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ON SECOND HANKEL DETERMINANT FOR TWO NEW SUBCLASSES OF ANALYTIC FUNCTIONS

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Abstract. In this paper, we obtain sharp upper bounds for the functional $|a_2a_4 - a_3^2|$ for functions belonging to $S^*(\alpha,\beta)$ and $C(\alpha,\beta)$. Our results extend corresponding previously known results.

1 Introduction

Let S denote the class of normalized analytic univalent functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

where $z \in E : \{z : |z| < 1\}$.

In 1976, Noonan and Thomas [9] defined the q^{th} Hankel determinant for $q \geq 1$ and $n \geq 0$ by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & \cdot & \cdots & \\ \vdots & & \vdots & \\ a_{n+q-1} & \cdot & \cdots & a_{n+2q-2} \end{vmatrix}$$

This determinant has also been considered by several authors. For example, Noor in [10], determined the rate of growth of $H_q(n)$ as $n \to \infty$ for functions of the form (1.1) with bounded boundary. In particular, sharp bounds on $H_2(2)$ were obtained by the authors of articles [1], [3], [5], [6], [12] for different classes of functions.

One can observe that the Fekete-Szego functional is $H_2(1)$. Also they generalized the estimate $|a_3 - \mu a_2^2|$, where μ is real and $f(z) \in S$.

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In this paper, we consider the second Hankel determinant for $q = 2$ and $n = 2$, $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$ and obtain an upper bound for the functional $|a_2a_4 - a_3^2|$ for functions belonging to the classes $S^*(\alpha, \beta)$ and $C(\alpha, \beta)$ which are defined as follows:

Definition 1. *Let $f(z)$ be given by (1.1). Then $f(z) \in S^*(\alpha, \beta)$ if and only if $Re\left\{\frac{zf'(z)}{f(z)} + \alpha\frac{z^2f''(z)}{f(z)}\right\} > \beta$, $z \in E$ for some β $(0 \leq \beta < 1)$ and $\alpha \geq 0$.*

Remark 2. *The choice $\alpha = 0$ yields $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \beta$, $z \in E$, so that we get $S^*(0, \beta)$, the class of starlike functions of order β [11].*

Remark 3. *When $\alpha = 0$, $\beta = 0$, we get the class S^*, the class of starlike functions [11].*

Remark 4. *When $\beta = 0$, we get the corresponding result of Shanmugam [13].*

Definition 5. *Let $f(z)$ be given by (1.1). Then $f(z) \in C(\alpha, \beta)$ if and only if $Re\left\{\frac{[zf'(z)+\alpha z^2 f''(z)]'}{f'(z)}\right\} > \beta$, $z \in E$, for some β $(0 \leq \beta < 1)$ and $\alpha \geq 0$.*

Remark 6. *The choice $\alpha = 0$ yields $Re\left\{\frac{1+zf''(z)}{f'(z)}\right\} > \beta$, $z \in E$, so that we get $C(0, \beta)$, the class of convex functions of order β [11].*

Remark 7. *When $\alpha = 0$, $\beta = 0$, we get the class C, the class of convex functions [11].*

Remark 8. *When $\beta = 0$, we get the corresponding result of Shanmugam [13].*

2 Preliminary Results

Let P be the family of all functions $p(z)$ analytic in E for which $Re\{p(z)\} > 0$ and

$$p(z) = 1 + c_1z + c_2z^2 + \cdots \tag{2.1}$$

for $z \in E$.

To prove the main results we shall need the following lemmas. Throughout this paper, we assume that $p(z)$ is given by (2.1) and $f(z)$ is given by (1.1).

Lemma 9. *[2] If $p(z) \in P$, then $|c_k| \leq 2$ for each $k \in N$.*

Lemma 10. *([7, 8]) Let $p(z) \in P$, then*

$$2c_2 = c_1^2 + x(4 - c_1^2) \tag{2.2}$$

and

$$4c_3 = c_1^3 + 2(4 - c_1^2)c_1x - c_1(4 - c_1^2)x^2 + 2(4 - c_1^2)(1 - |x|^2)y \tag{2.3}$$

for some value of x, y such that $|x| \leq 1$ and $|y| \leq 1$.

