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EXPLICIT STABILITY CONDITIONS FOR NEUTRAL TYPE VECTOR FUNCTIONAL DIFFERENTIAL EQUATIONS. A SURVEY

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Abstract. This paper is a survey of the recent results of the author on the stability of linear and nonlinear neutral type functional differential equations. Mainly, vector equations are considered. In particular, equations whose nonlinearities are causal mappings are investigated. These equations include neutral type, ordinary differential, differential-delay, integro-differential and other traditional equations. Explicit conditions for the Lyapunov, exponential, input-to-state and absolute stabilities are derived. Moreover, solution estimates for the considered equations are established. They provide us the bounds for the regions of attraction of steady states. A part of the paper is devoted to the Aizerman type problem from the the absolute stability theory. The main methodology presented in the paper is based on a combined usage of the recent norm estimates for matrix-valued functions with the generalized Bohl - Perron principle, positivity conditions for fundamental solutions of scalar equations and properties of the so called generalized norm

1 Introduction

1. This paper is a survey of the recent results of the author on the stability of the neutral type linear and nonlinear vector functional differential equations. Functional differential equations naturally arise in various applications, such as control systems, mechanics, nuclear reactors, distributed networks, heat flows, neural networks, combustion, interaction of species, microbiology, learning models, epidemiology, physiology, and many others. The theory of functional differential equations has been developed in the works of V. Volterra, A.D. Myshkis, N.N. Krasovskii, B. Razumikhin, N. Minorsky, R. Bellman, A. Halanay, J. Hale and other mathematicians.

The problem of the stability analysis of neutral type equations continues to attract the attention of many specialists despite its long history. It is still one of the

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most burning problems of the theory of functional differential equations because of the absence of its complete solution. The basic method for the stability analysis is the method based on the Lyapunov type functionals. By that method many very strong results are obtained. However finding the Lyapunov type functionals for vector neutral type equations is often connected with serious mathematical difficulties, especially in regard to non-autonomous equations. To the contrary, the stability conditions presented in the suggested survey are mainly explicitly formulated in terms of the determinants and eigenvalues of auxiliary matrices dependent on a parameter. This fact allows us to apply the well-known results of the theory of matrices to the stability analysis.

2. Recall that the Bohl - Perron principle means that the homogeneous ordinary differential equation (ODE) $dy/dt = A(t)y$ $(t \geq 0)$ with a variable $n \times n$-matrix $A(t)$, bounded on $[0, \infty)$ is exponentially stable, provided the nonhomogeneous ODE $dx/dt = A(t)x + f(t)$ with the zero initial condition has a bounded solution for any bounded vector valued function f, cf. [7]. In [26, Theorem 4.15] the Bohl - Perron principle was generalized to a class of retarded systems with a scalar measure; besides the asymptotic (not exponential) stability was proved. Afterwards the result of the book [26] was improved under additional conditions and was effectively used for the stability analysis of the first and second order scalar equations, cf. [3, 4, 5] and references therein. In the book [23] the Bohl - Perron principle have been extended to differential delay equations in the general case. Moreover, in that book a result similar to the Bohl - Perron principle on the connections between homogeneous and non-homogeneous differential delay equations in the terms of the L^p -norm was derived.

In the present paper we suggest a generalization of the Bohl - Perron principle to a class of neutral type equations.

3. We also consider some classes of equations with nonlinear causal mappings and linear neutral parts. These equations include neutral type, differential, differential-delay, integro-differential and other traditional equations. The stability theory of equations with causal mappings is in an early stage of development, cf. [6, 33].

In this article we present conditions for the Lyapunov stability, L^2-absolute stability, input-to-state stability and the exponential stability of solutions of the pointed nonlinear equations.

The literature on the absolute stability of retarded and continuous systems is rather rich. The basic stability results for differential-delay equations are presented in the well-known books [29, 42].

4. Furthermore, in the paper [2] M.A. Aizerman conjectured that a single input-single output system is absolutely stable in the Hurwitzian angle. That hypothesis caused the great interest among the specialists. Counter-examples were set up that demonstrated it was not, in general, true. Therefore, the following problem arose: to find the class of systems that satisfy Aizerman's hypothesis. The author has showed in [11] that any system satisfies the Aizerman hypothesis if its impulse

function is non-negative. The similar result was proved for multivariable systems, distributed ones, and retarded systems (see [23] and references therein). In this paper we investigate the Aizerman's hypothesis for neutral type equations.

5. The paper consists of 11 sections. In Section 2, the main notations used in the paper are presented. Section 3 deals with linear time-invariant systems. Besides, estimates for various norms of fundamental solutions and characteristic matrices are derived. By the derived estimates we then obtain the stability conditions for equations with nonlinear causal mappings. The generalized Bohl - Perron principle is presented in Section 4. In Section 5 we illustrate the application of the generalized Bohl - Perron principle to linear time variant systems "close" to autonomous ones.

Sections 6-8 are devoted to vector nonlinear equations with separated linear parts and nonlinear causal mappings. Namely, in Section 6 we establish conditions providing the Lyapunov stability in the space of continuous vector valued functions. Section 7 is devoted to the L^2-absolute stability. The exponential stability of solutions to nonlinear equations is considered in Section 8. The results presented in Sections 6-8 generalize the stability criteria from [14, 16, 17] (see also [23]). The Aizerman type problem is discussed in Section 9.

In Section 10, by virtue of the generalized norm, we establish global stability conditions for nonlinear systems with diagonal linear parts. In Section 11 we present a test for the input-to-state stability.

2 Notations

Let $\mathbb{C}^n$ be the complex n-dimensional Euclidean space with the scalar product $(.,.)_{C^n}$ and the Euclidean norm $\|x\|_n = \sqrt{(x,x)_{C^n}}$ $(x \in \mathbb{C}^n)$. $C(a,b) = C([a,b],\mathbb{C}^n)$ is the space of continuous functions defined on a finite or infinite real segment $[a,b]$ with values in $\mathbb{C}^n$ and the norm $\|w\|_{C(a,b)} = \sup_{t\in[a,b]} \|w(t)\|_n$. $C^1(a,b) = C^1([a,b],\mathbb{C}^n)$ is the space of continuously differentiable functions defined on $[a,b]$ with values in $\mathbb{C}^n$ and the norm $\|w\|_{C^1(a,b)} = \|w\|_{C(a,b)} + \|\dot w\|_{C(a,b)}$, where $\dot w$ is the derivative of w. In addition, $L^p(a,b) = L^p([a,b],\mathbb{C}^n)$ $(p \ge 1)$ is the space of functions w defined on $[a,b]$ with values in $\mathbb{C}^n$ and the finite norm

$$\|w\|_{L^p(a,b)} = [\int_a^b \|w(t)\|_n^p dt]^{1/p} \ (1 \le p < \infty); \ \|w\|_{L^\infty(a,b)} = vrai \ sup_{t\in[a,b]}\|w(t)\|_n,$$

I is the unit operator in the corresponding space.

For an $n\times n$-matrix A, $\lambda_k(A)$ $(k = 1, ..., n)$ denote the eigenvalues of A numerated in an arbitrary order with their multiplicities, A^* is the adjoint to A and A^{-1} is the inverse to A; $\|A\|_n = \sup_{x\in\mathbb{C}^n} \|Ax\|_n/\|x\|_n$ is the spectral (operator) norm; $N_2(A)$ is the Hilbert-Schmidt (Frobenius) norm of A: $N_2^2(A) = Trace\ AA^*$, $A_I = (A-A^*)/2i$ is the imaginary component.

The following quantity plays an essential role in the sequel:

$$g(A) = (N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2)^{1/2}.$$

In Section 2.2 of [15] it is proved that $g^2(A) \le N^2(A) - |Trace\ A^2|$,

$$g^2(A) \le 2N_2^2(A_I) \text{ and } g(e^{i\tau}A + zI) = g(A) \tag{2.1}$$

for all $\tau \in \mathbb{R}$ and $z \in \mathbb{C}$. If A_1 and A_2 are commuting matrices, then

$$g(A_1 + A_2) \le g(A_1) + g(A_2). \tag{2.2}$$

From Corollary 2.1.2 [15], it follows

Lemma 1. *For any invertible $n \times n$-matrix A, the inequality*

$$\|A^{-1}\|_n \le \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A)} \tag{2.3}$$

is true, where $\rho(A)$ is the smallest absolute eigenvalue of A: $\rho(A) = \min_{k=1,\dots,n} |\lambda_k(A)|$.

3 Autonomous systems

3.1 Estimates for L^2- and $C-$ norms of fundamental solutions

For a positive constant $\eta < \infty$ consider the problem

$$\dot{y}(t) - \int_0^\eta d\tilde{R}(\tau)\dot{y}(t-\tau) = \int_0^\eta dR(\tau)y(t-\tau) \ (t \ge 0), \tag{1.1}$$

$$y(t) = \phi(t) \text{ for } -\eta \le t \le 0, \tag{1.2}$$

where $\phi \in C^1(-\eta, 0)$ is given; $R(s) = (r_{ij}(s))_{i,j=1}^n$ and $\tilde{R}(s) = (\tilde{r}_{ij}(s))_{i,j=1}^n$ are real $n \times n$-matrix-valued functions defined on $[0, \eta]$, whose entries have bounded variations $var(r_{ij})$ and $var(\tilde{r}_{ij})$, and finite numbers of jumps. In addition, $\tilde{R}(s)$ does not have a jump at $t = 0$. The integrals in (1.1) are understood as the Lebesgue - Stieltjes integrals. A solution of problem (1.1), (1.2) is an absolutely continuous vector function $y(t)$ defined on $[-\eta, \infty)$ and satisfying (1.1) and (1.2).

We define the variation of $R(.)$ as the matrix $Var(R) = (var(r_{ij}))_{i,j=1}^n$ and denote $V(R) := \|Var(R)\|_n$. So $V(R)$ is the spectral norm of matrix $Var\ (R)$. Similarly $V(\tilde{R})$ is defined. It is assumed that

$$V(\tilde{R}) < 1. \tag{1.3}$$

The matrix-valued function

$$K(z) = Iz - z\int_0^\eta exp(-zs)d\tilde{R}(s) - \int_0^\eta exp(-zs)dR(s) \ (z \in \mathbb{C})$$

is the characteristic matrix-valued function to equation (1.1) and the zeros of $det\ K(\lambda)$ are the characteristic values of $K(.)$; $\lambda \in \mathbb{C}$ is said a regular value of $K(.)$, if $det\ K(\lambda) \neq 0$. *Everywhere below it is assumed that all the characteristic values of* $K(.)$ *are in the open left half-plane* C_-. We also give some conditions that provide the location of the characteristic values in C_-.

Due to Theorem 3.1.1 from [29, p. 114], under condition (1.3) equation (1.1) is asymptotically stable and L^2-stable, if all the characteristic values of $K(.)$ are in C_-. Moreover, the integral

$$G(t) := \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{it\omega}K^{-1}(i\omega)d\omega \ (t \geq 0) \tag{1.4}$$

exists and the function $G(t)$ defined by (1.4) for $t \geq 0$ and by $G(t) = 0$ for $-\eta \leq t < 0$ is called the fundamental solution to (1.1). $G(t)$ is a solution to (1.1) and $G(0) = I$, cf. [29].

Finally, denote

$$v_0 := \frac{2\,V(R)}{1 - V(\tilde{R})}, \quad \theta(K) := \sup_{-v_0 \leq \omega \leq v_0} \|K^{-1}(i\omega)\|_n,$$

and

$$W(K) := \sqrt{2\theta(K)(1 + V(\tilde{R}))(1 + \theta(K)V(R))}.$$

Theorem 2. *Let condition (1.3) hold and all the zeros of* det $K(z)$ *be in* C_-. *Then the fundamental solution of (1.1) satisfies the inequalities,*

$$\|G\|_{L^2(0,\infty)} \leq W(K), \tag{1.5}$$

$$\|\dot{G}\|_{L^2(0,\infty)} \leq \frac{V(R)\|G\|_{L^2(0,\infty)}}{1 - V(\tilde{R})} \leq \frac{V(R)W(K)}{1 - V(\tilde{R})} \tag{1.6}$$

and

(1.7)

$$\|G\|^2_{C(0,\infty)} \leq 2\|\dot{G}\|_{L^2(0,\infty)}\|G\|_{L^2(0,\infty)} \leq a_0^2(K), \ \textit{where } a_0(K) := W(K)\sqrt{\frac{2V(R)}{1 - V(\tilde{R})}}.$$

The proof of this theorem is presented in the next subsection.

