in their first variable. For example, the category **Clos** of closure spaces may be obtained as a $(\mathsf{P},\mathbf{2})$-algebra via the canonical extension of P, and the category **Top** as a $(\mathsf{F},\mathbf{2})$-algebra via the op-canonical extension of F. Although all the examples mentioned in this introduction are isomorphic to canonical $(\mathsf{T},\mathbf{V})$-algebras, where T is one of I, U or P, they may also be described as either canonical or op-canonical $(\mathsf{T},\mathbf{V})$-algebras, where T is one of F or P (in all cases, $\mathbf{V}$ is either $\mathbf{2}$ or $\overline{\mathbf{R}}_+$). Moreover, a new category appears: the category **Clsn** of *closeness spaces* whose objects are the metric counterpart of closure spaces, in the same way that approach spaces are the metric counterpart of topological spaces.

Thus, denoting by $\mathbf{Alg}(\mathsf{T},\mathbf{V})$ the category of canonical $(\mathsf{T},\mathbf{V})$-algebras, and by $\mathbf{Alg}(\mathsf{T}',\mathbf{V})$ the category of op-canonical $(\mathsf{T},\mathbf{V})$-algebras, we obtain the following list of isomorphisms:

$$\begin{array}{rr}
\mathbf{Ord}\cong\mathbf{Alg}(\mathsf{I},\mathbf{2})\cong\mathbf{Alg}(\mathsf{P}',\mathbf{2}) & \mathbf{Met}\cong\mathbf{Alg}(\mathsf{I},\overline{\mathbf{R}}_+)\cong\mathbf{Alg}(\mathsf{P}',\overline{\mathbf{R}}_+)\\
\mathbf{Top}\cong\mathbf{Alg}(\mathsf{U},\mathbf{2})\cong\mathbf{Alg}(\mathsf{F}',\mathbf{2}) & \mathbf{App}\cong\mathbf{Alg}(\mathsf{U},\overline{\mathbf{R}}_+)\cong\mathbf{Alg}(\mathsf{F}',\overline{\mathbf{R}}_+)\\
\mathbf{Clos}\cong\mathbf{Alg}(\mathsf{P},\mathbf{2})\cong\mathbf{Alg}(\mathsf{F},\mathbf{2}) & \mathbf{Clsn}\cong\mathbf{Alg}(\mathsf{P},\overline{\mathbf{R}}_+)\cong\mathbf{Alg}(\mathsf{F},\overline{\mathbf{R}}_+)
\end{array}$$

From a general point of view, it is possible to determine a certain number of adjunctions between these categories which result in either embeddings or isomorphisms. In particular, the category of op-canonical $(\mathsf{P},\mathbf{V})$-algebras—which is isomorphic to $\mathbf{Alg}(\mathsf{I},\mathbf{V})$—embeds as a full coreflective subcategory into the category of op-canonical $(\mathsf{T},\mathbf{V})$-algebras, and under a suitable hypothesis, the category of canonical $(\mathsf{T},\mathbf{V})$-algebras embeds as a full coreflective subcategory into the category of canonical $(\mathsf{P}, \mathbf{V})$-algebras. For $\mathsf{T} = \mathsf{I}$ or U, the $(\mathsf{T}, \mathbf{V})$-algebras are both canonical and op-canonical, so the mentioned results are illustrated by the two horizontal lines in the following commutative diagram of coreflective embeddings:

$$\begin{array}{ccccc}
\mathbf{Ord} & \longrightarrow & \mathbf{Top} & \longrightarrow & \mathbf{Clos} \\
\downarrow & & \downarrow & & \downarrow \\
\mathbf{Met} & \longrightarrow & \mathbf{App} & \longrightarrow & \mathbf{Clsn}
\end{array}$$

where the vertical arrows are induced by the coreflective embedding $E : \mathbf{2} \to \overline{\mathbf{R}}_+$, as described in [5].

The general theory pertaining to these results is presented in Sections 2 to 4. Sections 5 and 6 contain the applications of the theory to the powerset and filter monads, although these are also used throughout the previous sections to illustrate the different definitions introduced therein.

2. Lax algebras

There are two major differences between the definition of lax algebras given in [5] or [6], and the weaker one given here. First, it is sufficient for our purpose to work with a lax extension T_M of a **Set**-functor $T : \mathbf{Set} \to \mathbf{Set}$ rather than with a $\mathbf{V}$-admissible monad; indeed, the close interplay occurring between **Set** and $\mathbf{Mat}(\mathbf{V})$ allows us to use the properties of the original monad, without reference to the op-laxness in $\mathbf{Mat}(\mathbf{V})$ of either its unit or multiplication (see for example the proof of the monotonicity of a lax algebra's structure matrix in 2.6, or the proof of Proposition 2.7). Second, in order to include closure spaces as models of lax algebras, the lax functor $T_M : \mathbf{Mat}(\mathbf{V}) \to \mathbf{Mat}(\mathbf{V})$ must not extend the functor $T : \mathbf{Set} \to \mathbf{Set}$ strictly, nor commute with the involution $^\circ$; the replacement conditions are given in (1) below. For the sake of completeness and in order to settle the notations, we begin by recalling the main definitions and results pertaining to $(\mathsf{T}, \mathbf{V})$-algebras.

2.1. Quantales. Throughout this article, $\mathbf{V}$ will denote a unital commutative quantale, *i.e.* a complete lattice provided with an associative and commutative binary operation $\otimes$ which preserves suprema in each variable:

$$a \otimes \bigvee_{i \in I} b_i = \bigvee_{i \in I} (a \otimes b_i),$$

and for which there is a neutral element k. The bottom and top elements of $\mathbf{V}$ are denoted by $\bot$ and $\top$ respectively.

For example, the two-element chain $\mathbf{2} = \{\bot, \top\}$ with $x \otimes y$ being the infimum of x and y, and $k = \top$ is a suitable candidate for $\mathbf{V}$.

The extended half-line $\overline{\mathbf{R}}_+ = [0,\infty]$, considered for our purpose with the order opposite to the natural order, with $\otimes$ being the addition (for which ∞ is an absorbing element) and k the top element 0, is another candidate for $\mathbf{V}$.

2.2. $\mathbf{V}$-MATRICES. The objects of the category $\mathbf{Mat}(\mathbf{V})$ are sets, and the morphisms $r : X \nrightarrow Y$ are functions $r : X \times Y \to \mathbf{V}$; these morphisms will often be referred to as $\mathbf{V}$-*matrices*, (or simply as *matrices*) since they may be viewed as matrices $(r(x,y))_{x\in X, y\in Y}$. Composition is given by

$$(sr)(x,z) = \bigvee_{y\in Y} r(x,y) \otimes s(y,z) \ ,$$

where $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$. The identity $1_X : X \nrightarrow X$ is defined by $1_X(x,y) = k$ if $x = y$ and $1_X(x,y) = \bot$ otherwise.

There is a partial order on the hom-sets of $\mathbf{Mat}(\mathbf{V})$ induced by the partial order on $\mathbf{V}$, and given by $r \le r'$ if and only if $r(x,y) \le r'(x,y)$ for all $x \in X, y \in Y$; this order is compatible with composition. There is also an order-preserving involution sending a morphism $r : X \nrightarrow Y$ to its *transpose* $r^\circ : Y \nrightarrow X$ defined by $r^\circ(y,x) = r(x,y)$. Note that $(1_X)^\circ = 1_X$ and $(sr)^\circ = r^\circ s^\circ$ by commutativity of $\otimes$.

Finally, there is a functor $M : \mathbf{Set} \to \mathbf{Mat}(\mathbf{V})$ which maps objects identically and sends a map $f : X \to Y$ to the matrix $f : X \nrightarrow Y$ given by

$$f(x,y) = \begin{cases} k & \text{if } f(x) = y \\ \bot & \text{otherwise.} \end{cases}$$

Naturally, the functor M sends the identity map to the identity matrix. Since it will always be possible to deduce from the context whether we are working with a $\mathbf{Set}$-map $f : X \to Y$ or its image $f : X \nrightarrow Y$, we will not use the notation $Mf : X \nrightarrow Y$. Thus, by composing a map $f : X \to Y$, a matrix $s : Y \nrightarrow Z$, and the transpose of a map $g : Y \to Z$, we get the convenient formula $(g^\circ sf)(x,y) = s(f(x), g(y))$. Notice also that $1_X \le f^\circ f$ and $ff^\circ \le 1_X$, so for $t : X \nrightarrow Z$ we have

$$t \le sf \iff tf^\circ \le s \qquad \text{and} \qquad gr \le t \iff r \le g^\circ t \ .$$

These properties may be used to obtain the pointwise notation of the various conditions presented further on (see [5] for details).

2.3. MONADS. A $\mathbf{Set}$-monad T is a triple (T, e, m), where $T : \mathbf{Set} \to \mathbf{Set}$ is a functor, and the *unit* $e : \mathrm{Id} \to T$ and *multiplication* $m : TT \to T$ of T are natural transformations satisfying

$$m(Te) = 1 = m(eT) \qquad \text{and} \qquad m(Tm) = m(mT) \ .$$

The *identity monad* I is simply the triple $(\mathrm{Id}, 1, 1)$.

The *powerset monad* $\mathsf{P} = (P, e, m)$ is defined as follows. The powerset functor P assigns to a set X the set PX of subsets of X, and sends a map $f : X \to Y$ to Pf :

$PX \to PY$ defined by $Pf(A) = \{f(x) \mid x \in A\}$, where $A \subseteq X$. For $x \in X$ and $\mathcal{A} \in PPX$, the maps e_X and m_X are given by

$$e_X(x) = \{x\} \qquad \text{and} \qquad m_X(\mathcal{A}) = \bigcup \mathcal{A} \ .$$

The *filter monad* $\mathsf{F} = (F, e, m)$ is defined as follows. The filter functor F assigns to a set X the set FX of filters on X, and sends a map $f : X \to Y$ to $Ff : FX \to FY$ defined by $A \in Ff(\mathfrak{f}) \iff f^{-1}(A) \in \mathfrak{f}$, where $\mathfrak{f} \in FX$. The maps e_X and m_X are given by

$$A \in e_X(x) \iff x \in A \qquad \text{and} \qquad A \in m_X(\mathfrak{F}) \iff A^\sharp \in \mathfrak{F} \ ,$$

where $A^\sharp = \{\mathfrak{f} \in FX \mid A \in \mathfrak{f}\}$, $x \in X$ and $\mathfrak{F} \in FFX$.

Finally, the *ultrafilter monad* $\mathsf{U} = (U, e, m)$ is defined similarly to the filter monad, by replacing the filter functor F by the ultrafilter functor U which assigns to a set X the set of ultrafilters on X.

2.4. Lax extensions of T. A *lax extension* of a **Set**-functor T is a map

$$\begin{aligned} T_{\!_M} : \mathbf{Mat}(\mathbf{V}) &\to \mathbf{Mat}(\mathbf{V}) \\ (r : X \nrightarrow Y) &\mapsto (T_{\!_M} r : TX \nrightarrow TY) \end{aligned}$$

which preserves the partial order on the hom-sets and satisfies

(1) $Tf \leq T_{\!_M} f$ and $(Tf)^\circ \leq T_{\!_M} f^\circ$,

(2) $(T_{\!_M} s)(T_{\!_M} r) \leq T_{\!_M}(sr)$

for all $f : X \to Y$, $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$. A **Set**-monad $\mathsf{T} = (T, e, m)$ equipped with a lax extension $T_{\!_M}$ of T will be called a *lax extension* of (T, e, m), and will be denoted slightly abusively by $\mathsf{T} = (T_{\!_M}, e, m)$. It should be stressed however that $(T_{\!_M}, e, m)$ is not a lax monad in the sense of [3]; in particular, e and m need not be op-lax in $\mathbf{Mat}(\mathbf{V})$ (see however Proposition 3.5 below).