Theorem 11. *[4] Let $f(z) \in S^*$. Then*

$$|a_2 a_4 - a_3^2| \leq 1.$$

The result obtained is sharp.

Theorem 12. *[4] Let $f(z) \in C$. Then*

$$|a_2 a_4 - a_3^2| \leq \frac{1}{8}.$$

The result obtained is sharp.

3 Main Results

Theorem 13. *Let $f(z) \in S^*(\alpha, \beta)$, then*

$$|a_2 a_4 - a_3^2| \leq \frac{(1-\beta)^2}{(1+3\alpha)^2}.$$

The result obtained is sharp.

Proof. Let $f(z) \in S^*(\alpha, \beta)$. Then there exists a $p(z) \in P$, such that

$$zf'(z) + \alpha z^2 f''(z) = f(z)[(1-\beta)p(z) + \beta] \tag{3.1}$$

for some $z \in E$.
Equating the coefficients in (3.1), we get

$$\begin{aligned} a_2 &= \frac{c_1(1-\beta)}{1+2\alpha} \\ a_3 &= \frac{c_2(1-\beta)}{2(1+3\alpha)} + \frac{c_1^2(1-\beta)^2}{2(1+2\alpha)(1+3\alpha)} \\ a_4 &= \frac{c_3(1-\beta)}{3(1+4\alpha)} + \frac{c_1 c_2(1-\beta)^2(3+8\alpha)}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1^3(1-\beta)^3}{6(1+2\alpha)(1+3\alpha)(1+4\alpha)}. \end{aligned} \tag{3.2}$$

From (3.2), it is easily established that

$$\begin{aligned} |a_2 a_4 - a_3^2| = \Bigg| &\frac{c_1 c_3(1-\beta)^2}{3(1+2\alpha)(1+4\alpha)} - \frac{c_2^2(1-\beta)^2}{4(1+3\alpha)^2} \\ &- \frac{c_1^4(1-\beta)^4(1+6\alpha)}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} - \frac{\alpha c_1^2 c_2(1-\beta)^3}{6(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \Bigg| \end{aligned} \tag{3.3}$$

Substituting for c_2 and c_3 from (2.2) and (2.3) and since $|c_1| \le 2$, by Lemma 9, let $c_1 = c$ and assume without restriction that $c \in [0, 2]$. We obtain

$$
\begin{aligned}
|a_2a_4 - a_3^2| = \Bigg| & \frac{(1-\beta)^2[c^4 + 2(4-c^2)c^2x - (4-c^2)c^2x^2 + 2c(4-c^2)(1-|x|^2)y]}{12(1+2\alpha)(1+4\alpha)} \\
& - \frac{(1-\beta)^2[c^4 + (4-c^2)^2x^2 + 2c^2x(4-c^2)]}{16(1+3\alpha)^2} \\
& - \frac{(1-\beta)^4[c^4(1+6\alpha)]}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} - \frac{(1-\beta)^2\alpha[c^4 + (4-c^2)xc^2]}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\Bigg| \qquad (3.4)
\end{aligned}
$$

By triangle inequality,

$$
\begin{aligned}
|a_2a_4 - a_3^2| \le & \frac{(1-\beta)^2[c^4 + 2(4-c^2)c^2\rho + 2c(4-c^2) + c(c-2)(4-c^2)\rho^2]}{12(1+2\alpha)(1+4\alpha)} \\
& + \frac{(1-\beta)^2[c^4 + (4-c^2)^2\rho^2 + 2c\rho(4-c^2)]}{16(1+3\alpha)^2} \\
& + \frac{(1-\beta)^4c^4(6\alpha+1)}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} + \frac{(1-\beta)^3\alpha[c^4 + c^2\rho(4-c^2)]}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\
& = F(\rho) \qquad (3.5)
\end{aligned}
$$

with $\rho = |x| \le 1$. Furthermore

$$
\begin{aligned}
F'(\rho) = & \frac{(1-\beta)^2[2c^2(4-c^2) + 2c\rho(c-2)(4-c^2)]}{12(1+2\alpha)(1+4\alpha)} \\
& + \frac{(1-\beta)^2[2(4-c^2)^2\rho + 2c(4-c^2)]}{16(1+3\alpha)^2} \\
& + \frac{(1-\beta)^3\alpha c^2(4-c^2)}{12(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}
\end{aligned}
$$

and with elementary calculus, we can show that $F'(\rho) > 0$ for $\rho > 0$.