Let us point estimates for $\theta(K)$. Recall that $N_2(A)$ is the Hilbert-Schmidt (Frobenius) norm and

$$g(A) = (N_2^2(A) - \sum_{k=1}^{n} |\lambda_k(A)|^2)^{1/2}$$

(see Section 2), and

$$\|A^{-1}\|_n \le \sum_{k=0}^{n-1} \frac{g^k(A)}{\sqrt{k!}\rho^{k+1}(A)} \tag{1.8}$$

for an invertible $n \times n$-matrix A. In addition,

$$g^2(A) \le 2N_2^2(A_I) \text{ and } g(e^{i\tau}A + zI) = g(A) \tag{1.9}$$

for all $\tau \in \mathbb{R}$ and $z \in \mathbb{C}$. Put

$$B(z) = z\int_0^\eta exp(-zs)dR(s) + \int_0^\eta exp(-zs)dR(s).$$

So $K(z) = zI - B(z)$. By (1.9) $g(B(z)) = g(K(z))$. Thanks to (1.8), for any regular value z of $K(.)$, the inequality

$$\|[K(z)]^{-1}\|_n \le \Gamma(K(z)) \ (z \in \mathbb{C}) \tag{1.10}$$

is valid, where

$$\Gamma(K(z)) = \sum_{k=0}^{n-1} \frac{g^k(B(z))}{\sqrt{k!}\rho^{k+1}(K(z))}$$

and $\rho(K(z))$ is the smallest absolute value of the eigenvalues of $K(z)$:

$$\rho(K(z)) = \min_{k=1,\dots,n} |\lambda_k(K(z))|.$$

If $B(z)$ is a normal matrix, then $g(B(z)) = 0$, and $\|[K(z)]^{-1}\|_n \le \rho^{-1}(K(z))$. For example, that inequality holds, if $K(z) = zI - \tilde{A}ze^{-z\eta} - Ae^{-z\eta}$, where A and $\tilde{A}$ are commuting Hermitian matrices. Due to (1.10) we arrive at

Lemma 3. *One has*

$$\theta(K) \le \Gamma_0(K), \ \textit{where } \Gamma_0(K) := \sup_{-v_0 \le \omega \le v_0} \Gamma(K(i\omega)).$$

Furthermore, from [15, Theorem 2.11] it follows that

$$\|A^{-1} \ det\ (A)\|_n \le \frac{N_2^{n-1}(A)}{(n-1)^{(n-1)/2}}$$

for any invertible $n \times n$-matrix A. Hence, for any regular point z of $K(.)$, one has

$$\|K^{-1}(z)\|_n \le \frac{N_2^{n-1}(K(z))}{(n-1)^{(n-1)/2}|det\ (K(z))|},$$

and thus

$$\theta(K) \le \theta_d(K) \text{ where } \theta_d(K) := \sup_{-v_0\le\omega\le v_0} \frac{N_2^{n-1}(K(i\omega))}{(n-1)^{(n-1)/2}|det\ (K(i\omega))|}. \tag{1.11}$$

3.2 Proof of Theorem 3.1.1

Below the meaning of the integral

$$\int_0^\eta w(s)|dr(s)|$$

for a scalar continuous function w and a real function r of bounded variation is the following: since $r(s)$ is of bounded variation, we have $r(s) = r_+(s) - r_-(s)$, where $r_+(s)$ and $r_-(s)$ are nondecreasing functions. Then

$$\int_0^\eta w(s)|dr(s)| := \int_0^\eta w(s)dr_+(s) + \int_0^\eta w(s)dr_-(s).$$

In particular, put

$$vd(r) := \int_0^\eta s|dr(s)| = \int_0^\eta sdr_+(s) + \int_0^\eta sdr_-(s)$$

and

$$vd\ (R) := \|(vd\ (r_{jk}))_{j,k=1}^n\|_n.$$

So $vd\ (R)$ is the spectral norm of the matrix $(vd\ (r_{jk}))_{j,k=1}^n$. Clearly $vd(R) \le \eta V\ (R)$.

For a continuous scalar function $w(s)$ denote

$$Vd(R,w) := \|(\int_0^\eta |w(\tau)||dr_{jk}|)_{j,k=1}^n\|_n.$$

So $Vd\ (R,w)$ is the spectral norm of the matrix whose entries are $\int_0^\eta |w(\tau)||dr_{jk}|$ and $Vd(R,w) = vd(R)$ for $w(s) = s$.

Lemma 4. *Let $w(s)$ be a continuous scalar function defined on $[0,\eta]$. Then*

$$\left\|\int_0^\eta w(\tau)dR(\tau)f(t-\tau)\right\|_{L^2(0,T)} \le Vd\ (R,w)\|f\|_{L^2(-\eta,T)}\ \ (T>0;\ f\in L^2(-\eta,T)). \tag{2.1}$$

For the proof see [21, Lemma 1].

Furthermore, for an $f \in L^2([-\eta, T], \mathbb{C}^n), T \le \infty$, put

$$Ef(t) = \int_0^\eta dR(s) f(t-s), \tilde{E}f(t) = \int_0^\eta d\tilde{R}(s) f(t-s) \ \ (0 \le t \le T).$$

Now the previous lemma implies.

Corollary 5. *We have* $\|E\|_{L^2(-\eta,T)\to L^2(0,T)} \le V\ (R)$ *and*

$$\left\| \int_0^\eta \tau dR(\tau) f(t-\tau) \right\|_{L^2(0,T)} \le vd\ (R) \|f\|_{L^2(-\eta,T)}\ (T > 0;\ f \in L^2(-\eta, T)).$$

Note that in [20, Section 3], the inequality

$$\|Ef\|_{C(-\eta,T)\to C(0,T)} \le \sqrt{n}\ \ V(R)\ \ (T > 0)$$

is proved.

We need also the following result

Lemma 6. *The equality* $\sup_{-\infty\le\omega\le\infty} \|K^{-1}(i\omega)\|_n = \theta(K)$ *is valid.*

For the proof see Lemma 2 [20].

Let us consider the non-homogeneous equation

$$\dot{x} - \tilde{E}\dot{x} = Ex + f \ \ (f \in L^2(0, \infty)) \tag{2.2}$$

with the zero initial condition

$$x(t) = 0, t \le 0. \tag{2.3}$$

Applying the Laplace transform to problem (2.2), (2.3), we get $\hat{x}(z) = K^{-1}(z)\hat{f}(z)$, where $\hat{x}(z)$ and $\hat{f}(z)$ are the Laplace transforms of $x(t)$ and $f(t)$, respectively. Consequently,

$$\|\hat{x}(i\omega)\|_{L^2(-\infty,\infty)} \le \sup_{-\infty\le\omega\le\infty} \|K^{-1}(i\omega)\|_n \|\hat{f}(i\omega)\|_{L^2(-\infty,\infty)}.$$

Now from Lemma 3.2.3 and the Parseval equality we arrive at

Lemma 7. *Let condition (1.3) hold and all the zeros of* $\det K(z)$ *be in* C_-*. Then*

$$\|x\|_{L^2(0,\infty)} \le \theta(K) \|f\|_{L^2(0,\infty)}. \tag{2.4}$$

Further, for a constant $\nu > 0$, put $Z(t) = G(t) - e^{-\nu t}I$. Substitute this equality into (1.1). Then we obtain

$$\dot{Z} - \tilde{E}\dot{Z} - EZ = f_\nu \ , \tag{2.5}$$

where

$$f_\nu = -\nu e^{-\nu t}I - \nu\tilde{E}(Ie^{-\nu t}) + E(Ie^{-\nu t}).$$

Clearly, $Z(t) = 0, t \le 0$. For the brevity in the rest of this section we sometimes put $\|.\|_{L^2(0,\infty)} = |.|_{L^2}$. Due to (2.4) we obtain $|Z|_{L^2} \le \theta(K)|f_\nu|_{L^2}$. But $|e^{-\nu t}|^2_{L^2} = 1/2\nu$ and by Corollary 3.2.2,

$$|\tilde{E}e^{-\nu t}|_{L^2} \le V(\tilde{R})|e^{-\nu t}|_{L^2} = \frac{V(\tilde{R})}{\sqrt{2\nu}} \text{ and } |Ee^{-\nu t}|_{L^2} \le \frac{V(R)}{\sqrt{2\nu}}.$$

Thus making use (2.4) and (2.5), we obtain

$$|G|_{L^2} \le |Z|_{L^2} + |e^{-\nu t}|_{L^2} \le \theta(K)|f_\nu|_{L^2} + \frac{1}{\sqrt{2\nu}} \le w(\nu), \tag{2.6}$$

where

$$w(\nu) = \frac{\theta(K)(1+V(\tilde{R}))\sqrt{\nu}}{\sqrt{2}} + \frac{1+\theta(K)V(R)}{\sqrt{2\nu}}.$$

Put $x = \sqrt{\nu}$,

$$a = \frac{\theta(K)(1+V(\tilde{R}))}{\sqrt{2}} \text{ and } b = \frac{1+\theta(K)V(R)}{\sqrt{2}};$$

then $w(\nu) = ax+b/x$. The minimum of the right-hand part is attained at $x_0 = \sqrt{b/a}$. Besides,

$$ax_0 + \frac{b}{x_0} = 2\sqrt{ab} = 2\sqrt{\frac{\theta(K)(1+V(\tilde{R}))}{\sqrt{2}}\frac{(1+\theta(K)V(R))}{\sqrt{2}}} = W(K).$$

Now (2.6) yields inequality (1.5).

Furthermore Corollary 3.2.2 and (1.1) imply $|\dot{G}|_{L^2} \le V(\tilde{R})|\dot{G}|_{L^2} + V(R)|G|_{L^2}$. Hence,

$$\|\dot{G}\|_{L^2(0,\infty)} \le (1 - V(\tilde{R}))^{-1}V(R)\|G\|_{L^2(0,\infty)} \tag{2.7}$$

and therefore (1.5) yields inequalities (1.6).

Now we need the following simple result, cf. [23, Lemma 4.4.6].

Lemma 8. *Let $f \in L^2(0,\infty)$ and $\dot{f} \in L^2(0,\infty)$. Then $|f|^2_{C(0,\infty)} \le 2|f|_{L^2(0,\infty)}|\dot{f}|_{L^2(0,\infty)}$.*

This result and (2.7) imply

$$\|G\|^2_{C(0,\infty)} \le 2(1 - V(\tilde{R}))^{-1}V(R)\|G\|^2_{L^2(0,\infty)}.$$

Consequently, (1.5) proves inequalities (1.7). □

3.3 Lower estimates for quasi-polynomials

In this subsection we present estimates for quasi-polynomials which will be used below. Consider the function

$$k(z) = z\left(1 - \int_0^\eta e^{-\tau z} d\tilde{\mu}\right) + \int_0^\eta e^{-\tau z} d\mu \ \ (z \in \mathbb{C}), \tag{3.1}$$

where $\mu = \mu(\tau)$ and $\tilde{\mu} = \tilde{\mu}(\tau)$ are nondecreasing functions defined on $[0, \eta]$, with

$$0 < var(\tilde{\mu}) < 1 \text{ and } var(\mu) < \infty. \tag{3.2}$$

Put

$$v_1 = \frac{2\ var(\mu)}{1 - var(\tilde{\mu})}.$$

The following two lemmas are proved in [20] (Lemmas 7 and 9).

Lemma 9. *The equality* $\inf_{-\infty \le \omega \le \infty} |k(i\omega)| = \inf_{-v_1 \le \omega \le v_1} |k(i\omega)|$ *is valid.*

Lemma 10. *Let the conditions (3.2),*

$$\eta v_1 < \pi/2 \tag{3.3}$$

and

$$d_0 := \int_0^\eta cos(v_1\tau) d\mu - v_1 \int_0^\eta sin(v_1 \tau) d\tilde{\mu} > 0 \tag{3.4}$$

hold. Then all the zeros of $k(.)$ *are in* C_- *and*

$$\inf_{-\infty \le \omega \le \infty} |k(i\omega)| \ge d_0 > 0. \tag{3.5}$$

For instance consider the function

$$k_1(z) = z(1 - \tilde{a}e^{-\tilde{h}z}) + ae^{-hz} + b$$

with $a, b, h, \tilde{h} = const \ge 0$, and $0 < \tilde{a} < 1$. Then $v_1 = 2(a+b)(1-\tilde{a})^{-1}$. Furthermore, due to Lemma 3.3.2 we arrive at the following result

Corollary 11. *Assume that the conditions*

$$hv_1 < \pi/2, \tilde{h}v_1 < \pi/2 \tag{3.6}$$

and

$$d_1 := a\ cos(v_1 h) + b - v_1 \tilde{a}\ \ sin(v_1 \tilde{h}) > 0. \tag{3.7}$$

Then all the zeros of $k_1(.)$ *are in* C_- *and* $\inf_{-\infty \le \omega \le \infty} |k(i\omega)| \ge d_1 > 0.$

3.4 Scalar equations with positive fundamental solutions

Consider the linear equation

$$\dot{y}(t) - a\dot{y}(t-\tilde{h}) + by(t-h) = 0, \tag{4.1}$$

where $a, b, h, \tilde{h}$ are positive constants.

The following lemma and corollary are proved in [24].

Lemma 12. *Let the equation*

$$s = se^{\tilde{h}s}a + e^{hs}b \tag{4.2}$$

have a positive root ζ. Then the Green function (the fundamental solution) $G_1(t)$ to (4.1) is nonnegative. Moreover,

$$G_1(t) \geq e^{-\zeta t} \ (t \geq 0), \tag{4.3}$$

$\dot{G}_1(t) \leq 0$ *and*

$$\int_0^\infty G_1(t)dt = \frac{1}{b}. \tag{4.4}$$

Note that, if (4.2) has a positive root, then necessarily $a < 1$. Note also that in [1, p. 26] (see also the references given therein) the positivity of Green's function was proved but the relations of the type (4.4), which we use below, were not considered.

Corollary 13. *Let (4.2) have a positive root. Then $\inf_{\omega\in\mathbb{R}} |k(i\omega)| = b$.*

Remark 14. *If there is a positive number λ, such that $ae^{\tilde{h}\lambda}\lambda + be^{h\lambda} \leq \lambda$, then due to the well-known Theorem 38.1 [31] equation (4.2) has a positive root $\zeta \leq \lambda$.*

Substitute $s = \omega c$ with a positive c into (4.2). Then

$$\omega e^{\tilde{h}c\omega}a + \frac{1}{c}e^{ch\omega}b = \omega. \tag{4.5}$$

If

$$e^{\tilde{h}c}a + \frac{1}{c}e^{ch}b \leq 1, \tag{4.6}$$

then due to the mentioned Theorem 38.1 [31] (4.5) has a positive root which not more than one. So (4.2) has a positive root $\zeta \leq c$. For example, if $c = 1/h$, then condition (4.5) takes the form

$$e^{\tilde{h}/h}a + hbe \leq 1.$$

This condition is the direct generalization of the corresponding result for equations with delay [1].

Minimaze the function

$$f(c) = e^{\tilde{h}c}a + \frac{1}{c}e^{ch}b$$

with respect to c. It is simple to show that

$$c_0 = -\frac{bh}{2a\tilde{h}} + \sqrt{\left(\frac{bh}{2a\tilde{h}}\right)^2 + \frac{b}{a\tilde{h}}} \tag{4.7}$$

is the zero of $f'(c)$. We thus have proved

Lemma 15. *If*

$$e^{\tilde{h}c_0}a + \frac{1}{c_0}e^{c_0 h}b \le 1, \tag{4.8}$$

then (4.2) has a positive root $\zeta \le c_0$.

