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OPERATOR SPLITTING METHODS FOR COMPUTATION OF EIGENVALUES OF REGULAR STURM-LIOUVILLE PROBLEMS

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Abstract. The purpose of this paper is to compute the highest eigenvalues of regular Sturm-Liouville problems with Dirichlet boundary conditions using symmetrical weighted sequential splitting method. The accuracy of the higher eigenvalues of the problem is demonstrated by some classical examples.

1 Introduction

We discuss the computation of higher eigenvalues of regular Sturm-Liouville problem (SLP) in canonical Liouville normal form

$$-y''(t) + q(t)y(t) = \lambda y(t) \tag{1.1}$$

with Dirichlet boundary conditions

$$y(0) = 0, \quad y(1) = 0 \tag{1.2}$$

for $q(t) \in C[0,1]$ and $t \in [0,1]$.

It is well known that, the analysis of an eigenvalue problem as its Liouville form has considerable advantage interms of a theoritical and numerical studies.

An alternative approaches on the same problem SLP with different boundary conditions has been suggested by a number of authors (see, [1, 2, 4, 5]). An annoying aspects of most numerical methods is that the accuracy of the approximation of the k^{th} eigenvalue decreases as k increases. In [3, 6, 8, 9, 10, 13], the asymptotic correction techniques are used to produce significant improvement for approximate eigenvalues, but not for all eigenvalues.

In this paper, we apply the symmetrical weighted sequential splitting method given in [11] to compute the higher eigenvalues of SLP (1.1) and (1.2). The operator splitting methods are based on the splitting of variable coefficient system into simpler

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constant and variable coefficient subproblems. So, instead of the original problem, we deal with simpler constant and variable coefficient system using their exact solutions.

In Section 2, we examine the symmetrical weighted sequential splitting method for the approximate determination of the eigenvalues of SLP. In section 3, the asymptotic formula for the error of eigenvalues is given. Our approach to deriving the asymptotic formula is built for $q(t) = q$. In section 4, some numerical experiments for regular problems are presented by confirming the effectiveness of the suggested approach.

2 Application the Symmetrical Weighted Sequential Splitting Method to Regular SLP

We consider the regular Sturm-Liouville problem (SLP) in canonical Liouville normal form

$$-y''(t) + q(t)y(t) = \lambda y(t), \qquad 0 \le t \le 1 \tag{2.1}$$

with Dirichlet boundary conditions

$$y(0) = 0, \quad y(1) = 0,$$

where $q(t) \in C[0,1]$. Equation (2.1) is equivalent with the first order system by $y' = z$

$$\frac{dY(t)}{dt} = A(t)Y(t), \qquad 0 \le t \le 1, \tag{2.2}$$

$$C_1 Y(0) + C_2 Y(1) = \mathbf{0}, \tag{2.3}$$

where

$$Y(t) = \begin{bmatrix} y(t) \\ z(t) \end{bmatrix}, \qquad A(t) = \begin{bmatrix} 0 & 1 \\ q(t) - \lambda & 0 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad C_2 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The matrix $A(t)$ is splitted as a sum of M and $q(t)N$ such that

$$A(t) = M + q(t)N,$$

where

$$M = \begin{bmatrix} 0 & 1 \\ -\lambda & 0 \end{bmatrix} \quad \text{and} \quad N = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

We consider the partition of the interval $[0,1]$ as

$$t_i = ih, \qquad i = 0,1,\dots,n, \qquad h = \frac{1}{n}.$$

The symmetrical weighted sequential splitting of the system on time interval $[t_i, t_{i+1}]$ is defined as in the following algorithm,