In the presence of a **Set**-map, (2) may become an equality and allow part of (1) to be treated as such. Indeed, if $f : X \nrightarrow Y$ and $g : Y \nrightarrow Z$ come from **Set**-maps, then

$$T_{\!_M}(sf) = (T_{\!_M} s)(Tf) = (T_{\!_M} s)(T_{\!_M} f) \qquad \text{and} \qquad T_{\!_M}(g^\circ r) = (Tg)^\circ (T_{\!_M} r) = (T_{\!_M} g^\circ)(T_{\!_M} r) \ .$$

The first set of equalities follows from

$$\begin{aligned} T_{\!_M}(sf) &\leq T_{\!_M}(sf)(Tf)^\circ(Tf) \leq T_{\!_M}(sf)(T_{\!_M} f^\circ)(Tf) \\ &\leq (T_{\!_M}(sff^\circ))(Tf) \leq (T_{\!_M} s)(Tf) \leq (T_{\!_M} s)(T_{\!_M} f) \leq T_{\!_M}(sf) \ , \end{aligned}$$

and the second is obtained similarly.

Notice that if $T_{\!_M} : \mathbf{Mat}(\mathbf{V}) \to \mathbf{Mat}(\mathbf{V})$ is a lax extension of T, then $T'_{\!_M} : \mathbf{Mat}(\mathbf{V}) \to \mathbf{Mat}(\mathbf{V})$ given by

$$T'_{\!_M} r := (T_{\!_M} r^\circ)^\circ \ ,$$

is also a lax extension of T. As we will show further on, these two extensions are not necessarily equal.

2.5. THE INDUCED PREORDER. A lax extension $T_{\!_M}$ of a **Set**-functor T induces the following preorder on the set TX:

$$\mathfrak{x} \leq \mathfrak{y} \iff k \leq T_{\!_M} 1_X(\mathfrak{y}, \mathfrak{x}) ,$$

where $\mathfrak{x}, \mathfrak{y} \in TX$. Indeed, the condition (1) yields reflexivity, while (2) yields transitivity. As a consequence, if $\mathfrak{x} \leq \mathfrak{x}'$ and $\mathfrak{y}' \leq \mathfrak{y}$, then $T_{\!_M} r(\mathfrak{x}, \mathfrak{y}) \leq T_{\!_M} r(\mathfrak{x}', \mathfrak{y}')$, which means that $T_{\!_M} r$ preserves this preorder in the first variable and reverses it in the second. Similarly, $T'_{\!_M} r$ reverses it in the first variable and preserves it in the second.

If $f : X \to Y$ is a **Set**-map, and $\mathfrak{x}, \mathfrak{y}$ are elements of TX such that $\mathfrak{x} \leq \mathfrak{y}$, then

$$T_{\!_M} 1_Y(Tf(\mathfrak{y}), Tf(\mathfrak{x})) = (Tf)^\circ(T_{\!_M} 1_Y)(Tf)(\mathfrak{y}, \mathfrak{x}) = T_{\!_M}(f^\circ f)(\mathfrak{y}, \mathfrak{x}) \geq 1_X(\mathfrak{y}, \mathfrak{x}) \geq k ,$$

so that $Tf(\mathfrak{x}) \leq Tf(\mathfrak{y})$. This shows that Tf preserves the preorder on TX (and T may be seen as a functor $T : \mathbf{Set} \to \mathbf{Ord}$).

2.6. LAX ALGEBRAS. For a **Set**-monad $\mathsf{T} = (T, e, m)$ equipped with a lax extension $T_{\!_M}$ of T, the category $\mathbf{Alg}(\mathsf{T}, \mathbf{V})$ of $(\mathsf{T}, \mathbf{V})$-*algebras*, also called *lax algebras*, has as objects pairs (X, r) with X a set and $r : TX \nrightarrow X$ a *structure matrix* satisfying the *reflexivity* and *transitivity* laws:

(3) $1_X \leq r e_X$,

(4) $r(T_{\!_M} r) \leq r m_X$.

A morphism $f : (X, r) \to (Y, s)$ is a **Set**-map $f : X \to Y$ satisfying:

(5) $r \leq f^\circ s(Tf)$,

and composing as in **Set**.

A crucial property of the structure matrix of a lax algebra (X, r) is the preservation of the preorder on TX (in the first variable):

$$\mathfrak{x} \leq \mathfrak{y} \implies r(\mathfrak{x}, z) \leq r(\mathfrak{y}, z) .$$

Indeed, reflexivity of r yields $1_{TX} \leq T_{\!_M} 1_X \leq (T_{\!_M} r)(Te_X)$, so that if $\mathfrak{x}, \mathfrak{y} \in TX$ are such that $\mathfrak{x} \leq \mathfrak{y}$, we have

$$r(\mathfrak{x}, z) \leq T_{\!_M} 1_X(\mathfrak{y}, \mathfrak{x}) \otimes r(\mathfrak{x}, z) \leq T_{\!_M} r(Te_X(\mathfrak{y}), \mathfrak{x}) \otimes r(\mathfrak{x}, z) \leq r(\mathfrak{y}, z) ,$$

by transitivity of r. Moreover, if the previous monad T is replaced by the monad $\mathsf{T}' = (T'_{\!_M}, e, m)$ (but the preorder on TX is still defined via $T_{\!_M}$), then a similar argument yields that r reverses the preorder on TX:

$$\mathfrak{x} \leq \mathfrak{y} \implies r(\mathfrak{y}, z) \leq r(\mathfrak{x}, z) .$$

If k is the top element of $\mathbf{V}$, then a morphism $f : (X, r) \to (Y, s)$ of $(\mathsf{T}, \mathbf{V})$-algebras may be defined by using $T_{\!_M}$ in place of T. Indeed, in this case $s(T_{\!_M} 1_Y)(\mathfrak{x}, z) = \bigvee_{\mathfrak{y} \leq \mathfrak{x}} s(\mathfrak{y}, z) = s(\mathfrak{x}, z)$, so that $f^\circ s(Tf) = f^\circ s(T_{\!_M} 1_Y)(Tf) = f^\circ s(T_{\!_M} f)$. The same argument holds in the case where $f : (X, r) \to (Y, s)$ is a morphism of $(\mathsf{T}', \mathbf{V})$-algebras.

2.7. Proposition. *The category of* $(\mathsf{T},\mathbf{V})$*-algebras is topological. In fact, for a family of* $(Y_i,s_i)_{i\in I}$ *of* $(\mathsf{T},\mathbf{V})$*-algebras and* $(f_i : X \to Y_i)_{i\in I}$ *of* **Set***-maps, the initial structure* r *on* X *can be described by* $r = \bigwedge_{i\in I} f_i^\circ s_i(Tf_i)$*, or*

$$r(\mathfrak{x},y) = \bigwedge_{i\in I} s_i(Tf_i(\mathfrak{x}), f_i(y))$$

in pointwise notation, where $\mathfrak{x}\in TX$ *and* $y \in X$*.*

Proof. This result may be proved as in [5]. ∎

2.8. Examples of $(\mathsf{T},\mathbf{V})$-algebras. Since a $(\mathsf{T},\mathbf{V})$-algebra in the sense of [5] is a $(\mathsf{T},\mathbf{V})$-algebra in the above sense, we have the following examples.

The category $\mathbf{Alg}(\mathsf{I},\mathbf{2})$ is the category **Ord** of preordered sets, and $\mathbf{Alg}(\mathsf{I},\overline{\mathbf{R}}_+)$ is the category **Met** of premetric spaces.

For the ultrafilter monad $\mathsf{U} = (U, e, m)$, we can define the lax extension

$$U_M r(\mathfrak{x},\mathfrak{y}) := \bigwedge_{\substack{A\in\mathfrak{x}\\ B\in\mathfrak{y}}} \bigvee_{\substack{x\in A\\ y\in B}} r(x,y) \ ,$$

where $r : X \nrightarrow Y$, $\mathfrak{x}\in UX$ and $\mathfrak{y}\in UY$. Then $\mathbf{Alg}(\mathsf{U},\mathbf{2})$ is isomorphic to the category **Top** of topological spaces, and $\mathbf{Alg}(\mathsf{U},\overline{\mathbf{R}}_+)$ to the category **App** of approach spaces.

2.9. Taut monads. A functor $T : \mathbf{Set}\to\mathbf{Set}$ is *taut* if it preserves inverse images, *i.e.* for any map $f : X \to Y$ and subset B of Y, the pullback $(Tf)^{-1}(TB)$ is isomorphic to $T(f^{-1}(B))$. As a consequence, if $\iota : A \to X$ is an injection, then $T\iota : TA \to TX$ is one too (this allows us to avoid certain technical difficulties related to injections of empty sets into non-empty sets). In order to simplify notations, if A is a subset of X we will consider TA as a subset of TX, and similarly $(Tf)^{-1}(TB)$ will be identified with $T(f^{-1}(B))$. In the same way, we will write $Tf(TA) \subseteq T(f(A))$ for any $A \subseteq X$. Finally, note that a taut functor preserves finite intersections.

Let $\mathsf{T} = (T, e, m)$ be a **Set**-monad with taut T. If the unit e and multiplication m are *taut*:

$$e_X(y)\in TA \iff y\in A \qquad\text{and}\qquad m_X(\mathfrak{X})\in TA \iff \mathfrak{X}\in TTA$$

for any set X and $A \subseteq X$, then the monad T itself is said to be *taut.* Of course, e is taut if and only if e_X is injective for all X. Moreover, if T is taut, then so is e (see [12], Proposition 2.3).

The identity, powerset, filter and ultrafilter monads are all taut, and the previous identification convention yields natural results for their functors. Indeed, in the case of the powerset functor, we have for $A, B \in PX$ that $A \in PB \iff A \subseteq B$. In the case of the filter functor, we have for $\mathfrak{f}\in FX$ that $\mathfrak{f}\in FA \iff A\in\mathfrak{f}$, so with the notations of 2.3 we may write $A^\sharp = FA$.

2.10. Remark. Since the Beck-Chevalley condition (BC) of [2] has several important consequences in the theory of lax algebras, it is worth mentioning that if a functor satisfies (BC), then it it is naturally taut.

2.11. Completely distributive lattices. Let $\mathbf{V}$ be a complete lattice, and $a, b \in \mathbf{V}$. Define $a \prec b$ whenever the following condition holds:

$$\text{for any subset } S \subseteq \mathbf{V} \text{ with } b \leq \bigvee S, \text{ there exists } s \in S \text{ satisfying } a \leq s \ .$$

The lattice $\mathbf{V}$ is *completely distributive* (see [14]) if for any $b \in \mathbf{V}$, we have

$$b = \bigvee \{a \in \mathbf{V} \mid a \prec b\} \ .$$

It follows immediately from its definition that the relation $\prec$ has the following properties:

i) $a \prec b$ implies $a \leq b$;

ii) $a \leq a' \prec b' \leq b$ implies $a \prec b$;

iii) $a \prec \bigvee S$ implies there exists $s \in S$ with $a \prec s$.