3.5 Autonomous systems with discrete delays

Let $\tilde{A} = (\tilde{a}_{jk}), A = (a_{jk})$ and $C = (c_{jk})$ be real $n \times n$-matrices. Consider the equation

$$\dot{y}(t) - \tilde{A}\dot{y}(t - \tilde{h}) + Ay(t - h) + Cy(t) = 0 \ \ (t \ge 0), \tag{5.1}$$

assuming that $\|\tilde{A}\|_n < 1$. So $K(z) = z(I - \tilde{A}e^{-\tilde{h}z}) + Ae^{-hz} + C$. The entries of K are

$$k_{jk}(z) = z(1 - \tilde{a}_{jk}e^{-\tilde{h}z}) + a_{jk}e^{-hz} + c_{jk} \ \ (j, k = 1, ..., n).$$

As it was shown by Ostrowski [40], for any $n \times n$-matrix $M = (m_{jk})$ the inequality

$$|det\ M| \ge \prod_{j=1}^{n}(|m_{jj}| - \sum_{i=1, i\ne j}^{n} |m_{ji}|) \tag{5.2}$$

is valid, provided

$$|m_{jj}| > \sum_{i=1, i\ne j}^{n} |m_{ji}| \quad (j = 1, ..., n).$$

Hence,

$$|det\ K(z)| \ge \prod_{j=1}^{n}(|k_{jj}(z)| - \sum_{m=1, m\ne j}^{n} |k_{jm}(z)|), \tag{5.3}$$

provided the right-hand part is positive.

For equation (5.1) we have $V(\tilde{R}) = \|\tilde{A}\|_n$, $V(R) = \|A\|_n + \|C\|_n$ and

$$v_0 = 2(\|A\|_n + \|C\|_n)(1 - \|\tilde{A}\|_n)^{-1}.$$

In addition,

$$N_2(K(i\omega)) \le N_0(K) \quad (|\omega| \le v_0), \tag{5.4}$$

where $N_0(K) := v_0(\sqrt{n} + N_2(\tilde{A})) + N_2(A) + N_2(C)$. Now according to (1.11) we obtain an estimate for $\theta(K)$. For instance, (5.1) can take the form

$$\dot{y}_j(t) - \tilde{a}_{jj}\dot{y}_j(t - \tilde{h}) + \sum_{k=1}^{n}(a_{jk}y_k(t-h) + c_{jk}y_k(t)) = 0, \tag{5.5}$$

$(j = 1, ..., n;\ t \ge 0)$; suppose that

$$a_{jj}, c_{jj} \ge 0;\ 0 < \tilde{a}_{jj} < 1. \tag{5.6}$$

So $\tilde{A} = diag\ (\tilde{a}_{jj})$. Put

$$w_j = \frac{2(a_{jj} + c_{jj})}{1 - \tilde{a}_{jj}}$$

and assume that

$$w_j\ max\{h, \tilde{h}\} < \frac{\pi}{2} \text{ and } d_j := a_{jj}cos(w_jh) + c_{jj} - w_j\tilde{a}sin(w_j\tilde{h}) > 0 \tag{5.7}$$

$(j = 1, ..., n)$. Then by Corollary 3.3.3, all the zeros of $k_{jj}(.)$ are in C_- and

$$\inf_{-\infty \le \omega \le \infty} |k_{jj}(i\omega)| \ge d_j > 0.$$

In addition, let

$$\rho_j := d_j - \sum_{m=1, m\ne j}^{n} (|a_{jm}| + |c_{jm}|) > 0\ (j = 1, ..., n). \tag{5.8}$$

According to (5.3) we get

$$|det\ K(i\omega)| \ge \prod_{j=1}^{n} \rho_j.$$

Thus by (5.4) and (1.11), we arrive at the following result.

Corollary 16. *Let conditions (5.6)-(5.8) be fulfilled. Then system (5.5) is asymptotically stable and*

$$\theta(K) \le \frac{N_0^{n-1}(K)}{(n-1)^{(n-1)/2}\prod_{j=1}^{n}\rho_j}.$$

Additional estimates for $\theta(K)$ are given in Subsection 7.3 below.

3.6 Systems with commuting Hermitian matrices

Let $\tilde{A}$ and A be positive definite Hermitian commuting $n \times n$-matrices. Consider the equation

$$\dot{y}(t) - \tilde{A}\dot{y}(t-\tilde{h}) + Ay(t-h) = 0 \quad (t \geq 0), \tag{6.1}$$

assuming that $\|\tilde{A}\|_n < 1$. In this subsection we suggest an L^1-norm estimate for solutions of (6.1).

We have $K(z) = z(I - \tilde{A}e^{-\tilde{h}z}) + Ae^{-hz}$. Rewrite (6.1) as the system in the basis of the eigenvalues of A and $\tilde{A}$, which are coincide since the matrices commute:

$$\dot{y}_j(t) - \lambda_j(\tilde{A})\dot{y}(t-\tilde{h}) + \lambda_j(A)y(t-h) = 0 \quad (t \geq 0), \tag{6.2}$$

Assume that each of the equations

$$s = s(1 - \lambda_k(\tilde{A}))e^{s\tilde{h}} + \lambda_k(A)e^{\tilde{s}h}, k = 1, ..., n \tag{6.3}$$

have a positive root ζ_j. Then due to Lemma 3.4.1 the Green function $G_j(t)$ to each of equations (6.2) is nonnegative. Moreover,

$$\int_0^\infty G_j(t)dt = \frac{1}{\lambda_j(A)}. \tag{6.4}$$

Besides, $G(t)$ is the vector with coordinates $G_j(t)$. Put

$$x_j(t) = \int_0^t G_j(t-s)f_j(s)ds$$

for a scalar continuous function f_j. Then

$$\sup_t (\sum_{k=1}^n |x_k(t)|^2)^{1/2} = \sup_t (\sum_{k=1}^n |\int_0^t G_j(t-s)f_j(s)ds|^2)^{1/2} \leq$$

$$\sqrt{n} \sup_{t,k} |f_k(t)| \int_0^\infty G_k(t)dt \leq \sqrt{n}\|f\|_{C(0,\infty)} \sup_k \frac{1}{\lambda_k(A)}.$$

Thus we have proved the following

Theorem 17. *Let $\tilde{A}$ and A be positive definite Hermitian commuting $n \times n$-matrices and equations (6.3) have positive roots. Then the fundamental solution to (6.1) satisfies the inequality*

$$\|\hat{G}\|_{C(0,\infty)} \leq \frac{\sqrt{n}}{\min_k \lambda_k(A)}.$$

4 The Generalized Bohl-Perron Principle

4.1 Statement of the result

In the present section we extend the Bohl - Perron principle to a class of neutral type functional differential equations.

Let $A_j(t)$ $(t \geq 0;\ j = 1, ..., m_1)$ be continuously differentiable $n \times n$-matrices; $B_k(t)$ $(t \geq 0;\ k = 1, ..., m_0)$, continuous $n \times n$-matrices. In addition, $\eta < \infty$ is a positive constant, $A(t, \tau)$ $(t \geq 0;\ \tau \in [0, \eta])$ is an $n \times n$-matrix continuously differentiable in t for each τ; $B(t, \tau)$ $(t \geq 0;\ \tau \in [0, \eta])$ is an $n \times n$-matrix continuous in t for each τ; $A(t, \tau)$, $A'_t(t, \tau)$ and $B(t, \tau)$ are integrable in τ on $[0, \eta]$.

Define the operators $E_0, E_1 : C(-\eta, \infty) \to C(0, \infty)$ by

$$(E_0 f)(t) = \sum_{k=1}^{m_0} B_k(t) y(t - v_k(t)) + \int_0^\eta B(t, s) y(t - s) ds$$

and

$$(E_1 f)(t) = \sum_{k=1}^{m_1} A_k(t) y(t - h_k) + \int_0^\eta A(t, s) y(t - s) ds \quad (t \geq 0),$$

where $0 < h_1 < ... < h_{m_1} \leq \eta$ $(m_1 < \infty)$ are constants, $v_j(t)$ are real continuous functions, such that $0 \leq v_j(t) \leq \eta$.

Our main object in this section is the equation

$$\frac{d}{dt}[y(t) - (E_1 y)(t)] = (E_0 y)(t) \tag{1.1}$$

with the initial condition

$$y(t) = \phi(t) \quad (-\eta \leq t \leq 0) \tag{1.2}$$

for a given $\phi \in C^1(-\eta, 0)$. We consider also the non-homogeneous equation

$$\frac{d}{dt}[x(t) - (E_1 x)(t)] = (E_0 x)(t) + f(t) \quad (t \geq 0) \tag{1.3}$$

with a given vector function $f \in C(0, \infty)$ and the zero initial condition

$$x(t) \equiv 0 \ (-\eta \leq t \leq 0). \tag{1.4}$$

It is assumed that

$$V_1 := \sup_{t \geq 0} \left[\sum_{k=1}^{m_1} \|A_k(t)\|_n + \int_0^\eta \|A(t, s)\|_n ds \right] < 1 \tag{1.5a}$$

and
(1.5b)
$$V_0 := \sup_{t\geq 0}\left[\sum_{k=1}^{m_1}\|A_k'(t)\|_n + \int_0^\eta \|A_t'(t,s)\|_n ds + \sum_{k=1}^{m_0}\|B_k(t)\|_n + \int_0^\eta \|B(t,s)\|_n ds\right] < \infty.$$

A solution of problem (1.1), (1.2) is a continuous function, satisfying the problem

$$(1.6a) \qquad y(t) - (E_1 y)(t) = \phi(0) - (E_1\phi)(0) + \int_0^t (E_0 y)(t_1)dt_1 \ \ (t \geq 0),$$

$$(1.6b) \qquad y(t) = \phi(t) \ \ (-\eta \leq t \leq 0).$$

A solution of problem (1.3), (1.4) is defined as a continuous function $x(t)$, which satisfies the equation

$$(1.7) \qquad x(t) - (E_1 x)(t) = \int_0^t (E_0 x)(t_1)dt_1 + \int_0^t f(t_1)dt_1 \ \ (t \geq 0)$$

and condition (1.4).

The existence and uniqueness of solutions of problems (1.1), (1.2) and (1.3), (1.4) under conditions (1.5) is due to [27, p. 256, Theorem 9.1.1].

Now we are in a position to formulate the main result of the section.

Theorem 18. *Let conditions (1.5) hold. If, in addition, a solution $x(t)$ of problem (1.3), (1.4) is bounded on $[0,\infty)$ (that is, $x \in C(0,\infty)$) for any $f \in C(0,\infty)$, then equation (1.1) is exponentially stable.*

This theorem is proved in the next subsection.

4.2 Proof of Theorem 4.1.1

Rewrite (1.1) as

$$(2.1) \qquad \dot y(t) - (E_1 \dot y)(t) = (E_1' y)(t) + (E_0 y)(t) \ \ (t \geq 0),$$

where
$$(E_1' y)(t) = \sum_{k=1}^{m_1} A_k'(t) y(t-h_k) + \int_0^\eta A_t'(t,s) y(t-s)ds.$$

Lemma 19. *For any $T > 0$ one has $\|E_1 u\|_{C(0,T)} \leq V_1 \|u\|_{C(-\eta,T)}$ and*
$$\|(E_1' + E_0)u\|_{C(0,T)} \leq V_0 \|u\|_{C(-\eta,T)} \ (u \in C(-\eta,T)).$$

Proof. Let $u \in C(-\eta, T)$. We have

$$\|(E_1 u)(t)\|_n \le \sum_{k=1}^{m_1} \|A_k(t)u(t-h_k)\|_n + \int_0^\eta \|A(t,s)u(t-s)\|_n ds \le$$

$$\|u\|_{C(-\eta,T)}(\sum_{k=1}^{m_1} \|A_k(t)\|_n + \int_0^\eta \|A(t,s)\|_n ds) \le V_1 \|u\|_{C(-\eta,T)} \;\; (0 \le t \le T).$$

Similarly the second inequality can be proved. This proves the lemma. □

Lemma 20. *If for any $f \in C(0,\infty)$ a solution of problem (1.3), (1.4) is in $C(0,\infty)$, and conditions (1.5) hold, then any solution of problem (1.1), (1.2) is in $C(-\eta,\infty)$.*

Proof. Let $y(t)$ be a solution of problem (1.1), (1.2). Put

$$\zeta(t) = \begin{cases} \phi(0) & \text{if } t \ge 0, \\ \phi(t) & \text{if } -\eta \le t < 0 \end{cases}$$

and $x_0(t) = y(t) - \zeta(t)$. We can write $d\zeta(t)/dt = 0 \;\; (t \ge 0)$ and

$$\frac{d}{dt}[x_0(t) - (E_1 x_0)(t)] = (E_0 x_0)(t) + \psi(t) \;\; (t > 0),$$

where

$$\psi(t) = \frac{d(E_1\zeta)(t)}{dt} + (E_0\zeta)(t) = (E_1\dot{\zeta})(t) + (E_1'\zeta)(t) + (E_0\zeta)(t).$$

Besides, (1.4) holds with $x(t) = x_0(t)$. Since $\zeta \in C^1(-\eta,\infty)$, by the previous lemma we have $\psi \in C(-\eta,\infty)$. Due to the hypothesis of this lemma, $x_0 \in C(0,\infty)$. Thus $y = x_0 + \zeta \in C(-\eta,\infty)$. As claimed. □

Lemma 21. *Let conditions (1.5) hold. Then for any solution of problem (1.3), (1.4) and all $T > 0$, one has*

$$\|\dot{x}\|_{C(0,T)} \le (1 - V_1)^{-1}(V_0\|x\|_{C(0,T)} + \|f\|_{C(0,T)}).$$

Proof. By Lemma 4.2.1, from (2.1) we have

$$\|\dot{x}\|_{C(0,T)} \le V_1\|\dot{x}\|_{C(0,T)} + V_0\|x\|_{C(0,T)} + \|f\|_{C(0,T)}.$$

Hence the condition $V_1 < 1$ implies the required result. □

Proof of Theorem 4.1.1: Substituting

$$y(t) = y_\epsilon(t)e^{-\epsilon t} \tag{2.2}$$

with an $\epsilon > 0$ into (2.1), we obtain the equation

$$\dot{y}_\epsilon - \epsilon y_\epsilon - E_{\epsilon,1}\dot{y}_\epsilon + \epsilon E_{\epsilon,1} y_\epsilon = (E'_{\epsilon,1} + E_{\epsilon,0}) y_\epsilon, \tag{2.3}$$

where

$$(E_{\epsilon,1} f)(t) = \sum_{k=1}^{m_1} e^{h_k \epsilon} A_k(t) f(t-h_k) + \int_0^\eta e^{s\epsilon} A(t,s) f(t-s) ds,$$

$$(E_{\epsilon,0} f)(t) = \sum_{k=1}^{m_0} B_k(t) e^{v_k(t)\epsilon} f(t-v_k(t)) + \int_0^\eta B(t,s) e^{s\epsilon} f(t-s) ds$$

and

$$(E'_{\epsilon,1} f)(t) = \sum_{k=1}^{m_1} e^{h_k \epsilon} A'_k(t) f(t-h_k) + \int_0^\eta e^{s\epsilon} A'_t(t,s) f(t-s) ds.$$

Rewrite (2.3) as

$$\frac{d}{dt}[y_\epsilon - E_{\epsilon,1} y_\epsilon] = Z_\epsilon y_\epsilon, \tag{2.4}$$

where

$$Z_\epsilon := \epsilon I + \epsilon E_{\epsilon,1} + E_{\epsilon,0}.$$

Furthermore, introduce in $C(0,\infty)$ the operator $\hat{G} : f \to x$ where $x(t)$ is the solution of problem (1.3), (1.4). That is, $\hat{G}$ solves problem (1.3), (1.4).