$$\frac{dU_1(t)}{dt} = M\,U_1(t), \qquad U_1(t_i) = Y_{sp,i}, \tag{2.4}$$

$$\frac{dV_1(t)}{dt} = q(t)N\,V_1(t), \qquad V_1(t_i) = U_1(t_{i+1}) \tag{2.5}$$

and

$$\frac{dU_2(t)}{dt} = q(t)N\,U_2(t), \qquad U_2(t_i) = Y_{sp,i}, \tag{2.6}$$

$$\frac{dV_2(t)}{dt} = M\,V_2(t), \qquad V_2(t_i) = U_2(t_{i+1}), \tag{2.7}$$

for $i = 0,1,\dots,n-1$ and $Y_{sp,0}$ is a vector to be determined. The approximate split solution at the point t_{i+1} is defined as

$$Y_{sp,i+1} = \frac{1}{2}\left\{V_1(t_{i+1}) + V_2(t_{i+1})\right\}.$$

The exact solution of (2.4) is

$$U_1(t) = e^{(t-t_i)M} Y_{sp,i}.$$

Since N is nilpotent matrix of index 2 ($N^k = 0,\ k \geq 2$), $q(t)N$ and $(\int^t q(\xi)\,d\xi)N$ are commutative for all $t \in [0,1]$. Therefore, the solution of the system of differential equation (2.5) is

$$\begin{aligned} V_1(t) &= e^{w(t)N} U_1(t_{i+1}) \\ &= e^{[w(t)-w(t_i)]N} e^{hM} Y_{sp,i}, \end{aligned}$$

where $w(t) = \int^t q(\xi)d\xi$, $t \in [t_i, t_{i+1}]$.
By the same consideration, we get

$$\begin{aligned} U_2(t) &= e^{[w(t)-w(t_i)]N} Y_{sp,i}, \\ V_2(t) &= e^{(t-t_i)M} U_2(t_{i+1}) \\ &= e^{(t-t_i)M} e^{[w(t_{i+1})-w(t_i)]N} Y_{sp,i}. \end{aligned}$$

Therefore, we can write the approximate split solution as

$$Y_{sp,i+1} = \frac{1}{2}\left\{e^{[w(t_{i+1})-w(t_i)]N}e^{hM} + e^{hM}e^{[w(t_{i+1})-w(t_i)]N}\right\}Y_{sp,i} \tag{2.8}$$

$$= \frac{1}{2}\left\{e^{s_{i+1}N}e^{hM} + e^{hM}e^{s_{i+1}N}\right\}Y_{sp,i}, \tag{2.9}$$

where $s_{i+1} = s(t_{i+1}) = w(t_{i+1}) - w(t_i) = \int_{t_i}^{t_{i+1}} q(\xi)d\xi$, $i = 0, 1, \ldots, n-1$.

Finally, we can write the approximate split solution of (2.2) at $t_n = 1$ as

$$Y(1) \approx Y_{sp,n} = KY_{sp,0},$$

where K is 2×2 matrix obtained from the recurrence relation (2.8)

$$K = \frac{1}{2^n}\left\{\prod_{i=0}^{n-1}[e^{s_{n-i}N}e^{hM} + e^{hM}e^{s_{n-i}N}]\right\}.$$

The solution $Y_{sp,n}$ will be the solution of (2.2) and (2.3) if and only if

$$\begin{aligned}\mathbf{0} &= C_1 Y_{sp,0} + C_2 Y_{sp,n} \\ &= C_1 Y_{sp,0} + C_2 K Y_{sp,0} \\ &= (C_1 + C_2 K) Y_{sp,0}.\end{aligned}$$

For a non-trivial solution $Y_{sp,0}$, the determinant of $C_1 + C_2K$ must be zero. It follows that

$$Q(\lambda) = \det(C_1 + C_2K) \tag{2.10}$$

is the approximate characteristic equation of SLP (2.2). So, the approximation to eigenvalue is obtained solving $Q(\lambda) = 0$. Now, we task e^{hM} and $e^{s(t)N}$ to construct the matrix K.