For elements u and v of a completely distributive lattice, if $a \prec v$ for any $a \in \mathbf{V}$ with $a \prec u$, then we can conclude that $u \leq v$ by taking the join of all elements $a \prec u$. This argument will be used systematically in the following without necessarily explicit mention.

Notice that the lattice $\mathbf{2}$ is completely distributive (in which case $\prec$ is $\leq$), and the extended real half-line $\overline{\mathbf{R}}_+$ is too (with $\prec$ being $<$).

3. Canonical constructions

3.1. Canonical extensions. For a $\mathbf{V}$-matrix $r : X \nrightarrow Y$, define

$$r_a[A] := \{y \in Y \mid \text{ there exists } x \in A \text{ with } a \leq r(x, y)\}$$

$$\text{and} \qquad T_{\scriptscriptstyle M} r(\mathfrak{x}, \mathfrak{y}) := \bigvee \{a \in \mathbf{V} \mid \mathfrak{y} \in T(r_a[A]) \text{ for all } A \text{ with } TA \ni \mathfrak{x}\} \ ,$$

where $a \in \mathbf{V}$, $A \subseteq X$, $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$. Since the lax extension $T'_{\scriptscriptstyle M}$ obtained from $T_{\scriptscriptstyle M}$ (see 2.4) will be of importance in the rest of this article, we explicit the corresponding definitions:

$$r^\circ_a[B] := \{x \in X \mid \text{ there exists } y \in B \text{ with } a \leq r(x, y)\}$$

$$\text{and} \qquad T'_{\scriptscriptstyle M} r(\mathfrak{x}, \mathfrak{y}) := \bigvee \{a \in \mathbf{V} \mid \mathfrak{x} \in T(r^\circ_a[B]) \text{ for all } B \text{ with } TB \ni \mathfrak{y}\} \ ,$$

where $a \in \mathbf{V}$, $B \subseteq Y$, $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$. In case r is a $\mathbf{2}$-matrix, we will prefer to write $r[A]$ and $r^\circ[A]$ in place of $r_\top[A]$ and $r^\circ_\top[A]$ respectively. It will be proved further on that both $T_{\scriptscriptstyle M}$ and $T'_{\scriptscriptstyle M}$ form lax extensions of T. The $T_{\scriptscriptstyle M}$ defined above is called the *canonical extension* of T, and $T'_{\scriptscriptstyle M}$ is the *op-canonical extension* of T. Lemma 3.3 shows that if $\mathbf{V}$ is completely distributive, then the sets $r_a[A]$ used to define $T_{\scriptscriptstyle M}$ can be replaced by the smaller sets $r_{\overline{a}}[A]$ defined therein.

Remark that $a \leq b$ implies $r_b[A] \subseteq r_a[A]$, and that $A \subseteq B$ implies $r_a[A] \subseteq r_a[B]$.

3.2. The canonical induced preorder. In the case where $T_{_M}$ is the canonical extension of T, the preorder on TX described in 2.5 is given by

$$\mathfrak{x} \le \mathfrak{y} \iff \text{for any } A \subseteq X, \text{ if } \mathfrak{y} \in TA \text{ then } \mathfrak{x} \in TA ,$$

where $\mathfrak{x}, \mathfrak{y} \in TX$. From now on, this will be the preorder used on TX. Note that it is preserved by m_X whenever m is taut.

This preorder yields the natural order induced by the functors considered in this article. Indeed, if T is the powerset functor P and A, B are subsets of X, then $A \le B \iff A \subseteq B$. If T is the filter functor F and $\mathfrak{x}, \mathfrak{y}$ are filters on X, then $\mathfrak{x} \le \mathfrak{y} \iff \mathfrak{x}$ is finer than $\mathfrak{y}$, since $\mathfrak{x} \in TA$ may be interpreted as $A \in \mathfrak{x}$ (see 2.9). Finally, if T is the identity functor I or the ultrafilter functor U, then $\mathfrak{x} \le \mathfrak{y} \iff \mathfrak{x} = \mathfrak{y}$.

3.3. Lemma. *Let $r : X \nrightarrow Y$ be a $\mathbf{V}$-matrix, and suppose that $\mathbf{V}$ is completely distributive. For $a \in \mathbf{V}$, define*

$$r_{\overline{a}}[A] := \{y \in Y \mid \text{ there exists } x \in A \text{ with } a \prec r(x, y)\} .$$

Then for $\mathfrak{x} \in TX$ and $\mathfrak{y} \in TY$, we have

$$T_{_M} r(\mathfrak{x}, \mathfrak{y}) = \bigvee\{a \in \mathbf{V} \mid \mathfrak{y} \in T(r_{\overline{a}}[A]) \text{ for all } A \text{ with } TA \ni \mathfrak{x}\} .$$

Proof. Define

$$S := \{a \in \mathbf{V} \mid \mathfrak{y} \in T(r_a[A]) \text{ for all } A \text{ with } TA \ni \mathfrak{x}\} \text{ and}$$
$$\overline{S} := \{a \in \mathbf{V} \mid \mathfrak{y} \in T(r_{\overline{a}}[A]) \text{ for all } A \text{ with } TA \ni \mathfrak{x}\} .$$

Since $r_{\overline{a}}[A] \subseteq r_a[A]$, we naturally have $\bigvee \overline{S} \le \bigvee S$.

Let $b \in S$ and $a \prec b$. This yields $r_b[A] \subseteq r_{\overline{a}}[A]$, so that if $\mathfrak{y} \in T(r_b[A])$, then $\mathfrak{y} \in T(r_{\overline{a}}[A])$. Therefore, $a \le \bigvee \overline{S}$ and $b \le \bigvee \overline{S}$ because $\mathbf{V}$ is completely distributive, and we can conclude that $\bigvee S \le \bigvee \overline{S}$. ■

3.4. Lemma. *Let $r : X \nrightarrow Y$ be a $\mathbf{V}$-matrix, and suppose that $\mathbf{V}$ is completely distributive. For any $A \subseteq X$ and $a \in \mathbf{V}$, we have*

$$(T_{_M} r)_{\overline{a}}[TA] \subseteq T(r_{\overline{a}}[A]) ,$$

Proof. If $\mathfrak{y} \in (T_{_M} r)_{\overline{a}}[TA]$, then there exists $\mathfrak{x} \in TA$ with $a \prec T_{_M} r(\mathfrak{x}, \mathfrak{y})$. Thus, there is an element $b \in \mathbf{V}$ with $a \prec b$ such that for all B with $TB \ni \mathfrak{x}$, we have $\mathfrak{y} \in T(r_b[B])$. In particular, $\mathfrak{y} \in T(r_b[A]) \subseteq T(r_{\overline{a}}[A])$, which yields the conclusion. ■

3.5. Proposition. *If T is a taut functor, then $T_{_M}$ and $T'_{_M}$ defined in 3.1 are lax extensions of T. Moreover, if $\mathsf{T} = (T, e, m)$ is a taut monad and* **V** *is completely distributive, then e and m are both op-lax in* **Mat(V)** *with respect to $T_{_M}$, i.e. for any* **V***-matrix $r : X \nrightarrow Y$, we have*

$$r \leq e_Y^\circ (T_{_M} r) e_X \qquad \text{and} \qquad T_{_M}^2 r \leq m_Y^\circ (T_{_M} r) m_X \ ,$$

and it follows that e and m are also op-lax with respect to $T'_{_M}$.

Proof. Let us first prove that $T_{_M}$ is a lax extension. If $r, r' : X \nrightarrow Y$ are two **V**-matrices such that $r \leq r'$, then we have $r_a[A] \subseteq r'_a[A]$ for any $A \subseteq X$ and $a \in \mathbf{V}$, so that $T_{_M} r \leq T_{_M} r'$. Thus, $T_{_M}$ preserves the partial order on the hom-sets.

(1) Consider a map $f : X \to Y$. To show that $Tf(\mathfrak{x}, \mathfrak{y}) \leq T_{_M} f(\mathfrak{x}, \mathfrak{y})$, it is sufficient to consider the case $\mathfrak{y} = Tf(\mathfrak{x})$. Let $A \subseteq X$ be such that $TA \ni \mathfrak{x}$, so $Tf(\mathfrak{x}) \in Tf(TA) \subseteq T(f(A))$ and $\mathfrak{y} \in T(f(A))$. Since $f(A) \subseteq f_a[A]$ for any $a \leq k$, we have $k \leq T_{_M} f(\mathfrak{x}, \mathfrak{y})$ as required.

    To verify that $(Tf)^\circ \leq T_{_M} f^\circ$, suppose as before that $Tf(\mathfrak{x}) = \mathfrak{y}$. For any $B \subseteq Y$ with $\mathfrak{y} \in TB$, we have $\mathfrak{x} \in (Tf)^{-1}(TB) = T(f^{-1}(B))$. Remarking that $f^{-1}(B) \subseteq f_a^\circ[B]$ for all $a \leq k$, we may conclude that $k \leq T_{_M} f^\circ(\mathfrak{y}, \mathfrak{x})$.

(2) Consider two **V**-matrices $r : X \nrightarrow Y$ and $s : Y \nrightarrow Z$. Let $a, b \in \mathbf{V}$ be such that $\mathfrak{y} \in T(r_a[A])$ for all A with $TA \ni \mathfrak{x}$, and $\mathfrak{z} \in T(s_b[B])$ for all B with $TB \ni \mathfrak{y}$. For these A, we have $\mathfrak{z} \in T(s_b[r_a[A]])$. Moreover,

$$\begin{aligned} s_b[r_a[A]] &= \{z \in Z \mid \text{ there exist } x \in A, y \in Y \text{ with } a \leq r(x,y) \text{ and } b \leq s(y,z)\} \\ &\subseteq \{z \in Z \mid \text{ there exists } x \in A \text{ with } a \otimes b \leq (sr)(x,z)\} = (sr)_{a \otimes b}[A] \ . \end{aligned}$$

    Since $\otimes$ preserves suprema in each variable, we get $(T_{_M} s)(T_{_M} r) \leq T_{_M}(sr)$ by taking the join of all $a, b \in \mathbf{V}$ chosen as above.

Therefore, $T_{_M}$ is a lax extension of T. But it also follows that $T'_{_M}$ is a lax extension of T (see the concluding remark of 2.4).

To check that e is op-lax, consider a **V**-matrix $r : X \nrightarrow Y$, $x \in X$, $y \in Y$ and $a = r(x, y)$. Thus, $y \in r_a[A]$ for all $A \ni x$. If $e_X(x) \in TA$, then $x \in A$ by injectivity of e_X, so that $y \in r_a[A]$. This implies that $a \leq T_{_M} r(e_X(x), e_Y(y))$, and we can conclude that $r \leq e_Y^\circ (T_{_M} r) e_X$ as required. It also follows directly that $r \leq e_Y^\circ (T'_{_M} r) e_X$.