This implies that F is an increasing function and thus the upper bound for (3.4) corresponds to $\rho = 1$ and $c = 0$ gives

$$|a_2a_4 - a_3^2| \le \frac{(1-\beta)^2}{(1+3\alpha)^2}.$$

It follows from (2.3) that if $c_1 = c = 0$ and $|x| = \rho = 1$ then $c_3 = 0$.
If $p(z) \in P$ with $c_1 = 0$, $c_2 = 2$ and $c_3 = 0$ then we obtain
$p(z) = \dfrac{1+z^2}{1-z^2} = 1 + 2z^2 + 2z^4 + \cdots \in P$,
which shows that the result is sharp. □

Remark 14. *When we replace β by 0, we get the corresponding result of Shanmugam et al. [13].*

Remark 15. *When we replace β by 0 and α by 0, then we get the corresponding result of Janteng et al. [4].*

Theorem 16. *Let $f(z) \in C(\alpha, \beta)$, then*

$$|a_2a_4 - a_3^2| \leq \frac{1}{144}\left|\frac{M}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\right|,$$

where $M = (1-\beta)^2(280\alpha^3 + 332\alpha^2 + 128\alpha + 16) + (1-\beta)^4(1+7\alpha) + (1-\beta)^3(8\alpha^2 + 3\alpha + 1)$. The result obtained is sharp.

Proof. Let $f(z) \in C(\alpha, \beta)$
Then there exists a $p(z) \in P$, such that

$$f'(z) + zf''(z) + \alpha z^2 f'''(z) + 2\alpha z f''(z) = f'(z)[(1-\beta)p(z) + \beta] \tag{3.6}$$

for some $z \in E$.

Equating the coefficients in (3.6), we get

$$\begin{aligned} a_2 &= \frac{c_1(1-\beta)}{2(1+2\alpha)} \\ a_3 &= \frac{c_1^2(1-\beta)^2}{6(1+2\alpha)(1+3\alpha)} + \frac{c_2(1-\beta)}{6(1+3\alpha)} \\ a_4 &= \frac{c_1^3(1-\beta)^3}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_1c_2(1-\beta)^2(3+8\alpha)}{24(1+2\alpha)(1+3\alpha)(1+4\alpha)} + \frac{c_3(1-\beta)}{12(1+4\alpha)}. \end{aligned} \tag{3.7}$$

From (3.7),

$$\begin{aligned} |a_2a_4 - a_3^2| = \frac{1}{144}\bigg| &\frac{6c_1c_3(1-\beta)^2}{(1+2\alpha)(1+4\alpha)} - \frac{4c_2^2(1-\beta)^2}{(1+3\alpha)^2} \\ &- \frac{c_1^4(1-\beta)^4(1+7\alpha)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} + \frac{c_1^2c_2(1-\beta)^3(8\alpha^2+3\alpha+1)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\bigg| \end{aligned} \tag{3.8}$$

Now assuming $c_1 = c$ $(0 \leq c \leq 2)$ and using (2.2) and (2.3), we get

$$\begin{aligned} = \frac{1}{144}\bigg| &\frac{(1-\beta)^2[6c^4 + 12c(4-c^2)cx - 6c^2(4-c^2)x^2 + 12c(4-c^2)(1-|x|^2)y]}{4(1+2\alpha)(1+4\alpha)} \\ &- \frac{(1-\beta)^2[c^2 + x(4-c^2)]^2}{(1+3\alpha)^2} - \frac{(1-\beta)^4c^4(1+7\alpha)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\ &+ \frac{(1-\beta)^3c^2[c^2 + x(4-c^2)](8\alpha^2+3\alpha+1)}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}\bigg| \end{aligned}$$