It is apparent that

$$M^{2j} = (-1)^j \lambda^j I\ , \tag{2.11}$$

$$M^{2j+1} = (-1)^j \lambda^j M \qquad \text{for} \quad j = 0, 1, \ldots \tag{2.12}$$

Using (2.11) and (2.12), we have

$$e^{tM} = \left\{1 - \frac{\lambda t^2}{2!} + \cdots + (-1)^n \frac{\lambda^n t^{2n}}{(2n)!} + \cdots \right\} I$$

$$+ \left\{t - \frac{\lambda t^3}{3!} + \cdots + (-1)^n \frac{\lambda^n t^{2n+1}}{(2n+1)!} + \cdots \right\} M \tag{2.13}$$

$$= \cos(\sqrt{\lambda}t)I + \frac{1}{\sqrt{\lambda}} \sin(\sqrt{\lambda}t)M. \tag{2.14}$$

Since N is nilpotent matrix of index 2, it is clear that

$$e^{s_{n-i}N} = I + s_{n-i}N. \tag{2.15}$$

Using (2.14) and (2.15), we obtain that

$$K = \frac{1}{2^n}\left\{\prod_{i=0}^{n-1}[2e^{hM} + s_{n-i}(Ne^{hM} + e^{hM}N)]\right\},$$

where

$$e^{hM}N + Ne^{hM} = \begin{bmatrix} b(\lambda) & 0 \\ 2a(\lambda) & b(\lambda) \end{bmatrix} = b(\lambda)I + 2a(\lambda)N,$$

and $\quad a(\lambda) = \cos(\sqrt{\lambda}h), \quad b(\lambda) = \frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}h).$
Then K will be

$$K = \frac{1}{2^n}\left\{\prod_{i=0}^{n-1}[2e^{hM} + s_{n-i}[b(\lambda)I + 2a(\lambda)N]]\right\}.$$

If $q(t) = 0$, then $s_i = 0$. Since $nh = 1$, we have

$$K = \frac{1}{2^n}\prod_{i=0}^{n-1} 2e^{hM}$$
$$= e^M.$$

From $\det(C_1 + C_2K) = 0$, we get the characteristic equation of the original SLP

$$\frac{1}{\sqrt{\lambda}}\sin(\sqrt{\lambda}) = 0$$

and we get the exact eigenvalues of SLP (1.1) and (1.2), $\lambda_k = k^2\pi^2$, $k = 1, 2, \ldots$ using the proposed method.

Now, we consider the case $q(t)$ is constant that is $q(t) = q$,

$$\begin{aligned} w(t) &= \int^t q(\xi)\, d\xi \\ &= qt. \end{aligned}$$

From $t_i = ih$, we get

$$\begin{aligned} s(t_{i+1}) &= w(t_{i+1}) - w(t_i) \\ &= qh. \end{aligned}$$

Then K will be

$$\begin{aligned} K &= \frac{1}{2^n}[2e^{hM} + qh(bI + 2aN)]^n \\ &= \frac{1}{2^n}L^n, \end{aligned}$$

where $L = \begin{bmatrix} 2a + qhb & 2b \\ -2\lambda b + 2aqh & 2a + qhb \end{bmatrix}$ and $a(\lambda) := a$, $b(\lambda) := b$ for simplicity.

By similarity transformation, it follows that

$$L = PDP^{-1},$$

where

$$D = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, \quad P = \begin{bmatrix} \frac{\sqrt{b(-\lambda b + aqh)}}{-\lambda b + aqh} & -\frac{\sqrt{b(-\lambda b + aqh)}}{-\lambda b + aqh} \\ 1 & 1 \end{bmatrix},$$

μ_1 and μ_2 are eigenvalues of L,

$$\mu_{1,2} = 2a + qhb \mp 2\sqrt{b(-\lambda b + aqh)}. \tag{2.16}$$

Using $L^n = PD^nP^{-1}$, (2.10) and (2.16), we have the characteristic function $Q(\lambda)$ as follows

$$\begin{aligned} Q(\lambda) &= \det(C_1 + \frac{1}{2^n}C_2L^n) && (2.17) \\ &= \frac{-1}{2^{n+1}} \frac{b\sqrt{n}}{\sqrt{aqb - b^2 n\lambda}}(\mu_2^n - \mu_1^n), && (2.18) \end{aligned}$$

where

$$\mu_1^n = \left(\frac{1}{n}\right)^n [2an + qb + 2\sqrt{bn(-\lambda bn + aq)}]^n,$$

$$\mu_2^n = \left(\frac{1}{n}\right)^n [2an + qb - 2\sqrt{bn(-\lambda bn + aq)}]^n,$$