To check that m is op-lax, let $\mathfrak{X} \in TTX$, $\mathfrak{Y} \in TTY$ and $a \in \mathbf{V}$ such that $a \prec T_{_M}^2 r(\mathfrak{X}, \mathfrak{Y})$. Thus, if $TTA \ni \mathfrak{X}$, then $\mathfrak{Y} \in T((T_{_M} r)_{\overline{a}}[TA]) \subseteq TT(r_{\overline{a}}[A])$ by Lemma 3.4. This implies that $m_Y(\mathfrak{Y}) \in T(r_{\overline{a}}[A])$ for all A with $TTA \ni \mathfrak{X}$, or equivalently for all A with $TA \ni m_X(\mathfrak{X})$ because m is taut. Lemma 3.3 then yields that $a \leq T_{_M} r(m_X(\mathfrak{X}), m_Y(\mathfrak{Y}))$, so $T_{_M}^2 r \leq m_Y^\circ (T_{_M} r) m_X$ by complete distributivity of **V**. The corresponding inequality for $T'_{_M}$ easily follows. ∎

3.6. Canonical and op-canonical $(\mathsf{T},\mathbf{V})$-algebras. Consider the canonical and op-canonical extensions $T_{_M}$ and $T'_{_M}$ of T. If $\mathsf{T}=(T_{_M},e,m)$ and $\mathsf{T}'=(T'_{_M},e,m)$, then a $(\mathsf{T},\mathbf{V})$-algebra will be called a *canonical* $(\mathsf{T},\mathbf{V})$*-algebra*, and a $(\mathsf{T}',\mathbf{V})$-algebra will be called an *op-canonical* $(\mathsf{T},\mathbf{V})$*-algebra*. Recall from 2.6 that the structure matrix of a canonical $(\mathsf{T},\mathbf{V})$-algebra preserves the preorder on TX in its first variable, and the structure matrix of an op-canonical $(\mathsf{T},\mathbf{V})$-algebra reverses it.

Examples of canonical and op-canonical $(\mathsf{T},\mathbf{V})$-algebras will be studied for the powerset and the filter monads in the last two sections. Notice that it may happen that a lax algebra is at the same time a canonical and op-canonical $(\mathsf{T},\mathbf{V})$-algebra: this is the case for the identity monad for which $\mathrm{Id}_{_M}=\mathrm{Id}'_{_M}=\mathrm{Id}$, or the ultrafilter monad for which $U'_{_M}r=U_{_M}r$ (see Lemma 6.2); these situations are particular cases of the next proposition. Notice also that there exist $(\mathsf{T},\mathbf{V})$-algebras that are neither canonical nor op-canonical: this is the case for example for the lax algebras corresponding to the extension of Id considered in [5], Remark 3.2.

3.7. Proposition. *Suppose that* $T\emptyset=\emptyset$ *and for* $\mathfrak{x}\in TX$,

$$TB\cap TA\neq\emptyset \text{ for all } A \text{ with } TA\ni\mathfrak{x} \implies \mathfrak{x}\in TB\ ,$$

where $B\subseteq X$. *Then the canonical and op-canonical extensions of* T *are equal.*

Proof. Consider a $\mathbf{V}$-matrix $r:X\nrightarrow Y$, $x\in TX$, $\mathfrak{y}\in TY$, and $a\in\mathbf{V}$. Suppose that a is such that $\mathfrak{y}\in T(r_a[A])$ for all A with $TA\ni\mathfrak{x}$. If moreover $TB\ni\mathfrak{y}$, then $\mathfrak{y}\in T(r_a[A]\cap B)$ because T preserves intersections, so that $T\emptyset=\emptyset$ implies $r_a[A]\cap B\neq\emptyset$. Thus, for all A and B with $TA\ni\mathfrak{x}$ and $TB\ni\mathfrak{y}$, there exist $x\in A$ and $y\in B$ such that $a\leq r(x,y)$, or $A\cap r^\circ_a[B]\neq\emptyset$. We then have $TA\cap T(r^\circ_a[B])\neq\emptyset$ because T preserves inclusions. Since this inequality holds for all A with $TA\ni\mathfrak{x}$, we have $\mathfrak{x}\in T(r^\circ_a[B])$ by hypothesis. But this is true for all B with $TB\ni\mathfrak{y}$, so that $a\leq T'_{_M}r(\mathfrak{x},\mathfrak{y})$. Therefore, we have $T_{_M}r(\mathfrak{x},\mathfrak{y})\leq T'_{_M}r(\mathfrak{x},\mathfrak{y})$. The same argument applied to $r^\circ:Y\nrightarrow X$ shows that $T'_{_M}r(\mathfrak{x},\mathfrak{y})\leq T_{_M}r(\mathfrak{x},\mathfrak{y})$, and we are done. ∎

3.8. Corollary. *If* T *preserves finite coproducts, then the canonical and op-canonical extensions of* T *are equal.*

Proof. Let us verify the hypotheses of the proposition. Since T preserves finite coproducts, we have $T\emptyset=\emptyset$. Suppose now that $\mathfrak{x}\in TX$ and $B\subseteq X$ satisfy $TB\cap TA\neq\emptyset$ for all A with $TA\ni\mathfrak{x}$, and set $B'=X\setminus B$. If $\mathfrak{x}\notin TB$, then $\mathfrak{x}\in TX\setminus TB=TB'$ because T preserves finite coproducts. But then $TB\cap TB'=\emptyset$, a contradiction. The proposition then yields the desired result. ∎

To conclude this section, we exhibit an interesting property of the op-canonical $(\mathsf{T},\mathbf{V})$-algebras, which does not seem to have a counterpart in the canonical case.

3.9. Proposition. *Let* $r:TX\nrightarrow X$ *be the structure matrix of an op-canonical* $(\mathsf{T},\mathbf{V})$*-algebra, and* $\mathcal{A}$ *a subset of* TX. *Then for* $\mathfrak{X}\in T\mathcal{A}$ *and* $z\in X$, *we have*

$$\bigwedge_{\mathfrak{x}\in\mathcal{A}} r(\mathfrak{x},z)\leq r(m_X(\mathfrak{X}),z)\ .$$

Moreover, if all $\mathfrak{x} \in \mathcal{A}$ *satisfy* $\mathfrak{x} \leq m_X(\mathfrak{X})$*, then* $\bigwedge_{\mathfrak{x}\in\mathcal{A}} r(\mathfrak{x}, z) = r(m_X(\mathfrak{X}), z)$.

Proof. Let $a \in \mathbf{V}$ be such that $a \leq \bigwedge_{\mathfrak{x}\in\mathcal{A}} r(\mathfrak{x}, z)$. Thus, $a \leq r(\mathfrak{x}, z)$ for all $\mathfrak{x} \in \mathcal{A}$, which implies that $\mathfrak{x} \in r_a^\circ[\{z\}]$ for all $\mathfrak{x} \in \mathcal{A}$, and $T\mathcal{A} \subseteq T(r_a^\circ[\{z\}])$. So $\mathfrak{X} \in T(r_a^\circ[\{z\}])$ by hypothesis, and we can conclude that $a \leq T'_{_M} r(\mathfrak{X}, e_X(z)) \leq r(m_X(\mathfrak{X}), z)$ because e_X is injective and r transitive.

The last equality naturally follows because r is preorder-reversing in its first variable. ∎

3.10. Remark. From now on, we will suppose that the lattice $\mathbf{V}$ is completely distributive.

4. Isomorphisms and embeddings

4.1. Extension conditions. Let T be a **Set**-functor and $\mathcal{A}$ a collection of subsets of X. The following condition will be called the *extension condition for* $\mathcal{A}$:

(E) for every **2**-matrix $r : X \nrightarrow Y$, and $\mathfrak{y} \in TY$ such that $\mathfrak{y} \in T(r[B])$ for all $B \in \mathcal{A}$, there exists $\mathfrak{x} \in \bigcap_{B\in\mathcal{A}} TB$ such that $\mathfrak{y} \in T(r[A])$ for all $A \subseteq X$ with $TA \ni \mathfrak{x}$.

For example, I and P both satisfy the extension condition for any $\mathcal{A} = \{B\}$ with $B \subseteq X$. Similarly, U and F satisfy (E) for any filter or filter base $\mathcal{A}$ (this is true for U by the Extension Lemma, see for example [8], Corollary 2.3).

4.2. Restriction of a monad. Let $\mathsf{T} = (T, e, m)$ and $\mathsf{S} = (S, d, n)$ be **Set**-monads such that T and S are both taut, and suppose that the preorder induced on the sets SX make them into complete atomistic lattices. We say that T is a *restriction of* S *to atoms*, if there is a natural transformation $\iota : T \to S$, such that ι_X sends TX bijectively onto the set of atoms of SX, $d = \iota e$, and $n\iota^2 = \iota m$ (with $\iota^2 = (S\iota)(\iota T) = (\iota S)(T\iota)$). In particular, ι is a morphism of monads, so that T is a submonad of S.

For $\mathfrak{x} \in TX$ we have $\mathfrak{x} \in TA \iff \iota_X(\mathfrak{x}) \in SA$ so the preorder induced on the sets TX is also an order. Remark also that $T\emptyset = \emptyset$ because $S\emptyset$ contains a unique element.

For $\mathcal{B} \subseteq SX$, define

$$\mathcal{A}_{\mathcal{B}} := \{\mathfrak{x} \in TX \mid \text{ there exists } \mathfrak{f} \in \mathcal{B} \text{ with } \iota_X(\mathfrak{x}) \leq \mathfrak{f}\} .$$

A restriction T of S is *convenient* if there exists a natural transformation $\sigma : TS \to ST$ satisfying $\iota T = \sigma(T\iota)$, as well as the following two conditions:

$$\mathfrak{X} \in T\mathcal{B} \implies \sigma_X(\mathfrak{X}) \in S\mathcal{A}_{\mathcal{B}} \qquad \text{and} \qquad \sigma_X(\mathfrak{X}) \in STA \implies \mathfrak{X} \in TSA$$

for any $\mathfrak{X} \in TSX$, $\mathcal{B} \subseteq SX$, and $A \subseteq X$. Since $\mathcal{A}_{SA} = TA$ for any $A \subseteq X$, these conditions imply in particular that $\mathfrak{X} \in TSA \iff \sigma_X(\mathfrak{X}) \in STA$, so $n(\iota S) = n(S\iota)\sigma$ whenever S is taut.

For example, the identity monad I is a restriction of P to singletons, and the ultrafilter monad U is a restriction of F to ultrafilters. Moreover, both these restrictions are convenient; this is immediate in the identity case since we may set $\sigma = 1$; the ultrafilter case will be considered in Proposition 6.3.