Using triangle inequality,

$$\begin{aligned}
&\leq \frac{(1-\beta)^2[6c^4+12c^2\rho(4-c^2)+6c(c-2)\rho^2(4-c^2)+12c(4-c^2)]}{4(1+2\alpha)(1+4\alpha)} \\
&+ \frac{(1-\beta)^2[c^4+\rho^2(4-c^2)^2+2c^2\rho(4-c^2)]}{(1+3\alpha)^2} \\
&+ \frac{(1-\beta)^4c^4(1+7\alpha)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} + \frac{(1-\beta)^3[c^4+c^2\rho(4-c^2)](8\alpha^2+3\alpha+1)}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\
&= F(\rho) \qquad (3.9)
\end{aligned}$$

with $\rho = |x| \leq 1$.
Furthermore,

$$\begin{aligned}
F'(\rho) &= \frac{(1-\beta)^2 3[c^2(4-c^2)+c(c-2)(4-c^2)]}{(1+2\alpha)(1+4\alpha)} \\
&+ \frac{(1-\beta)^3c^2(4-c^2)(8\alpha^2+3\alpha+1)}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\
&+ \frac{(1-\beta)^2[2\rho(4-c^2)^2+2c^2(4-c^2)]}{(1+3\alpha)^2}
\end{aligned}$$

Using elementary calculus, we can show that $F'(\rho) > 0$ for $\rho > 0$. This shows that F is an increasing function and $\max\limits_{\rho\leq 1} F(\rho) = F(1)$.
Now, let

$$\begin{aligned}
G(c) = F(1) &= \frac{3(1-\beta)^2[c^2(4-c^2)+c(c-2)(4-c^2)]}{(1+2\alpha)(1+4\alpha)} \\
&+ \frac{(1-\beta)^2c^2(4-c^2)(8\alpha^2+3\alpha+1)}{2(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\
&+ \frac{2(1-\beta)^2[c^2(4-c^2)+(4-c^2)^2]}{(1+3\alpha)^2}
\end{aligned}$$

Trivially, G attains its maximum at $c = 1$. Thus the upper bound for (3.9) corresponds to $\rho = 1$ and $c = 1$, gives

$$\begin{aligned}
&\left| \frac{(1-\beta)^2 6c_1c_3}{(1+2\alpha)(1+4\alpha)} - \frac{(1-\beta)^2 4c_2^2}{(1+3\alpha)^2} - \frac{(1-\beta)^4c_1^4(1+7\alpha)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \right. \\
&\qquad \left. + \frac{(1-\beta)^3(8\alpha^2+3\alpha+1)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \right| \\
&\leq \frac{15(1-\beta)^2}{(1+2\alpha)(1+4\alpha)} + \frac{(1-\beta)^2 16}{(1+3\alpha)^2} + \frac{(1-\beta)^4(1+7\alpha)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} \\
&+ \frac{2(8\alpha^2+3\alpha+1)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)}.
\end{aligned}$$

If $c_1 = 1$, $c_2 = -1$ and $c_3 = -2$ then we know
$p(z) = \dfrac{1-z^2}{1-z+z^2} = 1 + z - z^2 - 2z^3 + z^4 + \cdots \in P$,
which shows that the result is sharp. □

Remark 17. *When we replace β by 0, we get*

$$|a_2a_4 - a_3^2| \leq \frac{15}{(1+2\alpha)(1+4\alpha)} + \frac{2(8\alpha^2+3\alpha+1)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} + \frac{(1+7\alpha)}{(1+2\alpha)^2(1+3\alpha)^2(1+4\alpha)} + \frac{16}{(1+3\alpha)^2},$$

a result obtained by Shanmugam et al. [13].

Remark 18. *When we replace β by 0 and α by 0, we get*

$$|a_2a_4 - a_3^2| \leq \frac{1}{8},$$

the sharp result obtained by Janteng et al. [4].

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