$a = \cos(\frac{\sqrt{\lambda}}{n})$ and $b = \frac{1}{\sqrt{\lambda}}\sin(\frac{\sqrt{\lambda}}{n})$. We get limit of the characteristic equation $Q(\lambda)$ as

$$\lim_{n\to\infty} Q(\lambda) = \frac{1}{\sqrt{\lambda - q}}\left\{\frac{e^{i\sqrt{\lambda-q}} - e^{-i\sqrt{\lambda-q}}}{2i}\right\}$$

$$= \frac{1}{\sqrt{\lambda - q}}\sin\sqrt{\lambda - q},$$

where $\lambda - q > 0$. So, we conclude that $Q(\lambda)$ converges to the characteristic function of the original problem (2.1) for $q(t) = q$.

3 Asymptotic Behaviour for Eigenvalues of SLP

In order to derive the error estimate e_k, it is necessary to examine in some details of the asymptotic behaviour of error estimate for constant case $q(t) = q$. Let

$$\begin{aligned} |e_k| &= |\wedge_k - \lambda_k^{(p+1)}| \\ &= \left|\wedge_k - \left\{\lambda_k^{(p)} - F(\lambda_k^{(p)})\right\}\right|, \end{aligned}$$

where $\lambda_k^{(p)}$ is the k^{th} approximate eigenvalue to the k^{th} eigenvalues $\wedge_k$ of the original problem (2.1) that obtained by Newton method at p^{th} step, $F(\lambda)$ is the reduced rational function to $\frac{Q(\lambda)}{Q'(\lambda)}$, $Q(\lambda)$ formulated in (2.17) is approximate characteristic equation that obtained from the symmetrical weighted splitting method.

Then, from the convergence of Newton's iterations, we write

$$\begin{aligned} |e_k| &= |\wedge_k - \lambda_k^{(p+1)}| && (3.1)\\ &\le |\wedge_k - \lambda_k^{(1)}| && (3.2)\\ &\le \left|\wedge_k - \lambda_k^{(0)} + F(\lambda_k^{(0)})\right| && (3.3)\\ &\le \left|q + F(\lambda_k^{(0)})\right|, && (3.4) \end{aligned}$$

where $\lambda_k^{(0)} = k^2\pi^2$, $\wedge_k = k^2\pi^2 + q$.

We will discuss the asymptotic behaviour of error formula in two cases.

Case i : Let $k = nm + j$, $\lambda_k^{(0)} = (nm+j)^2\pi^2$ and $j = \frac{n}{2}$, n is even number of interval, then

$$a(\lambda) = 0, \qquad \sqrt{\lambda}b(\lambda) = (-1)^m, \qquad m = 0, 1, \dots .$$

Using the powers of $\frac{1}{\lambda}$ for asymptotic expansion of the $q + F(\lambda)$, we have

$$|e_k| = |e_{\frac{n}{2}(2m+1)}| \tag{3.5}$$

$$\leq \frac{|c_1|}{\lambda_k^{(0)}}, \tag{3.6}$$

where $c_1 = (q^2 - \frac{1}{12}q^3) + \mathcal{O}(\frac{1}{n})$. We obtain that $e_k \to 0$ as $k \to \infty$ for any constant q and the error is less than 1 if $c_1 < \lambda_k^{(0)}$, i.e,

$$k > \frac{\sqrt{|q^2 - \frac{1}{12}q^3|}}{\pi}, \tag{3.7}$$

for any even $n \geq 2$.