4.3. Lemma. *Let* T *be a taut monad satisfying the extension condition for all sets* $\mathcal{A} = \{TB \subseteq TX \,|\, \mathfrak{y} \in TB\}$*, where* $\mathfrak{y} \in TX$*. If* $r : TX \nrightarrow X$ *is the structure matrix of a canonical* $(\mathsf{T}, \mathbf{V})$*-algebra, then for any* $\mathfrak{x} \in TX$ *and* $z \in X$ *we have*

$$r(\mathfrak{x}, y) = \bigwedge_{TB \ni \mathfrak{x}} \bigvee_{\mathfrak{z} \in TB} r(\mathfrak{z}, z) \ .$$

Proof. Define $s(\mathfrak{x}, z) := \bigwedge_{TB \ni \mathfrak{x}} \bigvee_{\mathfrak{z} \in TB} r(\mathfrak{z}, z)$, and observe that $r(\mathfrak{x}, z) \leq s(\mathfrak{x}, z)$ naturally holds. Let $a \in \mathbf{V}$ be such that $a \prec s(\mathfrak{x}, z)$. Thus, for all B with $TB \ni \mathfrak{x}$, there exists $\mathfrak{z} \in TB$ with $a \leq r(\mathfrak{z}, z)$, or $z \in r_a[TB]$. This implies that $e_X(z) \in T(r_a[TB])$ for all B with $TB \ni \mathfrak{x}$. By the extension condition, there exists $\mathfrak{X} \in TTX$ such that $\mathfrak{X} \in TTB$ for all B with $TB \ni \mathfrak{x}$, and $e_X(z) \in T(r_a[\mathcal{B}])$ for all $\mathcal{B}$ with $T\mathcal{B} \supseteq \mathfrak{X}$. This allows us to conclude that $a \leq T_{M} r(\mathfrak{X}, e_X(z)) \leq r(m_X(\mathfrak{X}), z)$ by transitivity of r. Finally, $m_X(\mathfrak{X}) \leq \mathfrak{x}$ implies $a \leq r(\mathfrak{x}, z)$ because r is increasing in its first variable. Thus, $s(\mathfrak{x}, z) \leq r(\mathfrak{x}, z)$ as required. ∎

4.4. Lemma. *Let* T *be a taut monad and* $r : TX \nrightarrow X$ *the structure matrix of a canonical* $(\mathsf{T}, \mathbf{V})$*-algebra. Suppose that* T *preserves finite coproducts, and verifies the extension condition for all sets* $\{TB \subseteq TX \,|\, \mathfrak{x} \in TB\}$*, where* $\mathfrak{x} \in TX$*. If* $z \in X$*,* $\mathfrak{y} \in TX$ *and* $\mathcal{A} \subseteq TX$ *are such that* $\mathfrak{y} \in TA$ *for all* A *with* $TA \supseteq \mathcal{A}$*, then*

$$\bigwedge_{\mathfrak{x} \in \mathcal{A}} r(\mathfrak{x}, z) \leq r(\mathfrak{y}, z).$$

Proof. Remark first that if $B \subseteq X$ is such that $TB \cap TA \neq \emptyset$ for all A with $TA \supseteq \mathcal{A}$, then there exists $\mathfrak{x} \in \mathcal{A}$ with $\mathfrak{x} \in TB$ (see the proof of Corollary 3.8). Thus, if $TB \ni \mathfrak{y}$, then $TB \cap TA \neq \emptyset$ for all A with $TA \supseteq \mathcal{A}$ by hypothesis, so there exists $\mathfrak{x} \in \mathcal{A}$ with $\mathfrak{x} \in TB$. The previous lemma then implies

$$\bigwedge_{\mathfrak{x} \in \mathcal{A}} r(\mathfrak{x}, z) = \bigwedge_{\mathfrak{x} \in \mathcal{A}} \bigwedge_{TB \ni \mathfrak{x}} \bigvee_{\mathfrak{z} \in TB} r(\mathfrak{z}, z) \leq \bigwedge_{TB \ni \mathfrak{y}} \bigvee_{\mathfrak{z} \in TB} r(\mathfrak{z}, z) = r(\mathfrak{y}, z) \ .$$

∎

4.5. Proposition. *Let* $\mathsf{T} = (T, e, m)$ *and* $\mathsf{S} = (S, d, n)$ *be taut monads such that* T *is a convenient restriction of* S *to its atoms. Suppose furthermore that* T *preserves finite coproducts and satisfies the extension condition for all sets*

$$\{B \subseteq TX \,|\, \mathcal{A} \subseteq TB\} \qquad \text{and} \qquad \{TB \subseteq TX \,|\, \mathfrak{x} \in TB\}$$

where $\mathcal{A} \subseteq TX$ *and* $\mathfrak{x} \in TX$*. Then the category of op-canonical* $(\mathsf{S}, \mathbf{V})$*-algebras is isomorphic to the category of op-canonical* $(\mathsf{T}, \mathbf{V})$*-algebras.*

Proof. In this proof, elements $\mathfrak{x} \in TX$ will be considered as elements of SX via ι_X, and elements $\mathfrak{X} \in TTX$ will be considered as elements of SSX via $\iota_{SX}(T\iota_X) = (S\iota_X)\iota_{TX}$. The symbols $\mathfrak{x}, \mathfrak{y}$ and $\mathfrak{X}$ will be used to designate elements of TX and TTX respectively, whereas $\mathfrak{f}, \mathfrak{g}$ and $\mathfrak{F}$ will denote elements of SX and SSX that are not necessarily atoms; there will also be mention of elements $\mathfrak{X}'$ of TSX.

For a relation $r : SX \nrightarrow X$, let $\check{r} : TX \nrightarrow X$ be the restriction of r to elements of TX. If (X, r) is an op-canonical $(\mathsf{S}, \mathbf{V})$-algebra, then a routine verification shows that $S'_{_M} r(\mathfrak{X}, \mathfrak{y}) = T'_{_M} \check{r}(\mathfrak{X}, \mathfrak{y})$ and $(X, \check{r})$ is naturally an op-canonical $(\mathsf{T}, \mathbf{V})$-algebra.

Suppose that $f : (X, r) \to (Y, s)$ is a morphism of op-canonical $(\mathsf{S}, \mathbf{V})$-algebras. Then $\check{r}(\mathfrak{x}, y) \le s(Sf(\mathfrak{x}), f(y))$, and $Sf(\mathfrak{x}) = Tf(\mathfrak{x})$ yields that $f : (X, \check{r}) \to (Y, \check{s})$ is a morphism of $\mathbf{Alg}(\mathsf{T}', \mathbf{V})$. Thus, we can define a functor $R : \mathbf{Alg}(\mathsf{S}', \mathbf{V}) \to \mathbf{Alg}(\mathsf{T}', \mathbf{V})$ commuting with the underlying set functor, and sending (X, r) to $(X, \check{r})$.

We now proceed to verify that there is a functor $L : \mathbf{Alg}(\mathsf{T}', \mathbf{V}) \to \mathbf{Alg}(\mathsf{S}', \mathbf{V})$ commuting with the underlying set functor, and sending (X, r) to $(X, \hat{r})$, where $\hat{r} : SX \nrightarrow X$ is defined by

$$\hat{r}(\mathfrak{f}, y) = \bigwedge_{\mathfrak{x} \le \mathfrak{f}} r(\mathfrak{x}, y)$$

(with the symbols $\mathfrak{x}$ designating atoms of SX). To prove that $(X, \hat{r})$ is a $(\mathsf{S}', \mathbf{V})$-algebra, we only need to verify the transitivity condition for $\hat{r}$ (because $d_X(x) \in TX$ by hypothesis). Let $\mathfrak{F} \in SSX$, $\mathfrak{g} \in SX$, $z \in X$, and denote by $\mathfrak{X}'$ an element of TSX with $\mathfrak{X}' \le \mathfrak{F}$. Note that for $a \in \mathbf{V}$ we have

$$r_a^\circ[B] = \mathcal{A}_{\hat{r}_a^\circ[B]} \ .$$

Let $a \in \mathbf{V}$ be such that $a \prec S'_{_M} \hat{r}(\mathfrak{F}, \mathfrak{g})$, and $\mathfrak{X} \in TTX$ with $\mathfrak{X} \le \sigma_X(\mathfrak{X}')$. Thus, for all B with $SB \ni \mathfrak{g}$ we have $\mathfrak{F} \in S(\hat{r}_a^\circ[B])$, so that $\mathfrak{X} \in T(r_a^\circ[B])$. The extension condition for the set $\{B \subseteq TX \,|\, \mathfrak{g} \in SB\}$ yields an atom $\mathfrak{y} \le \mathfrak{g}$ with $\mathfrak{X} \in T(r_a^\circ[B])$ for all B with $TB \ni \mathfrak{y}$, and we have $a \le T'_{_M} r(\mathfrak{X}, \mathfrak{y})$. Thus, $a \otimes \hat{r}(\mathfrak{g}, z) \le T'_{_M} r(\mathfrak{X}, \mathfrak{y}) \otimes r(\mathfrak{y}, z) \le r(m_X(\mathfrak{X}), z)$, which allows us to conclude that $(S'_{_M} \hat{r})(\mathfrak{F}, \mathfrak{g}) \otimes \hat{r}(\mathfrak{g}, z) \le r(m_X(\mathfrak{X}), z)$ for all $\mathfrak{X} \le \sigma_X(\mathfrak{X}')$ and $\mathfrak{X}' \le \mathfrak{F}$ with $\mathfrak{X}' \in TSX$. Writing $\mathcal{A}' = \{\mathfrak{X}' \in TSX \,|\, \mathfrak{X}' \le \mathfrak{F}\}$ and $\mathcal{A} = \{\mathfrak{X} \in TTX \,|\, \text{ there exists } \mathfrak{X}' \in \mathcal{A}' \text{ with } \mathfrak{X} \le \sigma_X(\mathfrak{X}')\}$, we have for $\mathcal{B} \subseteq SX$:

$$\mathfrak{F} \in SSB \iff \mathcal{A}' \subseteq TSB \iff \sigma_X(\mathcal{A}') \subseteq STB \iff \mathcal{A} \subseteq TTB \ ,$$

so $n_X(\mathfrak{F}) \in SB \iff m_X(\mathcal{A}) \subseteq TB$. Since r is also the structure matrix of a canonical $(\mathsf{T}, \mathbf{V})$-algebra by Corollary 3.8, we can apply Lemma 4.4 to get for any $\mathfrak{y} \in TX$ with $\mathfrak{y} \le n_X(\mathfrak{F})$:

$$(S'_{_M} \hat{r})(\mathfrak{F}, \mathfrak{g}) \otimes \hat{r}(\mathfrak{g}, z) \le \bigwedge_{\mathfrak{x} \in m_X(\mathcal{A})} r(\mathfrak{x}, z) \le r(\mathfrak{y}, z) \ .$$

This allows us to conclude that $(S'_{_M} \hat{r})(\mathfrak{F}, \mathfrak{g}) \otimes \hat{r}(\mathfrak{g}, z) \le \hat{r}(m_X(\mathfrak{F}), z)$, as required.

If $f : (X, r) \to (Y, s)$ is a morphism of $\mathbf{Alg}(\mathsf{T}', \mathbf{V})$, then $\hat{r}(\mathfrak{f}, y) \le \bigwedge_{\mathfrak{x} \le \mathfrak{f}} s(Tf(\mathfrak{x}), f(y))$. Let $\mathfrak{y} \le Sf(\mathfrak{f})$. By the extension condition, there exists $\mathfrak{x} \in TX$ with $\mathfrak{x} \le \mathfrak{f}$ and $Tf(\mathfrak{x}) = \mathfrak{y}$.

This yields that

$$\bigwedge_{\mathfrak{x}\le\mathfrak{f}} s(Tf(\mathfrak{x}), f(y)) = \bigwedge_{\mathfrak{y}\le Sf(\mathfrak{f})} s(\mathfrak{y}, f(y)) = \hat{s}(Sf(\mathfrak{f}), f(y)) ,$$

so $f : (X, \hat{r}) \to (Y, \hat{s})$ is a morphism of $\mathbf{Alg}(\mathsf{S}', \mathbf{V})$.