By the hypothesis of the theorem, we have

$$x = \hat{G} f \in C(0,\infty) \text{ for any } f \in C(0,\infty).$$

So $\hat{G}$ is defined on the whole space $C(0,\infty)$. It is closed, since problem (1.3), (1.4) under conditions (1.5) has a unique solution. Therefore $\hat{G} : C(0,\infty) \to C(0,\infty)$ is bounded according to the Closed Graph Theorem [8, p. 57]. So the norm $\|\hat{G}\|_{C(0,\infty)}$ is finite. Consider now the equation

$$\frac{d}{dt}[x_\epsilon - E_{\epsilon,1} x_\epsilon] = Z_\epsilon x_\epsilon + f \tag{2.5}$$

with the zero initial conditions. Subtract (1.3) from (2.5), with $w(t) = x_\epsilon(t) - x(t)$, where x and x_ϵ are solutions of problems (1.3), (1.4) and (2.5), (1.4), respectively. Then

$$\frac{d}{dt}[w - E_1 w] = F_\epsilon \tag{2.6}$$

where

$$F_\epsilon = (Z_\epsilon - E_0) x_\epsilon + \frac{d}{dt}(E_{\epsilon,1} - E_1) x_\epsilon.$$

It is simple to check that $Z_\epsilon \to E_0$, $E'_{\epsilon,1} \to E'_1$ and $E_{\epsilon,1} \to E_1$ in the operator norm of $C(0,\infty)$ as $\epsilon \to 0$.

For the brevity in this proof put $\|.\|_{C(0,T)} = |.|_T$ for a finite $T > 0$. In addition,

$$\frac{d}{dt}(E_{\epsilon,1} - E_1)x_\epsilon = (E_{\epsilon,1} - E_1)\dot{x}_\epsilon + (E'_{\epsilon,1} - E'_1)x_\epsilon.$$

But according to Lemma 4.2.1, for a sufficiently small ϵ, we have

$$|E_{\epsilon,1}x_\epsilon|_T \le e^{\epsilon\eta}V_1|x_\epsilon|_T \text{ with } e^{\epsilon\eta}V_1 < 1.$$

Due Lemma 4.2.3, from (2.5), the inequality

$$|\dot{x}_\epsilon|_T \le (1 - e^{\epsilon\eta}V_1)^{-1}(|E'_{\epsilon,1} + Z_\epsilon|_T|x_\epsilon|_T + |f|_T)$$

follows. Since $Z_\epsilon \to E_0, E'_{\epsilon,1} \to E'_1$, $|E'_{\epsilon,1} + Z_\epsilon|_T$ is bounded uniformly with respect to ϵ and $T > 0$. So for a sufficiently small $\epsilon_0 > 0$, there is a constant c_1, such that

$$|\dot{x}_\epsilon|_T \le c_1(|x_\epsilon|_T + |f|_T) \ \ (\epsilon < \epsilon_0;\ T > 0).$$

Therefore

$$|F_\epsilon|_T \le a(\epsilon)(|x_\epsilon|_T + |f|_T),$$

where $a(\epsilon) \to 0$ as $\epsilon \to 0$ uniformly in $T > 0$. By (2.6) $x_\epsilon - x = \hat{G}F_\epsilon$. So

$$|x_\epsilon - x|_T \le \|\hat{G}\|_{C(0,\infty)}a(\epsilon)(|x_\epsilon|_T + |f|_T).$$

For a sufficiently small ϵ, we have $q(\epsilon) := \|\hat{G}\|_{C(0,\infty)}a(\epsilon) < 1$. Thus

$$|x_\epsilon|_T \le (1 - q(\epsilon))^{-1}(|x|_T + \|\hat{G}\|_{C(0,\infty)}a(\epsilon)|f|_T).$$

By the hypothesis of the present theorem, $x(t)$ is bounded on $[0,\infty)$. This gives us the inequality

$$|x_\epsilon|_T \le (1 - q(\epsilon))^{-1}(\|x\|_{C(0,\infty)} + a(\epsilon)\|\hat{G}\|_{C(0,\infty)}\|f\|_{C(0,\infty)}).$$

So, letting $T \to \infty$, we get $x_\epsilon \in C(0,\infty)$, since the right-hand part of the latter inequality does not depend on T.

Hence, by Lemma 4.2.2, a solution y_ϵ of (2.4) is bounded. Now (2.2) proves the exponential stability. As claimed. □

5 Nonautonomous systems with discrete delays

5.1 Statement of the result

In this section we present stability conditions which are based on Theorem 4.1.1. To this end consider the system

$$(1.1)\quad \dot y_j(t) - \tilde a_j \dot y_j(t-\tilde h_j) + a_j y_j(t-h_j) + \sum_{k=1}^n c_{jk}(t) y_k(t - v_{jk}(t)) = 0 \quad (j=1,...,n),$$

where $\tilde h_j, h_j, a_j$ and $\tilde a_j$ are positive constants; $c_{jk}(t)$ $(j,k=1,...,n)$ are continuous functions bounded on $[0,\infty)$ and $v_{jk}(t)$ are positive continuous functions, satisfying $v_{jk}(t) \le \eta$ $(t \ge 0)$, where

$$\eta = \max\{\max_j h_j, \max_j \tilde h_j\}.$$

Introduce the matrices

$$A = diag\ (a_j)_{j=1}^n, \tilde A = diag\ (\tilde a_j)_{j=1}^n \text{ and } C(t) = (c_{jk}(t))_{j,k=1}^n.$$

In the considered case we have

$$V(\tilde R) = \max_{j=1,\dots,n} \tilde a_j \text{ and } V(R) = \max_{j=1,\dots,n} a_j + \sup_{t\ge 0} \|C(t)\|_n.$$

Theorem 22. *Let each of the scalar equations*

$$(1.2)\qquad s = s e^{\tilde h_j s} \tilde a_j + e^{h_j s} a_j \quad (j=1,...,n)$$

have a positive root. In addition, let

$$(1.3)\qquad \sum_{k=1}^n \sup_{t\ge 0} |c_{jk}(t)| < a_j \quad (j=1,...,n).$$

Then system (1.1) is exponentially stable.

This theorem is proved in [22, Theorem 3.1]. Its proof is based on Theorem 4.1.1.

Note that from (1.2) it follows $V(\tilde R) = \max_{j=1,\dots,n} \tilde a_j < 1$.

As it was noted in Subsection 3.5, if there are positive numbers λ_j, such that

$$(1.4)\qquad \tilde a_j e^{\tilde h_j \lambda_j} \lambda_j + a_j e^{h_j \lambda_j} \le \lambda_j,$$

then due to the well-known Theorem 38.1 [31] equation (1.2) has a positive root $\zeta_j \le \lambda_j$. In particular, if

$$(1.5)\qquad e^{\tilde h_j} \tilde a_j + e^{h_j} a_j \le 1,$$

then (1.2) has a positive root $\zeta_j \le 1$.

5.2 Sharpness of the result

To investigate the level of conservatism of the sufficient condition of stability given in Theorem 5.1.1, consider the equation

$$\dot{y}(t) - a\dot{y}(t-\tilde{h}) + by(t-h) = by(t-\hat{h}),$$

where $h, \tilde{h}, \hat{h}, a, b$ are positive constants. This equation is not exponentially stable since its characteristic function $z - ze^{\tilde{h}z}a + (e^{hz} - e^{\hat{h}z})b$ has a root at $z = 0$. Similarly, considering the system

$$\dot{y}_j(t) - \tilde{a}_j\dot{y}_j(t-\tilde{h}_j) + a_j y(t-h_j) = c_j y(t-\hat{h}_j) \;\; (j = 1, ..., n),$$

with positive constants $\hat{h}_j$ and c_j, we can assert that its characteristic (diagonal) matrix has a characteristic value at $z = 0$, provided $c_j = a_j$ for at least one index j.

These examples show that condition (1.3) is sharp.

6 Lyapunov's stability of equations with nonlinear causal mappings

6.1 Solution estimates

Again use the operators $\tilde{E}$ and E defined on $C(0, \infty)$ by

$$Ef(t) = \int_0^\eta dR(s)f(t-s), \tilde{E}f(t) = \int_0^\eta d\tilde{R}(s)f(t-s) \;\; (t \geq 0).$$

For a positive $\varrho \leq \infty$ and an arbitrary $T > 0$ denote $\Omega(\varrho, T) = \{w \in C(-\eta, T) : \|w\|_{C(-\eta,T)} \leq \varrho\}$, and $\Omega(\varrho) = \Omega(\varrho, \infty)$. Consider the equation

$$\dot{x} - \tilde{E}\dot{x} - Ex = Fx + f \;\; (f \in C(0, \infty);\; t \geq 0), \tag{1.1}$$

where F is a continuous mapping of $\Omega(\varrho, T)$ into $C(-\eta, T)$ for each $T > 0$ and satisfying the condition

$$\|Fw\|_{C(0,T)} \leq q\|w\|_{C(-\eta,T)} \;\; (w \in \Omega(\varrho, T)), \tag{1.2}$$

where constant $q \geq 0$ does not depend on T. A (mild) solution of problem (1.1), (1.2) is a continuous function $x(t)$ defined on $[-\eta, \infty)$, such that

$$x(t) = z(t) + \int_0^t G(t-t_1)(Fx(t_1) + f(t_1))dt_1 \; (t \geq 0), \tag{1.3a}$$

$$x(t) = \phi(t) \in C^1(-\eta, 0) \;\; (-\eta \leq t \leq 0), \tag{1.3b}$$

where $G(t)$ is the fundamental solution of the linear equation

$$\dot{z} - \tilde{E}\dot{z} - Ez = 0, \tag{1.4}$$

and $z(t)$ is a solution of the problem (1.4), (1.3b). *It is assumed that the linear equation (1.4) is asymptotically stable.* Again use the Cauchy operator

$$\hat{G}w(t) = \int_0^t G(t-t_1)w(t_1)dt_1 \ \ (w \in C(0,\infty)),$$

and suppose that

$$\|\hat{G}\|_{C(0,\infty)} < \frac{1}{q} \tag{1.5}$$

and

$$\frac{\|z\|_{C(-\eta,\infty)} + \|\hat{G}f\|_{C(0,\infty)}}{1 - q\|\hat{G}\|_{C(0,\infty)}} < \varrho. \tag{1.6}$$

If $\varrho = \infty$, then (1.6) is automatically fulfilled.

Theorem 23. *Let conditions (1.2), (1.5) and (1.6) hold. Then problem (1.1), (1.2) has at least one solution $x(t)$, which satisfies the inequality*

$$\|x\|_{C(-\eta,\infty)} \le \frac{\|z\|_{C(-\eta,\infty)} + \|\hat{G}f\|_{C(0,\infty)}}{1 - q\|\hat{G}\|_{C(0,\infty)}}.$$

The proof of this lemma is a simple application of the Schauder Fixed Point Principle. About the existence results see for instance the very interesting paper [34] and references therein. That paper deals with the existence of solutions for a nonconvex functional differential inclusion with a compact-valued and upper semicontinuous set-valued mapping.

About estimates for $\|\hat{G}\|_{C(0,\infty)}$ see Subsection 3.6.

6.2 Stability conditions

Let $X(a,\infty) = X([a,\infty);Y)$ $(-\infty < a \le 0)$ be a normed space of functions defined on $[a,\infty)$ with values in a normed space Y and the unit operator I. For example $X(a,\infty) = C([a,\infty),\mathbb{C}^n)$ or $X(a,\infty) = L^p([a,\infty),\mathbb{C}^n)$. For any $\tau > 0$ and a $w \in X(-\eta,\infty)$ $(\eta \ge 0)$ put

$$w_\tau(t) = \begin{cases} w(t) & \text{if } -\eta \le t \le \tau, \\ 0 & \text{if } t > \tau \end{cases}.$$

Let Ω_X be a domain of $X(-\eta,\infty)$ containing zero. Consider a continuous mapping $F:\Omega_X\to X(0,\infty)$ and put

$$[F_\tau w](t)=\begin{cases}[Fw](t) & \text{if } 0\le t\le\tau,\\ 0 & \text{if } t>\tau\end{cases}.$$

for all $\tau>0$ and $w\in\Omega_X$.