Case ii : Let $k = nm + j$, $\lambda_k^{(0)} = (nm+j)^2\pi^2$ and $j \neq \frac{n}{2}$ by using the similar consideration as **Case i**, we get

$$|e_k| = |e_{nm+j}| \tag{3.8}$$

$$\leq \frac{|d_1|}{\sqrt{\lambda_k^{(0)}}}, \quad m = 0, 1, ..., \tag{3.9}$$

where

$$d_1 = \frac{\cos^3(\frac{j}{n}\pi)q^2}{4n\sin(\frac{j}{n}\pi)} + \mathcal{O}(\frac{1}{n^2}).$$

It is clear that

$$\left|\frac{\cos^3(\frac{j}{n}\pi)}{n\sin(\frac{j}{n}\pi)}\right| \leq \frac{1}{\pi} \quad \text{for all } j \neq \frac{n}{2}.$$

So, we get $d_1 = \mathcal{O}(\frac{q^2}{4\pi})$.

Therefore, the error of the eigenvalues is decreasing for $\frac{|d_1|}{\sqrt{\lambda_k^{(0)}}} < 1$, that is,

$$k > \frac{q^2}{4\pi^2}. \tag{3.10}$$

As a result, from the asymptotic behaviour of the error formulas in **Case i** and **Case ii**, we obtain that, for $m = 0, 1, ..., j = 1, 2, ...,$

$$| \wedge_k - \lambda_k^{(p+1)} | = \begin{cases} \mathcal{O}(\frac{1}{k^2}), & k = \frac{n}{2}(2m+1), \quad n : \text{even}, \\ \mathcal{O}(\frac{1}{k}), & k = nm + j, \quad j \neq \frac{n}{2}, \end{cases} \tag{3.11}$$

or

$$| \wedge_k - \lambda_k^{(p+1)} | = \begin{cases} \mathcal{O}(\frac{1}{\lambda_k}), & k = \frac{n}{2}(2m+1), \quad n : \text{even}, \\ \mathcal{O}(\frac{1}{\sqrt{\lambda_k}}), & k = nm + j, \quad j \neq \frac{n}{2}, \end{cases} \tag{3.12}$$

satisfying the conditions (3.7) and (3.10) corresponding to the choosen n.

4 Numerical Results

To illustrate the symmetrical weighted splitting method for Strum-Liouville problems in canonical normal form with Dirichlet boundary conditions, the problem (2.1) is considered with $q(t) = 5$, $q(t) = e^t$ and $q(t) = t^2$ as considered in [7] and [12]. Throughout tables, the computed k^{th} approximate eigenvalue for choosen number of intervals n are denoted by $\lambda_{k,n}$. Let $\wedge_k$ be the k^{th} exact eigenvalue, $\lambda_{k,n}^{(f)}$ be finite difference approximation for choosen n. For the numerical results, the observed orders are obtained the following formulas

$$order = \log\left(\frac{\lambda_{k,n} - \lambda_{k,l}}{\lambda_{r,n} - \lambda_{r,l}}\right) / \log\left(\frac{r}{k}\right), \tag{4.1}$$

$$order = \log\left(\frac{\wedge_s - \lambda_{s,n}}{\wedge_r - \lambda_{r,n}}\right) / \log\left(\frac{r}{s}\right), \tag{4.2}$$

where $\lambda_{k,n}$ and $\lambda_{k,l}$ are the approximate eigenvalues to $\wedge_k$ for different number of intervals n, l, respectively.

In Table 1 and 2, the errors and the observed orders are given by using the formula (4.1) and (4.1) for even and odd number of intervals n.

Table 3 shows the accuracy of the computed eigenvalues by splitting method $\lambda_{k,2}$ for $n = 2$ by comparing the results of finite difference method $\lambda_{k,20}^{(f)}$ for $n = 20$. It is observed that the results of the presented method for $n = 2$ are better than those of the finite difference method for $n = 20$.