Thus, we have two functors $R : \mathbf{Alg}(\mathsf{S}', \mathbf{V}) \to \mathbf{Alg}(\mathsf{T}', \mathbf{V})$ and $L : \mathbf{Alg}(\mathsf{T}', \mathbf{V}) \to \mathbf{Alg}(\mathsf{S}', \mathbf{V})$ commuting with the underlying set functors, and sending (X, r) to $(X, \check{r})$, and (X, s) to $(X, \hat{s})$ respectively. By noticing that $\sigma_X d_{SX}(\mathfrak{f}) \in T\mathcal{A}_{\{\mathfrak{f}\}}$ and $\mathfrak{f} = n_X d_{SX}(\mathfrak{f}) = n_X \sigma_X d_{SX}(\mathfrak{f})$, Proposition 3.9 implies $\hat{\check{r}}(\mathfrak{f}, y) = \bigwedge_{\mathfrak{x}\le\mathfrak{f}} r(\mathfrak{x}, y) = r(\mathfrak{f}, y)$. Moreover, we naturally have $\check{\hat{r}}(\mathfrak{x}, y) = r(\mathfrak{x}, y)$, so the functors R and L define an isomorphism between $\mathbf{Alg}(\mathsf{S}', \mathbf{V})$ and $\mathbf{Alg}(\mathsf{T}', \mathbf{V})$. ■

4.6. Corollary. *The category* $\mathbf{Alg}(\mathsf{P}', \mathbf{V})$ *is isomorphic to* $\mathbf{Alg}(\mathsf{I}, \mathbf{V})$.

Proof. As mentioned previously, the monad I is a convenient restriction of P to its atoms. Since it preserves finite coproducts, and also satisfies the extension condition for the sets given in the proposition, the desired result follows. Note however that in this simple case, the previous proof may be considerably simplified. ■

4.7. Corollary. *The category* $\mathbf{Alg}(\mathsf{P}', \mathbf{2})$ *is isomorphic to* $\mathbf{Ord}$*, and* $\mathbf{Alg}(\mathsf{P}', \overline{\mathbf{R}}_+)$ *is isomorphic to* $\mathbf{Met}$.

Proof. This is an immediate consequence of the previous corollary and the fact that $\mathbf{Alg}(\mathsf{I}, \mathbf{2}) \cong \mathbf{Ord}$ and $\mathbf{Alg}(\mathsf{I}, \overline{\mathbf{R}}_+) \cong \mathbf{Met}$ (see for example [5]). ■

For the rest of this section, the morphisms of $(\mathsf{P}, \mathbf{V})$-algebras will be denoted by $f : (X, c) \to (Y, d)$, while those of $(\mathsf{T}, \mathbf{V})$-algebras, will be denoted by $f : (X, r) \to (Y, s)$. The unit e and multiplication m of the powerset monad P will be given by their explicit formulation, whereas the monad T will be denoted $\mathsf{T} = (T, e, m)$.

4.8. Proposition. *Let* $\mathsf{T} = (T, e, m)$ *be a taut monad satisfying the extension condition for any* $\mathcal{A} = \{B\}$ *with* $B \subseteq X$*. The functors* $R : \mathbf{Alg}(\mathsf{P}, \mathbf{V}) \to \mathbf{Alg}(\mathsf{T}, \mathbf{V})$ *and* $L : \mathbf{Alg}(\mathsf{T}, \mathbf{V}) \to \mathbf{Alg}(\mathsf{P}, \mathbf{V})$ *commuting with the underlying set functor, and defined on objects by* $R(X, c) = (X, \hat{c})$*,* $L(X, r) = (X, \check{r})$*, where*

$$\hat{c}(\mathfrak{x}, y) := \bigwedge_{TB\ni\mathfrak{x}} c(B, y) \qquad \textit{and}$$

$$\check{r}(A, y) := \bigvee_{\mathfrak{y}\in TA} r(\mathfrak{y}, y) ,$$

(with $A \in PX$*,* $\mathfrak{x} \in TX$*, and* $y \in X$*) yield an adjunction* $L \dashv R$*. Moreover, if* T *satisfies the extension condition for all sets* $\{TB \subseteq TX \mid \mathfrak{y} \in TB\}$ *with* $\mathfrak{y} \in TX$*, then* $\hat{\check{r}} = r$ *and* L *is a full coreflective embedding.*

PROOF. We first notice that $\check{\hat{c}} \leq c$ and that $r \leq \hat{\check{r}}$ for any matrices $r : TX \nrightarrow X$ and $c : PX \nrightarrow X$.

Suppose that $r : TX \nrightarrow X$ is the structure matrix of a $(\mathsf{T}, \mathbf{V})$-algebra. The reflexivity of $\check{r}$ follows immediately from the definition: $k \leq r(e_X(x), y) \leq \check{r}(\{x\}, y)$. For the transitivity, let $\mathcal{A} \in PPX$, $B \in PX$, $z \in X$, and $a, b \in \mathbf{V}$ be two elements such that $a \prec P_{_M}\check{r}(\mathcal{A}, B)$ and $b \prec \check{r}(B, z)$. This last inequality yields an element $\mathfrak{y} \in TB$ such that $b \leq r(\mathfrak{y}, z)$. The first inequality implies that $B \subseteq \check{r}_{\overline{a}}[\mathcal{A}]$. Furthermore,

$$\check{r}_{\overline{a}}[\mathcal{A}] \subseteq \{y \in X \mid \text{ there exist } A \in \mathcal{A}, \mathfrak{x} \in TA \text{ with } a \leq r(\mathfrak{x}, y)\} \subseteq r_a[T(\textstyle\bigcup \mathcal{A})] ,$$

and we may write $\mathfrak{y} \in T(r_a[T(\bigcup \mathcal{A})])$. By the extension condition, there exists $\mathfrak{X} \in TT(\bigcup \mathcal{A})$ such that $\mathfrak{y} \in T(r_a[\mathcal{B}])$ for all $\mathcal{B}$ with $T\mathcal{B} \ni \mathfrak{X}$. This implies that $a \leq T_{_M}r(\mathfrak{X}, \mathfrak{y})$, and by using that $b \leq r(\mathfrak{y}, z)$, we get $a \otimes b \leq r(m_X(\mathfrak{X}), z)$ by transitivity of r. Moreover, $\mathfrak{X} \in TT(\bigcup \mathcal{A})$ implies $m_X(\mathfrak{X}) \in T(\bigcup \mathcal{A})$, so that $r(m_X(\mathfrak{X}), z) \leq \check{r}(\bigcup \mathcal{A}, z)$. Since $a \otimes b \leq \check{r}(\bigcup \mathcal{A}, z)$ for all elements $a, b \in \mathbf{V}$ with $a \prec P_{_M}\check{r}(\mathcal{A}, B)$ and $b \prec \check{r}(B, z)$, we may conclude that $P_{_M}\check{r}(\mathcal{A}, B) \otimes \check{r}(B, z) \leq \check{r}(\bigcup \mathcal{A}, z)$ as required. Consider now a morphism of $(\mathsf{T}, \mathbf{V})$-algebras $f : (X, r) \to (Y, s)$. The map $f : (X, \check{r}) \to (Y, \check{s})$ is a morphism of $(\mathsf{P}, \mathbf{V})$-algebras, since

$$\check{r}(A, y) \leq \bigvee_{\mathfrak{x} \in T(f^{-1}f(A))} r(\mathfrak{x}, y) \leq \bigvee_{\mathfrak{x} \in (Tf)^{-1}(T(f(A)))} s(Tf(\mathfrak{x}), f(y)) \leq \check{s}(Pf(A), f(y)) .$$

Suppose now that $c : PX \nrightarrow X$ is the structure matrix of a $(\mathsf{P}, \mathbf{V})$-algebra. The reflexivity of $\hat{c}$ follows from the monotonicity of c and the injectivity of e_X; indeed, $k \leq c(\{x\}, y) = \hat{c}(e_X(x), y)$. To prove the transitivity of $\hat{c}$, let $\mathfrak{X} \in TTX$, $\mathfrak{y} \in TX$, $z \in X$, and $a \in \mathbf{V}$ such that $a \prec T_{_M}\hat{c}(\mathfrak{X}, \mathfrak{y})$. This last condition implies that $\mathfrak{y} \in T(\hat{c}_a[TA])$ for all A with $TTA \ni \mathfrak{X}$. Furthermore, $\hat{c}_a[TA] \subseteq c_a[\{A\}]$, so that by setting $B = c_a[\{A\}]$, we naturally have $\mathfrak{y} \in TB$ and $B \subseteq c_a[\mathcal{A}]$ for all $\mathcal{A} \supseteq \{A\}$. This implies that $a \leq P_{_M}c(\{A\}, B)$, so that $T_{_M}\hat{c}(\mathfrak{X}, \mathfrak{y}) \otimes \hat{c}(\mathfrak{y}, z) \leq c(A, z)$ for all A with $TTA \ni \mathfrak{X}$. Since m is taut, we have $\hat{c}(m_X(\mathfrak{X}), z) = \bigwedge_{TTA \ni \mathfrak{X}} c(A, z)$, and the transitivity of $\hat{c}$ follows. Consider now a morphism of $(\mathsf{P}, \mathbf{V})$-algebras $f : (X, c) \to (Y, d)$. The map $f : (X, \hat{c}) \to (Y, \hat{d})$ is a morphism of $(\mathsf{T}, \mathbf{V})$-algebras, since

$$\hat{c}(\mathfrak{x}, y) \leq \bigwedge_{TA \ni \mathfrak{x}} d(Pf(A), f(y)) \leq \bigwedge_{TB \ni Tf(\mathfrak{x})} d(B, f(y)) = \hat{d}(Tf(\mathfrak{x}), f(y)) ,$$

by using the monotonicity of d and the fact that $\mathfrak{x} \in T(f^{-1}(B)) \iff Tf(\mathfrak{x}) \in TB$.

The last statement follows directly from Lemma 4.3. ∎

4.9. COROLLARY. *Let $\mathsf{T} = (T, e, m)$ be a taut monad satisfying the extension condition for all sets $\{\mathcal{B}\}$ with $B \subseteq X$, and $\{TB \subseteq TX \mid \mathfrak{y} \in TB\}$ with $\mathfrak{y} \in TX$. If for all $A \subseteq X$, there exists $\mathfrak{x}_A \in TA$ with $\mathfrak{x}_A \in TB \iff A \subseteq B$, then $\mathbf{Alg}(\mathsf{T}, \mathbf{V})$ is isomorphic to $\mathbf{Alg}(\mathsf{P}, \mathbf{V})$.*

Proof. With the notations of the previous proposition, it suffices to prove that $\check{\hat{c}} = c$ for any structure matrix $c : PX \nrightarrow X$. The definition of $\mathfrak{x}_A$ implies that $\mathfrak{x} \le \mathfrak{x}_A$ for all $\mathfrak{x} \in TA$, so by monotonicity of $\hat{c}$ and c,

$$\check{\hat{c}}(A, y) = \bigvee_{\mathfrak{x}\in TA} \bigwedge_{TB\ni\mathfrak{x}} c(B, y) = \bigwedge_{TB\ni\mathfrak{x}_A} c(B, y) = c(A, y) \ ,$$

and we are done. ∎

4.10. Proposition. *The category of op-canonical $(\mathsf{P}, \mathbf{V})$-algebras embeds as a full coreflective subcategory into the category of op-canonical $(\mathsf{T}, \mathbf{V})$-algebras. More precisely, the functors $R : \mathbf{Alg}(\mathsf{T}', \mathbf{V}) \to \mathbf{Alg}(\mathsf{P}', \mathbf{V})$ and $E : \mathbf{Alg}(\mathsf{P}', \mathbf{V}) \to \mathbf{Alg}(\mathsf{T}', \mathbf{V})$ commuting with the underlying set functor, and defined on objects by $R(X, r) = (X, \hat{r})$, $E(X, c) = (X, \check{c})$, where*

$$\hat{r}(A, y) := \bigwedge_{x\in A} r(e_X(x), y) \qquad \textit{and}$$
$$\check{c}(\mathfrak{x}, y) := \bigvee_{TB\ni\mathfrak{x}} c(B, y) \ ,$$

(with $A \in PX$, $\mathfrak{x} \in TX$, and $y \in X$) yield an adjunction $E \dashv R$ such that $\hat{\check{c}} = c$.