Definition 24. *Let F be a continuous mapping $F:\Omega_X$ into $X(0,\infty)$, having the following properties:*

$$F0\equiv 0, \tag{2.1}$$

and

$$F_\tau w=F_\tau w_\tau \textit{ for all } \tau>0\ \ (w\in\Omega_X). \tag{2.2}$$

Then F will be called a causal mapping (operator).

For all $\tau>0$ introduce the projections

$$[P(-\eta,\tau)w](t)=\begin{cases}w(t) & \text{if } -\eta\le t\le\tau,\\ 0 & \text{if } t>\tau\end{cases}\qquad (w\in X(-\eta,\infty))$$

and

$$[P(0,\tau)w](t)=\begin{cases}w(t) & \text{if } 0\le t\le\tau,\\ 0 & \text{if } t>\tau\end{cases}\qquad (w\in X(0,\infty)).$$

Then for the causal operator F we can write

$$P(0,\tau)F=P(0,\tau)FP(-\eta,\tau)\ \ (\tau>0).$$

Introduce also the subspace $X(a,\tau)$ $(a\le 0)$ by

$$X(a,\tau):=\{f\in X(a,\infty): f(t)\equiv 0,\ t>\tau\}.$$

Besides, we put

$$\|f\|_{X(a,\tau)}=\|f\|_{X(a,\infty)}$$

for all $f\in X(a,\tau)$.

We need the following result

Lemma 25. *Let F be a continuous causal mapping acting from Ω_X into $X(0,\infty)$, and*

$$\|Fw\|_{X(0,\infty)}\le q\|w\|_{X(-\eta,\infty)}\ (w\in\Omega_X).$$

Then for all $T>0$, one has

$$\|Fw\|_{X(0,T)}\le q\|w\|_{X(-\eta,T)}$$

and F is a continuous mapping in $X(-\eta,T)$.

Proof. Put

$$w_T(t) = \begin{cases} w(t) & \text{if } -\eta \le t \le T, \\ 0 & \text{if } t > T \end{cases}$$

and

$$F_T w(t) = \begin{cases} (Fw)(t) & \text{if } 0 \le t \le T, \\ 0 & \text{if } t > T. \end{cases}$$

Since F is causal, one has $F_T w = F_T w_T$. Consequently,

$$\|Fw\|_{X(0,T)} = \|F_T w\|_{X(0,\infty)} = \|F_T w_T\|_{X(0,\infty)} \le$$

$$\|F w_T\|_{X(0,\infty)} \le q\|w_T\|_{X(-\eta,\infty)} = q\|w\|_{X(-\eta,T)}.$$

Since F is continuous on $X(-\eta,\infty)$, the continuity of F on $X(-\eta,T)$ is obvious. This proves the result. □

Our definition of causal operators is somewhat different from the definition of the causal operator suggested in [6, 33], see also [23, Chapter 10]. In the paper [35] a deep investigation of a Cauchy problem with a causal operator in a separable Banach space is presented. Besides, sufficient conditions are given for the existence and uniqueness of solutions and some properties of set solutions are investigated. An example is given to illustrate the application of the main result to a Volterra integro-differential equation with delay.

Now let $X(a,\infty) = C(a,\infty)$ and

$$\Omega(\varrho) = \Omega_C(\varrho) = \{w \in C(-\eta,\infty) : \|w\|_{C(-\eta,\infty)} \le \varrho\}$$

for a positive $\varrho \le \infty$. The following condition often used below:

$$\|Fw\|_{C(0,\infty)} \le q\|w\|_{C(-\eta,\infty)} \ (w \in \Omega(\varrho)). \tag{2.2}$$

In the rest of the paper the uniqueness of the considered solutions is assumed.

Definition 26. *Let $F : C(-\eta,\infty) \to C(0,\infty)$ be a continuous mapping. Then the zero solution of (1.1) is said to be stable (in the Lyapunov sense), if for any $\epsilon > 0$, there exists a $\delta > 0$, such that the inequality $\|\phi\|_{C^1(-\eta,0)} \le \delta$ implies $\|x\|_{C(0,\infty)} \le \epsilon$ for any solution $x(t)$ of problem (1.1), (1.2).*

According to Lemma 6.2.2 and (2.2), F satisfies the hypothesis of Theorem 6.1.1. Hence, we get

Theorem 27. *Let $F : C(-\eta,\infty) \to C(0,\infty)$ be a continuous causal mapping satisfying conditions (2.2) and (1.5). Then the zero solution of (1.1) is stable. Moreover, a solution $x(t)$ of problem (1.1), (1.2) satisfies the inequality*

$$\|x\|_{C(-\eta,\infty)} \le \|z\|_{C(-\eta,\infty)}(1 - q\|\hat{G}\|_{C(0,\infty)})^{-1} \tag{2.3}$$

provided

$$\|z\|_{C(-\eta,\infty)} < (1 - q\|\hat{G}\|_{C(0,\infty)})\varrho. \tag{2.4}$$

Since the linear equation (1.4) is assumed to be stable, there is a constant c_0, such that

$$\|z\|_{C(-\eta,\infty)} \leq c_0\|\phi\|_{C(-\eta,0)}. \tag{2.5}$$

Due to (2.4), the inequality

$$c_0\|\phi\|_{C(-\eta,0)} \leq \varrho(1 - q\|\hat{G}\|_{C(0,\infty)})$$

gives us a bound for the region of attraction.

Furthermore, if : $C(-\eta,\infty) \to C(0,\infty)$ is causal and the condition

$$\lim_{\|w\|_{C(-\eta,\infty)}\to 0} \frac{\|Fw\|_{C(0,\infty)}}{\|w\|_{C(-\eta,\infty)}} = 0 \tag{2.6}$$

holds, then equation (1.1) will be called *a quasilinear causal equation.*

Theorem 28. *Let (1.1) be a quasilinear causal equation and the linear equation (1.4) be asymptotically stable. Then the zero solution to equation (1.1) is stable.*

Proof. From (2.6) it follows that for any $\varrho > 0$, there is a $q > 0$, such that (2.1) holds, and $q = q(\varrho) \to 0$ as $\varrho \to 0$. Take ϱ in such a way that the condition $q\|\hat{G}\|_{C(0,\infty)} < 1$ is fulfilled. Now the required result is due the to the previous theorem. □

For instance, if

$$\|Fw(t)\|_n \leq \sum_{k=1}^{m} \int_0^\eta \|w(t-s)\|_n^{p_k} d\mu_k(s) \ \ (t \geq 0;\ w \in C(-\eta,\infty)), \tag{2.7}$$

where $\mu_k(s)$ are nondecreasing functions, and $p_k = const \geq 1$. Then (2.1) holds. Indeed, we have

$$\|Fw\|_{C(0,T)} \leq \sum_{k=1}^{m} var\ (\mu_k)\|w\|^{p_k}_{C(-\eta,T)}.$$

So for any finite ϱ we obtain (2.2) with

$$q = q(\varrho) = \sum_{k=1}^{m} \varrho^{p_k-1} var\ (\mu_k).$$

Recall that that estimates for $\|\hat{G}\|_{C(0,\infty)}$ can be found in Section 3 (see also inequality (1.7)).

Note that differential delay equations with causal mappings were considered in [16, 17].

7 L^2-absolute Stability of Nonlinear Equations

7.1 Preliminaries

In this section, explicit conditions are established for the absolute stability of the considered systems in the terms of the L^2-norm of solutions.

First, consider the linear problem

$$\dot{y}(t) - \int_0^\eta d\tilde{R}(\tau)\dot{y}(t-\tau) = \int_0^\eta dR(\tau)y(t-\tau) \ (t \geq 0), \tag{1.1}$$

$$y(t) = \phi(t) \text{ for } -\eta \leq t \leq 0, \tag{1.2}$$

where $\phi(t) \in C^1(-\eta, 0)$ is given; $R(\tau)$ and $\tilde{R}(\tau)$ are $n \times n$-matrix-valued functions defined as above and satisfying

$$V(R) < \infty, \text{ and } V(\tilde{R}) < 1. \tag{1.3}$$

(see Section 3). Recall that

$$K(z) = Iz - z\int_0^\eta exp(-zs)d\tilde{R}(s) - \int_0^\eta exp(-zs)dR(s) \ (z \in \mathbb{C})$$

and $G(t)$ is the fundamental solution to (1.1).

For instance, (1.1) can take the form

$$\dot{y}(t) - \int_0^\eta \tilde{A}(\tau)\dot{y}(t-s)d\tau - \sum_{k=1}^{\tilde{m}} \tilde{A}_k y(t-\tilde{h}_k) = \int_0^\eta A(s)y(t-s)ds + \sum_{k=0}^{m} A_k y(t-h_k), \tag{1.4}$$

where $m, \tilde{m}$ are finite integers; $0 = h_0 < h_1 < ... < h_m \leq \eta$ and $0 < h_1 < ... < h_m \leq \eta$ are constants, $A_k, \tilde{A}_k$ are constant matrices and $A(s), \tilde{A}(s)$ are integrable on $[0, \eta]$. Besides,

$$V(R) \leq \left(\int_0^\eta \|A(s)\|_n ds + \sum_{k=0}^{m} \|A_k\|_n\right), V(\tilde{R}) \leq \left(\int_0^\eta \|\tilde{A}(s)\|_n ds + \sum_{k=0}^{\tilde{m}} \|\tilde{A}_k\|_n\right). \tag{1.5}$$

As it was mentioned, under condition (1.3), equation (1.1) is asymptotically stable and L^2-stable, if all the characteristic values of $K(.)$ are in the open left half-plane C_-.

Let $F : L^2(-\eta, \infty) \to L^2(0, \infty)$ be a continuous causal mapping. It is assumed that there is a constant q, such that

$$\|Fw\|_{L^2(0,\infty)} \leq q\|w\|_{L^2(-\eta,\infty)} \ \ (w \in L^2(-\eta, \infty)). \tag{1.6}$$

Consider the equation

$$\dot{x}(t) - \int_0^\eta d\tilde{R}(s)\dot{x}(t-s) - \int_0^\eta dR(s)x(t-s) = [Fx](t)\ (t \geq 0), \tag{1.7}$$

A solution of problem (1.7), (1.2) is a continuous function $x(t)$ defined on $[-\eta, \infty)$, such that

$$x(t) = z(t) + \int_0^t G(t-t_1)[Fx](t_1)dt_1\ (t \geq 0), \tag{1.8a}$$

$$x(t) = \phi(t)\ (-\eta \leq t \leq 0), \tag{1.8b}$$

where $z(t)$ is a solution of the problem (1.1), (1.2).

Let $\hat{G}$ be the operator defined on $L^2(0,\infty)$ by

$$\hat{G}f(t) = \int_0^t G(t-t_1)f(t_1)dt_1\ \ (f \in L^2(0,\infty)).$$

Furthermore, recall that due to Lemma 3.2.4

$$\|\hat{G}\|_{L^2(0,\infty)} \leq \theta(K) \tag{1.9}$$

and assume that

$$\theta(K) < \frac{1}{q}. \tag{1.10}$$

7.2 Stability conditions

We will say that equation (1.7) is *absolutely L^2-stable in the class of the nonlinearities satisfying* (1.6) if it has at least one solution and there is a positive constant $\hat{m}$ independent of the specific form of functions F (but dependent on q), such that

$$\|x\|_{L^2(0,\infty)} \leq \hat{m}\|\phi\|_{C^1(-\eta,0)}$$

for any solution $x(t)$ of problem (1.7), (1.2).

Lemma 29. *Let conditions (1.3) and (1.10) hold. Then equation (1.7) is absolutely L^2-stable in the class of the nonlinearities satisfying (1.6). Moreover, any solution $x(t)$ of problem (1.7), (1.2) satisfies the inequality*

$$\|x\|_{L^2(-\eta,\infty)} \leq (1 - q\theta(K)\|_{L^2(0,\infty)})^{-1}\|z\|_{L^2(-\eta,\infty)}.$$

Proof. Take a finite $T > 0$ and define the mapping Φ by

$$\Phi w(t) = z(t) + \int_0^t G(t-t_1)[Fw](t_1)dt_1 \ \ (0 \le t \le T; w \in L^2(0,T)),$$

and $\Phi w(t) = \phi(t)$ for $-\eta \le t \le 0$. Then by (1.6) and (1.10),

$$\|\Phi w\|_{L^2(-\eta,T)} \le \|\phi\|_{L^2(-\eta,0)} + \|z\|_{L^2(0,T)} + \theta(K)q\|w\|_{L^2(-\eta,T)}.$$

So Φ maps $L^2(-\eta,T)$ into itself. Taking into account that Φ is compact we prove the existence of solutions. Furthermore,

$$\|x\|_{L^2(-\eta,T)} = \|\Phi x\|_{L^2(-\eta,T)} \le \|z\|_{L^2(-\eta,T)} + \theta(K)q\|x\|_{L^2(-\eta,T)}.$$

Hence we easily we obtain (1.10). Since (1.1) is stable, there is a constant m_1, such that

$$\|z\|_{L^2(0,\infty)} \le m_1\|\phi\|_{C^1(-\eta,0)}.$$

This and (1.5), complete the proof. □

Recall that

$$g(A) = (N_2^2(A) - \sum_{k=1}^n |\lambda_k(A)|^2)^{1/2}.$$

It is not hard to check that $g^2(A) \le N^2(A) - |Trace\ A^2|$. Besides,

$$g^2(A) \le 2N_2^2(A_I) \text{ and } g(e^{i\tau}A + zI) = g(A) \tag{2.1}$$

for all $\tau \in \mathbb{R}$ and $z \in \mathbb{C}$ (see Section 2). Remind also that

$$B(z) = z\int_0^\eta z\ \ exp(-zs)d\tilde{R}(s) + \int_0^\eta exp(-zs)dR(s).$$

and for any regular value z of $K(.)$, the inequality

$$\|[K(z)]^{-1}\|_n \le \Gamma(K(z))\ (z \in \mathbb{C}) \tag{2.2}$$

is valid, where

$$\Gamma(K(z)) = \sum_{k=0}^{n-1} \frac{g^k(B(z))}{\sqrt{k!}\rho^{k+1}(K(z))}$$

and $\rho(K(z))$ is the smallest absolute value of the eigenvalues of $K(z)$:

$$\rho(K(z)) = \min_{k=1,\dots,n} |\lambda_k(K(z))|.$$

If $B(z)$ is a normal matrix, then $g(B(z)) = 0$, and $\|[K(z)]^{-1}\|_n \le \rho^{-1}(K(z))$. For example, that inequality holds, if $K(z) = zI - \tilde{A}ze^{-z\eta} - Ae^{-z\eta}$, where A and $\tilde{A}$ are commuting Hermitian matrices.