In Table 4, approximate eigenvalues obtained using presented method and finite difference method of the problem

$$-y''(t) + e^t y(t) = \lambda y(t), \qquad y(0) = y(1) = 0 \tag{4.3}$$

are given by comparing the results of the eigenvalues λ_k^* given in [12]. In Table 6, approximate eigenvalues obtained using presented method and finite difference method of the problem

$$-y''(t) + t^2 y(t) = \lambda y(t), \qquad y(0) = y(1) = 0 \tag{4.4}$$

are introduced by comparing the results of the eigenvalues λ_k^* given in [7]. In Table 5 and 7, the greater than 10^{th} eigenvalues are given for problems (4.3) and (4.4).

As a result, the accuracy of the computed eigenvalues by the symmetrical weighted sequential splitting method is better than the finite difference method for the higher eigenvalues without using correction and large n.

Table 1: Comparison of the eigenvalues for $n = 2$, $j = 1$ and $n = 6$, $j = 3$ and corresponding orders for $q(t) = 5$.

k	$\lambda_{k,2}$	$\|\lambda_{k,2} - \wedge_k\|$	$\lambda_{k,6}$	$\|\lambda_{k,6} - \wedge_k\|$	*order*
15	2225.657013825	3.97642E-3	2225.65805223	2.93801E-3	1.99442
39	15016.66770447	5.89583E-4	15016.6678586	4.35470E-4	1.99927
81	64759.47433883	1.36722E-4	64759.4743746	1.00979E-4	1.99983
219	473361.0966619	1.87049E-5	473361.096667	1.38147E-5	1.99989
411	1667188.445031	5.31063E-6	1667188.44503	3.92250E-6	2.00297
501	2477285.574274	3.57348E-6	2477285.57428	2.63983E-6	2.00277

Table 2: Comparison of the eigenvalues for $n = 3$, $j = 1$ and $n = 5$, $j = 1$ and corresponding orders for $q(t) = 5$.

k	$\lambda_{k,3}$	$\|\lambda_{k,3} - \wedge_k\|$	$\lambda_{k,5}$	$\|\lambda_{k,5} - \wedge_k\|$	*order*
16	2531.621430695	2.70402E-3	2531.638123758	1.93971E-2	1.01109
31	9489.692043667	2.21422E-3	9489.700596490	1.07607E-2	1.00718
61	36729.79932009	1.34364E-3	36729.80364782	5.67137E-3	1.0042
121	144505.8787701	7.33757E-4	144505.8809467	2.91039E-3	1.00188
301	894201.0286518	3.08740E-4	894201.0295255	1.18245E-3	1.00083
436	1876177.318445	2.15123E-4	1876177.319048	8.18115E-4	1.00063
541	2888650.685889	1.74061E-4	2888650.686375	6.59956E-4	1.00051

Table 3: Comparison of the errors of eigenvalues obtained using finite difference method and proposed method for $q(t) = 2$.

k	$\wedge_k$	$\mid \wedge_k - \lambda^{(f)}_{k,20} \mid$	$\mid \wedge_k - \lambda_{k,2} \mid$
1	11.869604401089	2.0277E-2	9.7745E-2
3	90.826439609804	1.6317	1.2824E-2
5	248.74011002723	12.4255	4.6873E-3
7	485.61061565337	46.8030	2.4017E-3
9	801.43795648823	124.5855	1.4554E-3
11	1196.2221325318	269.0746	9.7516E-4
13	1669.9631437841	504.7707	6.9855E-4
15	2222.6609902451	854.9756	5.2486E-4
17	2854.3156719148	1339.5105	4.0871E-4
19	3564.9271887932	1972.7765	3.2725E-4

Table 4: Comparison of the errors of first 10 eigenvalues obtained using finite difference method and proposed method for $q(t) = e^t$.