Proof. Proposition 3.9 implies that $\check{\hat{r}} \le r(\mathfrak{x}, y)$ and that $\hat{\check{c}} = c$ for any $(\mathsf{T}', \mathbf{V})$-algebra structure matrix $r : TX \nrightarrow X$ and $(\mathsf{P}', \mathbf{V})$-algebra structure matrix $c : PX \nrightarrow X$.

Corollary 4.9 states that $\mathbf{Alg}(\mathsf{P}', \mathbf{V})$ is isomorphic to $\mathbf{Alg}(\mathsf{I}, \mathbf{V})$ via the adjunction described in Proposition 4.5, so it is sufficient to prove that the matrix $\hat{r}(x, y) = r(e_X(x), y)$ is a structure of $\mathbf{Alg}(\mathsf{I}, \mathbf{V})$. In this case, reflexivity is immediate, and transitivity follows from

$$r(e_X(x), y) \otimes r(e_X(y), z) \le T'_{M} r(e_{TX}(e_X(x)), e_X(y)) \otimes r(e_X(y), z) \le r(e_X(x), z) \ .$$

If $f : (X, r) \to (Y, s)$ is a morphism of $(\mathsf{T}', \mathbf{V})$-algebras, we naturally have $r(e_X(x), y) \le s(Tf(e_X(x)), f(y))$, and since $Tf(e_X(x)) = e_Y(f(x))$, f is a morphism of the corresponding $(\mathsf{P}', \mathbf{V})$-algebras.

Let $c : PX \nrightarrow X$ be the structure matrix of a $(\mathsf{P}', \mathbf{V})$-algebra. The reflexivity of $\check{c}$ follows from the fact that c is order-reversing and e_X injective. To prove the transitivity, let $\mathfrak{X} \in TTX$, $\mathfrak{y} \in TX$, $z \in X$, and $a, b \in \mathbf{V}$ two elements such that $a \prec T'_{M}\check{c}(\mathfrak{X}, \mathfrak{y})$ and $b \prec \check{c}(\mathfrak{y}, z)$. First note that by setting $\mathcal{A}_B := c^\circ_a[B]$, we naturally have $a \le P'_{M} c(\mathcal{A}_B, B)$. Furthermore, there exists B with $TB \ni \mathfrak{y}$ and $b \le c(B, z)$, so that $\mathfrak{X} \in T(\check{c}^\circ_{\overline{a}}[B])$. Since

$$\check{c}^\circ_{\overline{a}}[B] \subseteq \{\mathfrak{x} \in TX \mid \text{ there exist } y \in B, A \subseteq X \text{ with } TA \ni \mathfrak{x} \text{ and } a \le c(A, y)\} \subseteq T(\textstyle\bigcup \mathcal{A}_B) \ ,$$

we have $m_X(\mathfrak{X}) \in T(\bigcup \mathcal{A}_B)$. Therefore,

$$T'_{M}\check{c}(\mathfrak{X}, \mathfrak{y}) \otimes \check{c}(\mathfrak{y}, z) \le P'_{M} c(\mathcal{A}_B, B) \otimes c(B, z) \le c(\textstyle\bigcup \mathcal{A}_B, z) \le \check{c}(m_X(\mathfrak{X}), z)$$

by transitivity of c. Consider now a morphism of $(\mathsf{P}', \mathbf{V})$-algebras $f : (X, c) \to (Y, d)$. Then

$$\check{c}(\mathfrak{x}, y) \le \bigvee_{TA \ni \mathfrak{x}} d(Pf(A), f(y)) \le \bigvee_{TB \ni Tf(\mathfrak{x})} d(B, f(y)) = \check{d}(Tf(\mathfrak{x}), f(y))$$

since $\{Pf(A) \in PY \,|\, TA \ni \mathfrak{x}\} \subseteq \{B \in PY \,|\, TB \ni Tf(\mathfrak{x})\}$. ■

5. The powerset monad and associated lax algebras

In this section, we give an alternate description of the categories of canonical and op-canonical $(\mathsf{P}, \mathbf{V})$-algebras; in particular, we show that $\mathbf{Alg}(\mathsf{P}, \mathbf{2})$ is isomorphic to the category of closure spaces. The following result gives another description of the powerset's canonical extension; of course, a similar formula may be obtained for its op-canonical extension (although it is of less interest here since the op-canonical algebras may be obtained via the identity monad by Corollary 4.6). This canonical extension also appears in [3], Example 6.3 as a lax functor $H : \mathbf{Mat}(\mathbf{2}) \to \mathbf{Mat}(\mathbf{2})$, and as its lax extension to $\mathbf{Mat}(\overline{\mathbf{R}}_+)$.

5.1. Proposition. *The canonical extension of P is given by*

$$P_{_M} r(A, B) = \bigwedge_{y \in B} \bigvee_{x \in A} r(x, y) \ ,$$

where $A \in PX$ and $B \in PY$.

Proof. Denote by $T_{_M}$ the canonical extension of P defined in 3.1. Let $A, B \in PX$ and $a \in \mathbf{V}$ such that $a \prec P_{_M} r(A, B)$. Thus, for every $y \in B$ there exists $x \in A$ with $a \le r(x, y)$, so $B \subseteq r_a[A]$ or equivalently $B \in P(r_a[A])$. Furthermore, if PC contains A, then we necessarily have $A \subseteq C$, so that $B \in P(r_a[C])$ and we can conclude that $a \le T_{_M} r(A, B)$.

Suppose now that $a \in \mathbf{V}$ is such that $a \prec T_{_M} r(A, B)$. This implies in particular that $B \in P(r_a[A])$, *i.e.* for each $y \in B$, there exists $x \in A$ with $a \le r(x, y)$, and we may conclude that $a \le P_{_M} r(A, B)$. ■

Let us recall the definition of a closure space.

5.2. Closure spaces. Let X be a set. An operator $c : PX \to PX$ is a *closure operator* if it is *extensive*, *monotone* and *idempotent*:

(C_1) $A \subseteq c(A)$;

(C_2) $B \subseteq A \implies c(B) \subseteq c(A)$;

(C_3) $c(c(A)) \subseteq c(A)$;

where $A, B \in PX$. A couple (X, c) is called a *closure space*. Closure spaces form the objects of the category $\mathbf{Clos}$, whose morphisms $f : (X, c) \to (Y, d)$ are the $\mathbf{Set}$-maps $f : X \to Y$ satisfying $f(c(A)) \subseteq d(f(A))$ for all $A \in PX$.

5.3. Proposition. *The category* $\mathbf{Alg}(\mathsf{P},\mathbf{2})$ *is isomorphic to* **Clos**. *In fact, a canonical* $(\mathsf{P},\mathbf{2})$*-algebra* (X,r) *and a closure space* (X,c) *determine each other via*

$$x\in c(A)\iff r(A,x)=\top\ .$$

Proof. This is a particular case of Proposition 5.6 which is proved further on. ∎

This result motivates the introduction of *closeness operators*, which might be seen as the metric counterpart of closure operators, since the former measure the distance between points and sets, rather than simply ascribing a true or false value to every such couple. As mentioned in the Introduction, closeness spaces are related to approach spaces in the same way that closure spaces are related to topological spaces.

5.4. Closeness spaces. The objects of the category $\mathbf{Clsn}(\mathbf{V})$ are couples (X,c), where X is a set and $c:PX\times X\to\mathbf{V}$ is a *closeness operator*, *i.e.* a map satisfying:

(C_1') $x\in A\implies k\le c(A,x)$;

(C_2') $B\subseteq A\implies c(B,x)\le c(A,x)$;

(C_3') $a\otimes c(A^{(a)},x)\le c(A,x)$;

where $x\in X$, $A\in PX$, $a\in\mathbf{V}$ and $A^{(a)}=\{x\in X\,|\,a\le c(A,x)\}$. The couple (X,c) is called a *closeness space*. A morphism of closeness spaces $f:(X,c)\to(Y,d)$ is a **Set**-map $f:X\to Y$ satisfying $c(A,y)\le d(Pf(A),f(y))$. If $\mathbf{V}=\mathbf{2}$, then we naturally have $\mathbf{Clsn}(\mathbf{2})=\mathbf{Clos}$. Moreover, if $\mathbf{V}=\overline{\mathbf{R}}_+$, we simply write **Clsn** instead of $\mathbf{Clsn}(\mathbf{V})$.

5.5. Remark. As in the context of approach spaces (see [10]), we observe that the conditions (C_1')–(C_3') are equivalent to (C_1'), (C_2') and

(C_3'') $\bigwedge_{y\in B}c(A,y)\otimes c(B,x)\le c(A,x)$ for all $A,B\subseteq X$ and $x\in X$.

Indeed, on one hand (C_3'') implies (C_3') by setting $B=A^{(a)}$. On the other hand, (C_2') and (C_3') imply (C_3'') by setting $a=\bigvee\{b\in\mathbf{V}\,|\,B\subseteq A^{(b)}\}$. Notice also that the set $A^{(a)}$ corresponds to the set $c_a[\{A\}]$ in the notations of 3.1.

5.6. Proposition. *The category* $\mathbf{Clsn}(\mathbf{V})$ *is isomorphic to* $\mathbf{Alg}(\mathsf{P},\mathbf{V})$ *via the following correspondence: a relation* $r:PX\times X\to\mathbf{V}$ *determines a closeness operator on* X *if and only if the associated matrix* $r:PX\nrightarrow X$ *is the structure of a canonical* $(\mathsf{P},\mathbf{V})$*-algebra.*

Proof. Suppose first that (X,r) is a canonical $(\mathsf{P},\mathbf{V})$-algebra. Since r is order-preserving in the first variable, reflexivity of r yields $k\le r(\{x\},x)\le r(A,x)$ whenever $x\in A$. It also follows that $B\subseteq A$ implies $r(A,x)\le r(B,x)$. To prove (C_3'), set $\mathcal{A}=\{A\}$. Then $\bigwedge_{y\in A^{(a)}}\bigvee_{B\in\mathcal{A}}r(B,y)\otimes r(A^{(a)},z)=\bigwedge_{y\in A^{(a)}}r(A,y)\otimes r(A^{(a)},z)\ge a\otimes r(A^{(a)},z)$. By transitivity, $a\otimes r(A^{(a)},z)\le r(A,z)$ as required.