Furthermore, from (2.2) the inequality

$$\theta(K) \le \Gamma_0(K), \text{ where } \Gamma_0(K) = \sup_{-v_0 \le \omega \le v_0} \Gamma(K(i\omega)) \tag{2.3}$$

follows. Thus due to Lemma 7.2.1 we arrive at the following result.

Theorem 30. *Let all the zeros of K be in C_- and the conditions (1.3), and*

$$q\Gamma_0(K) < 1 \tag{2.4}$$

hold. Then equation (1.7) is absolutely L^2-stable in the class of the nonlinearities satisfying (1.6).

Denote

$$\hat{g}(B) := \sup_{\omega \in [-v_0, v_0]} g(B(i\omega)) \text{ and } \hat{\rho}(K) := \inf_{\omega \in [-v_0, v_0]} \rho(K(i\omega)).$$

Then we have

$$\Gamma_0(K) \le \hat{\Gamma}(K), \text{ where } \hat{\Gamma}(K) := \sum_{k=0}^{n-1} \frac{\hat{g}^k(B)}{\sqrt{k!}\hat{\rho}^{k+1}(K)}.$$

Now Theorem 7.2.1 implies

Corollary 31. *Let all the zeros of K be in C_- and the conditions (1.3), and $q\hat{\Gamma}(K) < 1$ hold. Then equation (1.7) is absolutely L^2-stable in the class of the nonlinearities satisfying (1.6).*

Thanks to the definition of $g(A)$, for all $\omega \in \mathbb{R}$ one can write

$$g(B(i\omega)) \le N_2(B(i\omega)) \le \sqrt{n}\|B(i\omega)\|_n \le \sqrt{n}(|\omega|V(\tilde{R}) + V(R)). \tag{2.8}$$

The sharper estimates for $g(B(i\omega))$ under additional conditions are given below.

7.3 Nonlinear systems with discrete delays

Let $\tilde{A} = (\tilde{a}_{jk}), A = (a_{jk})$ and $C = (c_{jk})$ be $n \times n-$ matrices. Consider the equation

$$\dot{y}(t) - \tilde{A}\dot{y}(t - \tilde{h}) + Ay(t - h) + Cy(t) = [Fy](t) \quad (t \ge 0), \tag{3.1}$$

assuming that $\|\tilde{A}\|_n < 1$. So $K(z) = z(I - \tilde{A}e^{-\tilde{h}z}) + Ae^{-hz} + C$. The entries of K are

$$k_{jk}(z) = z(1 - \tilde{a}_{jk}e^{-\tilde{h}z}) + a_{jk}e^{-hz} + c_{jk.}$$

Thanks to the Hadamard criterion [36], any characteristic value λ of $K(z)$ satisfies the inequality

$$|k_{jj}(z)-\lambda| \leq \sum_{m=1,m\neq j}^{n} |k_{jm}(z)| \ \ (j=1,...,n).$$

Hence we have

$$\rho(K(z)) \geq \min_{j=1,\dots,n} (|k_{jj}(z)| - \sum_{m=1,m\neq j}^{n} |k_{jm}(z)|\,), \tag{3.2}$$

provided the right-hand part is positive. Furthermore, in the case (3.1) we have $V(\tilde{R}) = \|\tilde{A}\|_n$, $V(R) = \|A\|_n + \|C\|_n$, $v_0 = 2(\|A\| + \|C\|)(1-\|\tilde{A}\|)^{-1}$. In addition,

$$g(K(z)) = g(B(z)) = g(-z\tilde{A}e^{-\tilde{h}z} + Ae^{-hz} + C).$$

Hence, by (2.1)

$$g(B(i\omega)) \leq \frac{1}{\sqrt{2}} N_2(B(i\omega) - B^*(i\omega)) \leq$$

$$\frac{1}{\sqrt{2}}[|\omega| N_2(e^{-i\tilde{h}\omega}\tilde{A} + e^{i\tilde{h}\omega}\tilde{A}^*) + N_2(e^{-ih\omega}A - e^{i\tilde{h}\omega}A^*) + N_2(C - C^*)].$$

One can use also relation $g(B(i\omega)) = g(e^{is}B(i\omega))$ for all real s and ω. In particular, taking $s = \tilde{h} + \pi/2$, we have by (2.1)

$$g(B(i\omega)) \leq \frac{1}{\sqrt{2}}[|\omega| N_2(\tilde{A}-\tilde{A}^*) + N_2(e^{-i(h-\tilde{h})\omega}A + e^{i(h-\tilde{h})\omega}A^*) + N_2(Ce^{i\tilde{h}\omega} + e^{-i\tilde{h}\omega}C^*)].$$

If $\tilde{A}$ is self-adjoint, then

$$g(B(i\omega)) \leq \frac{1}{\sqrt{2}}[N_2(e^{-i(h-\tilde{h})\omega}A + e^{i(h-\tilde{h})\omega}A^*) + N_2(Ce^{i\tilde{h}\omega} + e^{-i\tilde{h}\omega}C^*)].$$

Hence,

$$g(B(i\omega)) \leq \sqrt{2}[N_2(A) + N_2(C)] \ \ (\omega \in \mathbb{R}). \tag{3.3}$$

For example, consider the system

$$\dot{y}_j(t) - \tilde{a}_{jj}\dot{y}_j(t-\tilde{h}) + \sum_{k=1}^{n}(a_{jk}y_k(t-h) + c_{jk}y_k(t)) = [F_j y](t), \tag{3.4}$$

$(j=1,...,n;\ t\geq 0)$, where $[F_j y](t)$ are coordinates of $[Fy](t)$, and suppose that

$$0 \leq \tilde{a}_{jj} < 1, a_{jj}, c_{jj} \geq 0 \ \ (j=1,...,n). \tag{3.5}$$

So $\tilde{A} = diag\ (\tilde{a}_{jj})$. Then (3.2) implies

$$\rho(K(i\omega)) \geq \min_{j=1,\dots,n} (|k_{jj}(i\omega)| - \sum_{m=1,m\neq j}^{n} (|a_{jm}| + |c_{jm}|)). \tag{3.6}$$

Taking into account that $|\tilde{a}_{jj}| \leq \|\tilde{A}\|_n < 1$, put

$$v_j = \frac{2(a_{jj} + c_{jj})}{1 - \tilde{a}_{jj}}$$

and assume that

$$v_j max\{h_j, \tilde{h}_j\} < \pi/2 \text{ and } d_j := a_{jj} cos(v_j h) + c_{jj} - v_j \tilde{a} sin(v_j \tilde{h}) > 0 \tag{3.7}$$

$(j = 1, ..., n)$. Then by Corollary 3.4.3 all the zeros of $k_{jj}(.)$ are in C_- and

$$\inf_{-\infty \leq \omega \leq \infty} |k_{jj}(i\omega)| \geq d_j > 0.$$

In addition, let

$$\hat{\rho}_d := \min_{j=1,\dots,n} (d_j - \sum_{m=1,m\neq j}^{n} (|a_{jm}| + |c_{jm}|)) > 0, \tag{3.8}$$

then according to (3.3) we get

$$\Gamma_0(K) \leq \Gamma_d := \sum_{k=0}^{n-1} \frac{(\sqrt{2}(N_2(A) + N_2(C)))^k}{\sqrt{k!}\hat{\rho}_d^{k+1}}.$$

Now Theorem 7.2.2 yields our next result.

Corollary 32. *Let conditions (3.5), (3.7) and (3.8) be fulfilled. Then system (3.4) is absolutely L^2-stable in the class of the nonlinearities satisfying (1.6), provided* $q\Gamma_d < 1$.

7.4 Nonlinear systems with distributed delays

Let us consider the equation

$$\dot{y}(t) - \tilde{A} \int_0^\eta \dot{y}(t-s) d\tilde{\mu}(s) + A \int_0^\eta y(t-s) d\mu(s) = [Fy](t) \ \ (t \geq 0), \tag{4.1}$$

where $\tilde{A}$ and A are $n \times n-$ matrices with $\|\tilde{A}\| < 1$, and $\mu, \tilde{\mu}$ are scalar nondecreasing functions with finite numbers of jumps, again, and $\tilde{\mu}$ does not have a jump at zero. Without loss of generality suppose that

$$var(\mu) = var(\tilde{\mu}) = 1. \tag{4.2}$$

So in the considered case $R(s) = \mu(s)A, \tilde{R}(s) = \tilde{\mu}(s)\tilde{A}$, $K(z) = zI - B(z)$ with

$$B(z) = z\tilde{A}\int_0^\eta e^{-zs}d\tilde{\mu}(s) - A\int_0^\eta e^{-zs}d\mu(s),$$

$V(R) = \|A\|_n$, $V(\tilde{R}) = \|\tilde{A}\|_n$ and $v_0 = 2\|A\|_n(1 - \|\tilde{A}\|_n)^{-1}$. Moreover,

$$\hat{g}(K) = \sup_{|\omega|<v_0} g(B(i\omega)) \le N_2(A) + v_0 N_2(\tilde{A}).$$

If

$$K(z) = zI - z\tilde{A}e^{-z\tilde{h}} + A\int_0^\eta e^{-zs}d\mu(s), \tag{4.3}$$

then by (2.1) $g(B(i\omega)) = g(ie^{i\omega\tilde{h}}B(i\omega)) \le$

$$\frac{1}{\sqrt{2}}[\,|\omega|N_2(\tilde{A} - \tilde{A}^*) + N_2(\int_0^\eta e^{-i\omega(s-\tilde{h})}d\mu(s)A + \int_0^\eta e^{i\omega(s-\tilde{h})}d\mu(s)A^*)].$$

Consequently, in the case (4.3) we get

$$\hat{g}(K) \le \frac{v_0}{\sqrt{2}}N_2(\tilde{A} - \tilde{A}^*) + \sqrt{2}N_2(A).$$

Now we can directly can apply Corollary 7.3.1.

In the rest of this subsection we suppose that $\tilde{A}$ and A commute, then the eigenvalues of K are

$$\lambda_j(K(z)) = z - z\int_0^\eta e^{-zs}d\tilde{\mu}(s)\lambda_j(\tilde{A}) + \int_0^\eta e^{-zs}d\mu(s)\lambda_j(A),$$

and, in addition, according to (2.2), $g(B(i\omega)) \le |\omega|g(\tilde{A}) + g(A)$. So

$$g(B(i\omega)) \le g(A, \tilde{A}) := v_0 g(\tilde{A}) + g(A) \quad (\omega \in [-v_0, v_0]).$$

Furthermore, suppose $\lambda_k(A)$ and $\lambda_k(\tilde{A})$ $(k = 1, ..., n)$ are positive and put

$$v_k = \frac{2\lambda_k(A)}{1 - \lambda_k(\tilde{A})}.$$

If

(4.4)

$$\eta v_k < \pi/2 \text{ and } d_k(\mu, \tilde{\mu}) := \lambda_k(A)\int_0^\eta cos\,(\tau v_k)d\mu - v_k\lambda_k(\tilde{A})\int_0^\eta sin\,(\tau v_k)d\tilde{\mu} > 0,$$

$(k = 1, ..., n)$, then by Corollary 3.4.3 all the characteristic values of K are in C_- and

$$\inf_{\omega\in\mathbb{R}} |\lambda_j(K(i\omega))| \ge \tilde{d}_{com} := \min_k d_k(\mu, \tilde{\mu}) \ \ (j = 1, \dots, n).$$

So

$$\hat{\Gamma}(K) \le \Gamma_{com}(K) := \sum_{k=0}^{n-1} \frac{\hat{g}^k(A,\tilde{A})}{\sqrt{k!}\tilde{d}_{com}^{k+1}}.$$

Now Corollary 7.2.3 implies

Corollary 33. *Let $\tilde{A}$ and A be commuting matrices with positive eigenvalues. Let the conditions (4.2), (4.4) and $q\Gamma_{com}(K) < 1$ be fulfilled. Then equation (4.1) is absolutely L^2-stable in the class of the nonlinearities satisfying (1.6).*

7.5 Example

Consider the system

$$\dot{y}_j(t) - a\dot{y}_j(t-h) + \sum_{k=1}^{2} c_{jk} y_k(t) = \int_0^t W_j(t,s) f_j(y_1(s-h), y_2(s-h))ds \tag{5.1}$$

($t \ge 0$), where $0 \le a < 1$, c_{jk} ($j = 1,2$) are real, $W_j(t,s)$ are real functions defined on $[0,\infty)^2$ with the property

$$\int_0^\infty \int_0^t |W_j(t,s)|^2 dsdt \le b_j^2 < \infty \ \ (b_j = const \ge 0)$$

and $f_j(z_1,z_2)$ are functions defined on $\mathbb{C}^2$ with the property

$$|f_j(z_1,z_2)| \le \tilde{q}_{j1}|z_1| + \tilde{q}_{j2}|z_2| \ \ (\tilde{q}_{jk} = const; \ z_j \in \mathbb{C}; j,k = 1,2).$$

So (5.1) has the form (3.4). Besides,

$$\int_0^\infty (\int_0^t W_j(t,s) f_j(y_1(s-h), y_2(s-h))ds)^2 dt \le$$

$$\int_0^\infty \int_0^t |W_j(t,s)|^2 dsdt \int_0^\infty |f_j(y_1(s-h), y_2(s-h))|^2 ds \le$$

$$b_j^2 \int_0^\infty (\tilde{q}_{j1}|y_1(s-h)| + \tilde{q}_{j2} y_2(s-h)|)^2 ds.$$

Then condition (1.6) holds with

$$q^2 = 2\sum_{k=1}^{2} b_j^2 \sum_{k=1}^{2} \tilde{q}_{jk}^2. \tag{5.2}$$

Furthermore, we have $K(z) = z(1 - ae^{-zh})I + C$ with $C = (c_{jk})$; $\tilde{A} = aI$, $B(z) = -ae^{-zh}I + C$ and by (2.1), Since $\tilde{A}$ and C commute, $g(B(z)) = g(C) \le g_C = |c_{12} - c_{21}|$ and the eigenvalues of K are

$$\lambda_j(K(z)) = z - z\, ae^{-zh} + \lambda_j(C).$$

Suppose $\lambda_k(C)$ $(k=1,2)$ are positive and put

$$v_k = \frac{2\lambda_k(C)}{1-a}.$$

If

(5.3) $$hv_k < \pi/2 \text{ and } d_k := \lambda_k(C) - v_k a\, sin\, (hv_k) > 0 \ \ (k=1,2),$$

then by Corollary 3.4.3, the characteristic values of K are in C_-, and

$$\inf_{\omega\in\mathbb{R}} |\lambda_j(K(i\omega))| \geq \hat{d} := \min_{k=1,2} d_k \ (j=1,2).$$

So

$$\hat{\Gamma}(K) \leq \tilde{\Gamma} := \frac{1}{\hat{d}}(1+\frac{g_C}{\hat{d}}).$$

Thanks to Corollary 7.2.3 we can assert that system (5.1) is absolutely L^2-stable provided the conditions (5.3) and $q\tilde{\Gamma} < 1$ hold, where q is defined by (5.2).