k	n	$\lambda_{k,n}$	λ^*_k	$\lvert\lambda_{k,n} - \lambda^*_k\rvert$	$\lvert\lambda^{(f)}_{k,39} - \lambda^*_k\rvert$
1	6	11.5269698092	11.5424	1.54302E-2	5.7E-3
2	4	41.1780319772	41.1867	8.66802E-3	8.13E-2
3	6	90.5364117453	90.5404	3.98825E-3	4.106E-1
4	6	159.621857116	159.6296	7.74288E-3	1.2954
5	2	248.454997266	248.4569	1.90273E-3	3.1544
6	4	357.021885511	357.023	1.11449E-3	6.5261
7	2	485.327159265	485.3281	9.40735E-4	12.0593
8	5	633.369784990	633.3724	2.61501E-3	20.5083
9	6	801.155299106	801.1558	5.00894E-4	32.7373
10	4	988.677943781	988.6783	3.56219E-4	49.7023

Table 5: The greater than 10^{th} eigenvalues for $n = 2$, $j = 1$ and $n = 6$, $j = 3$ for $q(t) = e^t$.

k	$\lambda_{k,2}$	$\lambda_{k,6}$	$order$
15	2222.37889240430	2222.37893348878	1.99823
21	4354.21362892151	4354.21364989546	1.99936
45	19987.6671518178	19987.6671563877	1.99981
69	46990.9048174579	46990.9048194018	1.99997
87	74704.7539823785	74704.7539836012	2.
129	164241.805115218	164241.805115775	2.00039
237	554367.527885094	554367.527885259	2.00442
351	1215946.85009974	1215946.85009981	1.99589
405	1618863.58016998	1618863.58017004	1.91204
513	2597375.63891178	2597375.63891182	2.20865

Table 6: Comparison of the errors of first 10 eigenvalues obtained using finite difference method and proposed method for $q(t) = t^2$.

k	n	$\lambda_{k,n}$	λ_k^*	$\lvert\lambda_{k,n} - \lambda_k^*\rvert$	$\lvert\lambda_{k,20}^{(f)} - \lambda_k^*\rvert$
1	7	10.1571617215	10.1511640305	5.99769E-3	2.0291E-2
2	7	39.8047902261	39.7993930037	5.39722E-3	3.2365E-1
3	5	89.1573512146	89.1543424563	3.00800E-3	1.6316885
4	6	158.242156672	158.243961707	1.80503E-3	5.1273118
5	2	247.073327812	247.071500228	1.82758E-3	12.425603
6	4	355.639012037	355.637743806	2.68230E-3	25.534059
7	2	483.943889994	483.942959280	9.30714E-4	46.803153
8	5	631.985860302	631.987257576	1.39727E-3	78.868467
9	2	799.771254125	799.770691532	5.62593E-4	124.58579
10	7	987.293213898	987.293288927	7.50294E-5	186.96079

Table 7: The greater than 10^{th} eigenvalues for $n = 2$, $j = 1$ and $n = 6$, $j = 3$ for $q(t) = t^2$.

k	$\lambda_{k,2}$	$\lambda_{k,6}$	*order*
21	4352.82886765494	4352.8288677639	1.99972
27	7195.27493775968	7195.2749378256	2.00004
33	10748.3325234635	10748.332523508	1.99971
45	19986.2822441108	19986.282244135	2.00125
51	25671.1743794546	25671.174379473	2.0018
63	39172.7932005282	39172.793200540	2.00679
81	64754.8078084397	64754.807808447	1.99191
87	74703.3690447965	74703.369044803	2.01201
105	108812.721855081	108812.72185509	2.0298
147	213272.614836339	213272.61483634	2.02787

5 Conclusions

In this paper, the symmetrical weighted sequential splitting method is applied to approximate the eigenvalues of regular Sturm-Liouville problem in Liouville normal form by converting them into a system of differential equation with Dirichlet boundary conditions. To illustrate the method, some classical examples are introduced. From the tables, it is seen that the high order accuracy can be succeeded without increasing the number of intervals n especially for large eigenvalues contrary to finite difference method.

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