Suppose now that (X, c) is a closeness space. It is clear that $k \leq c(\{x\}, x)$. If $B \in PX$ and $\mathcal{A} \in PPX$, then $\bigvee_{A\in\mathcal{A}} c(A, y) \leq c(\bigcup\mathcal{A}, y)$. Setting $a = \bigwedge_{y\in B} c(\bigcup\mathcal{A}, y)$, we observe that $B \subseteq (\bigcup\mathcal{A})^{(a)}$, so

$$\bigwedge_{y\in B}\bigvee_{A\in\mathcal{A}} c(A, y) \otimes c(B, z) \leq a \otimes c((\bigcup\mathcal{A})^{(a)}, z) \leq c(\bigcup\mathcal{A}, z) \ ,$$

and we are done.

The conclusion follows by noticing that the conditions for morphisms are equivalent. ■

6. The filter monad and associated lax algebras

As in the powerset case, the filter functor's canonical and op-canonical extensions may be described without the use of the sets $r_a[A]$ of Section 3. In this case however, we give the formula for the op-canonical extension.

6.1. Proposition. *The op-canonical extension of F is given by*

$$F'_{M} r(\mathfrak{f}, \mathfrak{g}) = \bigwedge_{B\in\mathfrak{g}}\bigvee_{A\in\mathfrak{f}}\bigwedge_{x\in A}\bigvee_{y\in B} r(x, y) \ ,$$

where $\mathfrak{f} \in FX$ and $\mathfrak{g} \in FY$.

Proof. Denote by T'_{M} the op-canonical extension of F. Let $\mathfrak{f}, \mathfrak{g} \in FX$ and $a \in \mathbf{V}$ be such that $a \prec F'_{M} r(\mathfrak{f}, \mathfrak{g})$. Thus, for every $B \in \mathfrak{g}$ there exists $A \in \mathfrak{f}$ satisfying $A \subseteq r^{\circ}_{a}[B]$. As a consequence, for every $B \in \mathfrak{g}$, we have $\mathfrak{f} \in F(r^{\circ}_{a}[B])$ and $a \leq T'_{M} r(\mathfrak{f}, \mathfrak{g})$. It follows that $F'_{M} r(\mathfrak{f}, \mathfrak{g}) \leq T'_{M} r(\mathfrak{f}, \mathfrak{g})$.

Suppose now that $a \in \mathbf{V}$ is such that $a \prec T'_{M} r(A, B)$. This implies that for every $B \in \mathfrak{g}$, we have $r^{\circ}_{a}[B] \in \mathfrak{f}$. Therefore, for every $B \in \mathfrak{g}$ there exists $A \in \mathfrak{f}$, namely $A = r^{\circ}_{a}[B]$, such that for every $x \in A$, there exists $y \in B$ satisfying $a \leq r(x, y)$. Thus, we may conclude that $a \leq F'_{M} r(\mathfrak{f}, \mathfrak{g})$. ■

6.2. Lemma. *The lax extension U_{M} of the ultrafilter functor given in 2.8, is equal to both the canonical and op-canonical extensions of U. Moreover, U_{M} is equal to the restriction of F'_{M} to ultrafilters, i.e. for all $\mathfrak{x}, \mathfrak{y} \in UX$ we have $U_{M} r(\mathfrak{x}, \mathfrak{y}) = F'_{M} r(\mathfrak{x}, \mathfrak{y})$.*

Proof. Corollary 3.8 yields that the canonical and op-canonical extensions of U are equal. The fact that the expression given in 2.8 describes these extensions may be seen as in the previous proposition. Again, the last claim may be proved with arguments similar to those in the proof of Proposition 3.7. ■

6.3. Proposition. *The category of op-canonical* $(\mathsf{F}, \mathbf{V})$*-algebras is isomorphic to the category of canonical* $(\mathsf{U}, \mathbf{V})$*-algebras.*

Proof. In order to apply Proposition 4.5, we only need to verify that U is a convenient restriction of F. Define $\sigma_X : UFX \to FUX$ by $\sigma_X(\mathfrak{X}) = \{\mathcal{A}_{\mathcal{B}} \,|\, \mathcal{B} \in \mathfrak{X}\}$ for $\mathfrak{X} \in UFX$, and consider a map $f : X \to Y$. In order to verify that σ is a natural transformation, it is useful to first show that $\mathcal{A}_{(Ff)^{-1}(\mathcal{B})} = (Uf)^{-1}(\mathcal{A}_{\mathcal{B}})$. On one hand, if $\mathfrak{x} \in \mathcal{A}_{(Ff)^{-1}(\mathcal{B})}$, then there exists $\mathfrak{f} \in (Ff)^{-1}(\mathcal{B})$ such that $\mathfrak{x} \leq \mathfrak{f}$, so $Uf(\mathfrak{x}) = Ff(\mathfrak{x}) \leq Ff(\mathfrak{f}) \in \mathcal{B}$ and $\mathcal{A}_{(Ff)^{-1}(\mathcal{B})} \subseteq (Uf)^{-1}(\mathcal{A}_{\mathcal{B}})$. On the other hand, if $\mathfrak{x} \in (Uf)^{-1}(\mathcal{A}_{\mathcal{B}})$, then there exists $\mathfrak{g} \in \mathcal{B}$ with $Ff(\mathfrak{x}) \leq \mathfrak{g}$. By definition of Ff, we have $f^{-1}(A) \in \mathfrak{x}$ for all $A \in \mathfrak{g}$. Thus, the sets $f^{-1}(A)$ for $A \in \mathfrak{g}$ form a filter $\mathfrak{f} \in (Ff)^{-1}(\mathcal{B})$ satisfying $\mathfrak{x} \leq \mathfrak{f}$, and $(Uf)^{-1}(\mathcal{A}_{\mathcal{B}}) \subseteq \mathcal{A}_{(Ff)^{-1}(\mathcal{B})}$.

Let $\mathcal{A}_{\mathcal{B}} \in \sigma_Y(UFf)(\mathfrak{X})$, where $\mathcal{B} \in UFf(\mathfrak{X})$. This means that $(Ff)^{-1}(\mathcal{B}) \in \mathfrak{X}$, so $\mathcal{A}_{(Ff)^{-1}(\mathcal{B})} \in \sigma(\mathfrak{X})$. By the previous point, we have $\mathcal{A}_{\mathcal{B}} \in (FUf)\sigma_X(\mathfrak{X})$, and $(FUf)\sigma_X(\mathfrak{X})$ is finer than $\sigma_Y(UFf)(\mathfrak{X})$.

We now show that $Uf(\mathcal{A}_{\mathcal{B}}) = \mathcal{A}_{Ff(\mathcal{B})}$. On one hand, if $\mathfrak{y} \in Uf(\mathcal{A}_{\mathcal{B}})$, there exist a filter $\mathfrak{f} \in \mathcal{B}$ and an ultrafilter $\mathfrak{x} \leq \mathfrak{f}$ such that $Uf(\mathfrak{x}) = \mathfrak{y}$, so $\mathfrak{y} \in \mathcal{A}_{Ff(\mathcal{B})}$ because $Uf(\mathfrak{x}) = Ff(\mathfrak{x}) \leq Ff(\mathfrak{f})$. On the other hand, if $\mathfrak{y} \in \mathcal{A}_{Ff(\mathcal{B})}$, there exists $\mathfrak{f} \in \mathcal{B}$ with $\mathfrak{y} \leq Ff(\mathfrak{f})$. By the Extension Lemma, there exists an ultrafilter $\mathfrak{x} \leq \mathfrak{f}$ with $Uf(\mathfrak{x}) = \mathfrak{y}$, and we have $\mathfrak{y} \in Uf(\mathcal{A}_{\mathcal{B}})$.

A basis for the filter $(FUf)\sigma_X(\mathfrak{X})$ is given by the sets $Uf(\mathcal{A}_{\mathcal{B}}) = \mathcal{A}_{Ff(\mathcal{B})}$ with $\mathcal{B} \in \mathfrak{X}$. But then $Ff(\mathcal{B}) \in UFf(\mathfrak{X})$, so naturally $\mathcal{A}_{Ff(\mathcal{B})} \in \sigma_Y(UFf)(\mathfrak{X})$, and $\sigma_Y(UFf)(\mathfrak{X})$ is finer than $(FUf)\sigma_X(\mathfrak{X})$. Therefore, we may conclude that σ is a natural transformation.

The other conditions that σ must verify follow immediately from its definition. ∎

6.4. Corollary. *The category* $\mathbf{Alg}(\mathsf{F}', \mathbf{2})$ *is isomorphic to* **Top***, and* $\mathbf{Alg}(\mathsf{F}', \overline{\mathbf{R}}_+)$ *is isomorphic to* **App***.*

Proof. The first assertion follows from the fact that $\mathbf{Alg}(\mathsf{U}, \mathbf{2})$ is isomorphic to **Top** (see [1]). The second from the fact that $\mathbf{Alg}(\mathsf{U}, \overline{\mathbf{R}}_+)$ is isomorphic to **App** (see [2]). ∎

6.5. Corollary. *The category* **Ord** *embeds as a full coreflective subcategory into* **Top***, and* **Met** *embeds as a full coreflective subcategory into* **App***. Similarly,* **Top** *embeds as a full coreflective subcategory into* **Clos***, and* **App** *embeds as a full coreflective subcategory into* **Clsn***.*

Proof. Since $\mathbf{Ord} \cong \mathbf{Alg}(\mathsf{P}', \mathbf{2})$ and $\mathbf{Met} \cong \mathbf{Alg}(\mathsf{P}', \overline{\mathbf{R}}_+)$, the first assertion is a consequence of Proposition 4.10. Moreover, the isomorphisms $\mathbf{Top} \cong \mathbf{Alg}(\mathsf{U}, \mathbf{2})$ and $\mathbf{App} \cong \mathbf{Alg}(\mathsf{U}, \overline{\mathbf{R}}_+)$ yield the second assertion via Proposition 4.8. Notice that the adjunction used in this last proposition is the one used in the original proofs of the isomorphisms $\mathbf{Top} \cong \mathbf{Alg}(\mathsf{U}, \mathbf{2})$ and $\mathbf{App} \cong \mathbf{Alg}(\mathsf{U}, \overline{\mathbf{R}}_+)$, where **Top** was described in terms of (additive) closure operators, and **App** in terms of (additive) closeness operators.

∎

6.6. Proposition. *The category of canonical* $(\mathsf{F}, \mathbf{V})$*-algebras is isomorphic to the category of canonical* $(\mathsf{P}, \mathbf{V})$*-algebras.*

Proof. This follows from Corollary 4.9: each set $A \subseteq X$ gives rise to the filter $\mathfrak{x}_A := \{B \subseteq X \mid A \subseteq B\}$, which satisfies the required hypothesis. ■

6.7. Corollary. *The category* $\mathbf{Alg}(\mathsf{F}, \mathbf{2})$ *is isomorphic to* **Clos***, and* $\mathbf{Alg}(\mathsf{F}, \overline{\mathbf{R}}_+)$ *is isomorphic to* **Clsn***.*

Proof. Again, this is immediate, since $\mathbf{Alg}(\mathsf{P}, \mathbf{2}) \cong \mathbf{Clos}$ and $\mathbf{Alg}(\mathsf{P}, \overline{\mathbf{R}}_+) \cong \mathbf{Clsn}$ by Corollary 4.7. ■

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School of Computer Science, McGill University
3480 University Street, Montreal, QC, Canada H3A 2A7
Email: gseal@fastmail.fm

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