8 Exponential Stability of Nonlinear Systems

8.1 Stability conditions

Recall that $\Omega(\varrho) = \{f \in C(-\eta,\infty) : \|f\|_{C(-\eta,\infty)} \leq \varrho\}$ for a given number $0 < \varrho \leq \infty$. Consider the equation

(1.1) $$\dot{x}(t) - \int_0^\eta d\tilde{R}(s)\dot{x}(t-s) - \int_0^\eta dR(t,s)x(t-s) = [Fx](t) \ (t \geq 0),$$

where F :is a continuous mapping $\Omega(\varrho) \to C(0,\infty)$, satisfying the following condition: there is a nondecreasing function $\nu(t)$ defined on $[0,\eta]$, such that

(1.2) $$\|[Ff](t)\|_n \leq \int_0^\eta \|f(t-s)\|_n d\nu(s) \ \ (t \geq 0; f \in \Omega(\varrho)).$$

A (mild) solution of equation (1.1) with an initial function $\phi \in C^1(-\eta,0)$ again is defined as a continuous function $x(t)$ defined on $[-\eta,\infty)$, such that

(1.3) $$x(t) = z(t) + \int_0^t G(t-t_1)[Fx](t_1)dt_1 \ (t \geq 0),$$

(1.4) $$x(t) = \phi(t) \ (-\eta \leq t \leq 0),$$

where $z(t)$ and G are defined as in Section 6. Recall that the uniqueness of solutions is assumed.

We will say that the zero solution to equation (3.1) is exponentially stable, if there are positive constants $r_0 \leq \varrho$, $\hat{m}$ and ϵ, such that for any ϕ with

$$\|\phi\|_{C^1(-\eta,0)} \leq r_0, \tag{1.5}$$

problem (1.1), (1.4) has a solution $x(t)$ and

$$\|x(t)\|_n \leq \hat{m}e^{-\epsilon t}\|\phi\|_{C^1(-\eta,0)} \ \ (t \geq 0).$$

Recall that $\theta(K)$ is defined in Section 3.

Theorem 34. *Let all the characteristic values of $K(.)$ be in C_- and the conditions (1.2),*

$$var(\nu)\theta(K) < 1 \tag{1.6}$$

and

$$V(\tilde{R}) < 1 \tag{1.7}$$

hold. Then the zero solution to (1.1) is exponentially stable.

This theorem is proved in the next subsection. About estimates for $\theta(K)$ see Section 3.

8.2 Proof of Theorem 8.1.1

Recall that for all $T > 0$, the inequalities

$$\|Ef\|_{C(-\eta,T)\to C(0,T)} \leq \sqrt{n} \ \ V\ (R) \tag{2.1}$$

and

$$\|Ef\|_{L^2(-\eta,T)\to L^2(0,T)} \leq V\ (R) \tag{2.2}$$

are valid (see Section 3).

The proof of Theorem 8.1.1 is based on the following lemmas which are proved in [20, Section 3].

Lemma 35. *Let condition (1.2) hold with $r = \infty$. Then $\|Fw\|_{L^2(-\eta,T)} \leq var(\nu)\|w\|_{L^2(-\eta,T)}$ $(w \in L^2(-\eta,T))$ for any $T > 0$.*

Furthermore, use the operator $\hat{G}$ defined on $L^2(0,\infty)$ by

$$\hat{G}f(t) = \int_0^t G(t-t_1)f(t_1)dt_1 \ \ (f \in L^2(0,\infty)),$$

and assume that

$$var\ (\nu)\ \|\hat{G}\|_{L^2(0,\infty)} < 1. \tag{2.3}$$

Lemma 36. *Let conditions (1.7), (2.3) and (1.2) with $\varrho = \infty$ hold. Then problem (1.1), (1.4) has a solution $x(t)$. Moreover,*

$$\|x\|_{L^2(-\eta,\infty)} \le (1 - var(\nu)\|\hat{G}\|_{L^2(0,\infty)})^{-1}\|z\|_{L^2(-\eta,\infty)}. \tag{2.4}$$

According to Lemma 3.2.4

$$\|\hat{G}\|_{L^2(0,\infty)} \le \theta(K) \tag{2.5}$$

Now the previous lemma implies the inequality

$$\|x\|_{L^2(-\eta,\infty)} \le (1 - var(\nu)\theta(K))^{-1}\|z\|_{L^2(-\eta,\infty)}.$$

Since the all the zeros of $det\, K(.)$ are in C_-, and (1.7) holds, we obtain

$$\|x\|_{L^2(-\eta,\infty)} \le c_2\|\phi\|_{C^1(-\eta,0)} \quad (c_2 = const). \tag{2.6}$$

From (1.1), (2.2) and Lemma 8.2.1 it follows that

$$\|\dot{x}\|_{L^2(0,\infty)} \le V(\tilde{R})\|\dot{x}\|_{L^2(-\eta,\infty)} + (V(R) + var(\nu))\|x\|_{L^2(-\eta,\infty)}.$$

Or

$$\|\dot{x}\|_{L^2(0,\infty)} \le V(\tilde{R})(\|\dot{x}\|_{L^2(0,\infty)} + \|\dot{x}\|_{L^2(-\eta,0)}) + (V(R) + var(\nu))\|x\|_{L^2(-\eta,\infty)}.$$

So due to (1.7) we obtain

Corollary 37. *Under the hypothesis of Lemma 8.2.2 we have*

$$\|\dot{x}\|_{L^2(0,\infty)} \le (1 - V(\tilde{R}))^{-1}[(V(R) + var(\nu))\|x\|_{L^2(0,\infty)} + V(\tilde{R})\|\dot{\phi}\|_{L^2(-\eta,0)}].$$

The previous lemma and (2.6) imply the inequality

$$\|\dot{x}\|_{L^2(0,\infty)} \le c_3\|\phi\|_{C^1(-\eta,0)} \quad (c_3 = const). \tag{2.7}$$

Due to Lemma 3.2.5, if $f \in L^2(0,\infty)$ and $\dot{f} \in L^2(0,\infty)$, then $\|f\|^2_{C(0,\infty)} \le 2\|f\|_{L^2(0,\infty)}\|\dot{f}\|_{L^2(0,\infty)}$. This inequality and (2.7) imply the next result.

Lemma 38. *Under the hypothesis of Lemma 8.2.2, the inequality*

$$\|x\|_{C(0,\infty)} \le c_4\|\phi\|_{C^1(-\eta,0)} \quad (c_4 = const) \tag{2.8}$$

is valid and therefore the zero solution of (1.1) is globally stable in the Lyapunov sense.

Proof of Theorem 8.1.1: Substituting

$$x(t) = y_\epsilon(t)e^{-\epsilon t} \tag{2.9}$$

with an $\epsilon > 0$ into (1.1), we obtain the equation

$$\dot{y}_\epsilon - \epsilon y_\epsilon - E_{\epsilon,\tilde{R}}\dot{y}_\epsilon + \epsilon E_{\epsilon,\tilde{R}} y_\epsilon = E_{\epsilon,R} y_\epsilon + F_\epsilon y_\epsilon, \tag{2.10}$$

where

$$(E_{\epsilon,\tilde{R}} f)(t) = \int_0^\eta e^{\epsilon\tau} d_\tau \tilde{R}(t,\tau) f(t-\tau), (E_{\epsilon,R} f)(t) = \int_0^\eta e^{\epsilon\tau} d_\tau R(t,\tau) f(t-\tau)$$

and $[F_\epsilon f](t) = e^{\epsilon t}[F(e^{-\epsilon t} f)](t)$. By (1.2) with $r = \infty$ we have

$$\|[F_\epsilon f](t)\|_n \le e^{\epsilon t} \int_0^\eta e^{-\epsilon(t-s)} \|f(t-s)\|_n d\nu \le e^{\epsilon\eta} \int_0^\eta \|f(t-s)\|_n d\nu.$$

Taking ϵ sufficiently small and applying our above arguments to equation (2.10), according to (2.8), we obtain

$$\|y_\epsilon\|_{C(0,\infty)} \le c_\epsilon \|\phi\|_{C^1(-\eta,0)} \quad (c_\epsilon = const). \tag{2.11}$$

Now (2.9) implies

$$\|x(t)\|_{C(0,\infty)} \le c_\epsilon \|\phi\|_{C^1(-\eta,0)} e^{-\epsilon t} \quad (t \ge 0). \tag{2.12}$$

So in the case $\varrho = \infty$, the theorem is proved.

Now let $\varrho < \infty$. By the Urysohn theorem [8, p. 15], there is a continuous scalar-valued function ψ_ϱ defined on $C(0,\infty)$, such that

$$\psi_\varrho(f) = 1 \quad (\|f\|_{C(0,\infty)} < \varrho) \text{ and } \psi_\varrho(f) = 0 \quad (\|f\|_{C(0,\infty)} \ge \varrho).$$

Put $[F_\varrho f](t) = \psi_\varrho(f)[Ff](t)$. Clearly, F_ϱ satisfies (2.2) for all $f \in C(-\eta,\infty)$. Consider the equation

$$\dot{x} - \tilde{E}\dot{x} = Ex + F_\varrho x. \tag{2.13}$$

The solution of (2.13) denote by x_ϱ. For a sufficiently small ϵ, according to (2.12), we have $\|x_\varrho(t)\|_{C(0,\infty)} \le c_\epsilon \|\phi\|_{C^1(-\eta,0)}$. If we take $\|\phi\|_{C^1(-\eta,0)} \le \varrho/c_\epsilon$, then $F_\varrho x_\varrho = Fx$ and equations (1.1) and (2.13) coincide. This and (2.12) prove the theorem. □

9 Aizerman's type problem

9.1 Statement of the result

Recall that $C_- = \{z \in \mathbb{C} : Re\ z < 0\}$. In this section $C(\Omega)$ is the space of continuous scalar-valued functions defined on a segment $\Omega \subset \mathbb{R}$ and equipped with the sup-norm. $L^2(\Omega)$ is the space of scalar-valued functions defined on Ω and equipped with the norm

$$\|v\|_{L^2(\Omega)} = [\int_\Omega |v(t)|^2 dt]^{1/2} \ (v \in L^2(\Omega)).$$

Consider the scalar neutral type functional differential equation

$$(1.1)\quad \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj} x^{(k)}(t-h_j) = f(t, x(t), x(t-h_1), x(t-h_2), ..., x(t-h_m)) \ \ (t > 0),$$

where $0 = h_0 < h_1 < ... < h_m$ are delays; a_{kj} are constant real coefficients with $a_{n0} = 1$ and $a_{nj} \neq 0$ for at least one $j \geq 1$. The function $f : [0,\infty) \times \mathbb{R}^{m+1} \to \mathbb{R}$ is continuously differentiable and

$$(1.2)\quad |f(t, z_0, z_1, ..., z_m)| \leq \sum_{k=0}^{m} q_k |z_k| \ (q_k = const \geq 0;\ z_k \in \mathbb{R}; k = 0, ..., m;\ \ t \geq 0).$$

Put $\eta = h_m = \max_{k=1,...,m} h_k$. With a given function ϕ having continuous derivatives up to n-th order take the initial conditions

$$(1.3)\qquad x(t) = \phi(t) \ (k = 0, ..., n;\ -\eta \leq t \leq 0).$$

Consider the linear equation

$$(1.4)\qquad \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj} \psi^{(k)}(t-h_j) = 0 \ (t \geq 0).$$

A solution of problem (1.4), (1.3) is n-times continuously differentiable function satisfying (1.4) and (1.3) for all $t > 0$.

Throughout this paper it is assumed that (1.4) is L^2-stable. Namely, all the zeros of the characteristic (transfer) function

$$(1.5)\qquad K(\lambda) = \sum_{k=0}^{n}\sum_{j=0}^{m} a_{kj} \lambda^k e^{-\lambda h_j}$$

are in C_-, and solutions of (1.4) are in $L^2(0,\infty)$, cf. [29, p. 114]. Introduce the Green function (fundamental solution, impulse response) of (1.4)

$$G(t) := \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{ti\omega} K^{-1}(i\omega) d\omega.$$

By the variation of constants formula [27, 29], equation (1.1) is equivalent to the following one:

$$x(t) = \psi(t) + \int_0^t G(t-s) f(s, x(s), x(s-h_1), ..., x(s-h_m)) ds \quad (t \ge 0), \tag{1.6}$$

where $\psi(t)$ is a solution of problem (1.4), (1.3).

A (mild) solution of problem (1.1), (1.3) is a function $x(t)$ defined on $[-\eta, \infty)$, which is continuous on $[0, \infty)$, satisfies (1.6) for $t \ge 0$, and satisfies (1.3) for $t \in [-\eta \le t \le 0]$. The existence of solutions is assumed (see the previous section).

Definition 39. *Equation (1.1) is said to be L^2-absolutely stable (L^2-a.s.) in the class of nonlinearities (1.2), if there is a positive constant c_0 independent of the specific form of function f (but dependent on q_j, $j = 0, ..., m$), such that*

$$\|x\|_{L^2(0,\infty)} \le c_0 \sum_{k=0}^{n} \|\phi^{(k)}\|_{C(-\eta,0)}$$

for any solution $x(t)$ of (1.1) with initial conditions (1.3).

Let A, b, c be an $n \times n$-matrix, a column-matrix and a row-matrix, respectively. In 1949 M. A. Aizerman conjectured the following hypothesis: for the absolute stability of the zero solution of the equation $\dot{x} = Ax + bf(cx)$ in the class of nonlinearities $f : \mathbb{R} \to \mathbb{R}$, satisfying $0 \le f(s)/s \le q$ ($q = const > 0, s \in \mathbb{R},\ s \ne 0$) it is necessary and sufficient that the linear equation $\dot{x} = Ax + q_1 bcx$ be asymptotically stable for any $q_1 \in [0, q]$ [2]. This hypothesis caused the great interest among the specialists. Counterexamples were set up that demonstrated it was not, in general, true, (see [44] and references therein). In connection with these results, A. A. Voronov [44] investigated the following problem: to find the class of systems that satisfy Aizerman's hypothesis. He also received the first important results in that direction. The author has showed that any system satisfies the Aizerman hypothesis, if its impulse function is non-negative [10]. The similar result was proved for multivariable systems and distributed ones, [11]). On the other hand, A.D. Myshkis, [38, Section 10] pointed out at the importance of consideration of the generalized Aizerman problem for retarded systems. That problem, called the Aizerman-Myshkis problem, was investigated under various assumptions, by the author in the papers [12, 14] and [17]. The very interesting results on the Aizerman-Myshkis problem can be found in the papers [41, 43]. For the classical results on absolute stability of retarded systems we refer the reader to the excellent book [42]. In that paper, absolute stability conditions for a wide class of functional and integro-differential equations have been established. These conditions are connected with the generalized Aizerman problem, which means the following: to separate a class of linear parts of nonlinear systems, such that there is a linear equations, whose stability provides the absolute stability of the considered nonlinear systems.

In the present paper we consider the following generalization of the Aizerman problem.

Problem 9.1: *To separate a class of systems (1.1), such that the L^2-stability of the linear system*

$$x^{(n)} + \sum_{k=0}^{n} \sum_{j=1}^{m} a_{kj} x^{(k)}(t - h_j) = \sum_{k=0}^{m} \tilde{q}_k x(t - h_k) \tag{1.7}$$

for some $\tilde{q}_k \in [0, q_k]$ $(k = 0, ..., m)$ provides the L^2-a.s. stability of system (1.1) in the class of nonlinearities (1.2).

It should be noted that Problem 1 is considerably more complicated than the Aizerman-Myshkis problem, since in the case of neutral type equations, the location of all the zeros of the characteristic function in C_- does not guarantee the exponential stability [28, 29].

Now we are in a position to formulate the main result of the paper.

Theorem 40. *Let Green's function of (1.4) be positive:*

$$G(t) \geq 0 \ (t \geq 0), \tag{1.8}$$

and the linear equation (1.7) be L^2-stable with $\tilde{q}_k = q_k$ $(k = 0, ..., m)$. Then (1.1) is L^2-a.s. in the class of nonlinearities (1.2).

So Theorem 9.1.2 separates a class of systems satisfying Problem 9.1. It is proved in the next section.

Remark 41. *Note that the notion of the L^2-stability (L^2-absolute stability) is stronger that the notion of the asymptotic stability (asymptotic absolute stability) at least under the natural condition*

$$d_n = \sum_{j=1}^{m} |a_{nj}| < 1. \tag{1.9}$$

Indeed, assume that a solution x of (1.1) is in $L^2(0, \infty)$ and note that from (1.1) and (1.2), for any finite $T > 0$, it follows that

$$\|x^{(n)}\|_{L^2(0,T)} \leq \sum_{k=0}^{n} d_k \|x^{(k)}\|_{L^2(-\eta,T)},$$

where

$$d_k = \sum_{j=1}^{m} |a_{kj}| \ \ (0 < k < n) \text{ and } d_0 = \sum_{k=0}^{m} \tilde{q}_k + \sum_{j=1}^{m} |a_{0j}|.$$

Hence,

$$\|x^{(n)}\|_{L^2(0,T)} \le \sum_{k=0}^{n} d_k \|x^{(k)}\|_{L^2(0,T)} + c_0 \text{ where } c_0 = \sum_{k=0}^{n} d_k \|x^{(k)}\|_{L^2(-\eta,0)}$$

and therefore

$$\|x^{(n)}\|_{L^2(0,T)} \le (1-d_n)^{-1} \sum_{k=0}^{n-1} d_k \|x^{(k)}\|_{L^2(0,T)} + (1-d_n)^{-1} c_0.$$

Put $w_k = \|x^{(k)}\|^{1/n}_{L^2(0,T)}, k = 1, ..., n$. By the moment inequality, cf. formula (5.41) from Section 5.9, Chapter I of the book [32], we have $w_k^n \le w_n^k w_0^{n-k}$. Thus,

$$w_n^n \le \sum_{k=0}^{n-1} \tilde{d}_k w_n^k w_0^{n-k} + \tilde{c}_0,$$

where $\tilde{c}_0 = (1-d_n)^{-1} c_0$ and $\tilde{d}_k = (1-d_n)^{-1} d_k$. By the well known estimates for the polynomial roots, we obtain

$$w_n \le 1 + \max\{\max_{k>0} \tilde{d}_k w_0^{n-k}, \tilde{d}_0 w_0^n + c_0\}.$$

Hence,

$$w_n \le 1 + \max\{\max_{k>0} \tilde{d}_k \|x\|^{(n-k)/n}_{L^2(0,\infty)}, \tilde{d}_0 \|x\|_{L^2(0,\infty)} + \tilde{c}_0\}.$$

The right-hand part of this inequality does not depend on T. This implies $x^{(n)} \in L^2(0,\infty)$ and therefore, $\dot{x} \in L^2(0,\infty)$. Thus

$$|x(t)|^2 = -\int_t^\infty \frac{d}{ds}|x(s)|^2 ds = -2\int_t^\infty |x(s)| \frac{d}{ds}|x(s)| ds \le$$

$$2(\int_t^\infty |x(s)|^2 ds)^{1/2} (\int_t^\infty |\dot{x}(s)|^2 ds)^{1/2} \to 0$$

as $t \to \infty$. As claimed.

9.2 Proof of Theorem 9.1.2

For an arbitrary n-times continuously differentiable function w defined on $[-\eta, \infty)$, let us introduce the operator $\hat{K}$ by

$$(\hat{K}w)(t) := \sum_{k=0}^{n} \sum_{j=0}^{m} a_{kj} w^{(k)}(t - h_j).$$

Put in (1.6) $y(t)=x(t)-\psi(t)$ where $\psi(t)$ is a solution of (1.4), (1.3). Then
(2.1)
$$y(t)=\int_0^t G(t-s)f(s,y(s)+\psi(s),y(s-h_1)+\psi(s-h_1),...,y(s-h_m)+\psi(s-h_m))ds.$$

Besides,

$$y^{(k)}(t)=0\ (k=0,...,n;\ -\eta\le t\le 0). \tag{2.2}$$

Due to the positivity of $G(t)$ and condition (1.2), we get

$$|y(t)|\le\int_0^t G(t-s)(F(s)+q_0|y(s)|+q_1|y(s-h_1)|+...+q_m|y(s-h_m)|)ds.$$

where

$$F(s)=q_0|\psi(s)|+q_1|\psi(s-h_1)|+...+q_m|\psi(s-h_m)|.$$

By the well-known Lemma 3.2.1 [7] (the comparison principle), we have

$$|y(t)|\le z(t)\ (t\ge 0), \tag{2.3}$$

where $z(t)$ is a solution of the equation

$$z(t)=\int_0^t G(t-s)(F(s)+q_0z(s)+...+q_mz(s-h_m))ds \tag{2.4}$$

with the conditions

$$z^{(k)}(t)=0\ (k=0,...,n;\ -\eta\le t\le 0). \tag{2.5}$$

But equation (2.4) is equivalent to the following one:

$$(\tilde{K}z)(t)=F(t)+q_0z(t)+q_1z(t-h_1)+...+q_mz(t-h_m). \tag{2.6}$$

Denote by $W(t)$ the Green function to (2.6). By the variation of constants formula, the latter equation with conditions (2.5) can be written as

$$z(t)=\int_0^t W(t-s)F(s)ds\ \ (t\ge 0).$$

Since (1.7) is L^2-stable, by the Schwarz inequality, we obtain the inequality

$$\|z\|_{L^2(0,\infty)}\le\|W\|_{L^2(0,\infty)}\|F\|_{L^2(0,\infty)}. \tag{2.7}$$

Let us check that $\|F\|_{L^2(0,\infty)}<\infty$. Indeed, since (1.4) is L^2-stable, we have $\|\psi\|_{L^2(0,\infty)}\le c_1\|\phi\|_{C,n}$, where

$$\|\phi\|_{C,n}:=\sum_{k=0}^n\|\phi^{(k)}\|_{C(-\eta,0)}.$$

But

$$\|F\|_{L^2(0,\infty)} \le q_0\|\psi\|_{L^2(0,\infty)} + q_1\|\psi\|_{L^2(-h_1,\infty)} + ... + q_m\|\psi\|_{L^2(-h_m,\infty)}.$$

Therefore, $\|F\|_{L^2(0,\infty)} \le c_2\|\psi\|_{L^2(-\eta,\infty)} \le c_3\|\phi\|_{C,n}$.

Taking into account (2.3), (2.7) and that $x(t) = y(t) + \psi(t)$, we arrive at the required result. □

9.3 Auxiliary results

In this section we prove some results, which will be used in the sequel. Consider the equation

$$u(t) - \int_d^\eta u(t-s)d\mu(s) = (Vu)(t) + f(t) \quad (t \ge 0) \tag{3.1}$$

where μ is a non-decreasing function, $d \in (0,\eta)$ is a constant, $f(t)$ is a continuous function such that $f(t) \ge 0$ $(t > 0)$ and $f(0) > 0$, V is a positive Volterra operator: $Vu(t) = \int_0^t K(t,s)u(s)ds$ where $K(t,s)$ is positive, continuous in t and integrable in s on any finite segment.

Consider the linear equation

$$\dot{y}(t) - a\dot{y}(t-\tilde{h}) + by(t-h) = 0, \tag{3.2}$$

where $a, b, h, \tilde{h}$ are positive constants. Due to Lemma 3.5.1, if the equation

$$s = se^{\tilde{h}s}a + e^{hs}b \tag{3.3}$$

has a positive root ζ, then the Green function $G_1(t)$ to (3.2) is nonnegative. Moreover,

$$G_1(t) \ge e^{-\zeta t} \ge 0 \ (t \ge 0), \tag{3.4}$$

$\dot{G}_1(t) \le 0$ and

$$\int_0^\infty G_1(t)dt = \frac{1}{b}. \tag{3.5}$$

In addition, Corollary 3.5.2 implies.

$$\inf_{\omega\in\mathbb{R}} |k(i\omega)| = b, \tag{3.6}$$

provided (3.3) has a positive root.

As it was mentioned in Subsection 3.5, if there is a positive number λ, such that $ae^{\tilde{h}\lambda}\lambda + be^{h\lambda} \le \lambda$, then due to the well-known Theorem 38.1 [31] equation (3.3) has a positive root $\zeta \le \lambda$. In particular, if

$$e^{\tilde{h}c}a + \frac{1}{c}e^{ch}b \le 1, \tag{3.7}$$

then equation (3.3) has a positive root $\zeta \le c$. Moreover, denote

$$c_0 = -\frac{bh}{2a\tilde{h}} + \sqrt{\left(\frac{bh}{2a\tilde{h}}\right)^2 + \frac{b}{a\tilde{h}}}$$

and suppose

$$e^{\tilde{h}c_0} a + \frac{1}{c_0} e^{c_0 h} b \le 1, \tag{3.8}$$

then as it is shown i (3.3) has a positive root $\zeta \le c_0$.

9.4 Particular cases

Consider the first order equation

$$\dot{x}(t) - a\dot{x}(t - \tilde{h}) + bx(t - h) = f(x(t), x(t - \tilde{h}), x(t - h)), \tag{4.1}$$

where $b, h, \tilde{h}$ are positive constants and $0 < a = const < 1$. The function $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable and

$$|f(t, z_0, z_1, z_2)| \le \sum_{k=0}^{2} q_k |z_k| \;(q_k = const \ge 0;\; z_k \in \mathbb{R}; k = 0, 1, 2;\;\; t \ge 0). \tag{4.2}$$

Let equation (3.3) have a positive root. Then by (3.4) $G_1(t) \ge 0$ and due to Theorem 9.1.2, equation (4.1) is L^2-a.s. provided the linear equation

$$\dot{x}(t) - a\dot{x}(t - \tilde{h}) + bx(t - h) = q_0 x(t) + q_1 x(t - \tilde{h}) + q_2 x(t - h) \tag{4.3}$$

is L^2-stable. Since $a < 1$, due to [29], equation (4.3) is L^2-stable, provided all the zeros of the characteristic function

$$k_2(z) := z - ze^{-\tilde{h}z} a + e^{-hz} b - (q_0 + q_1 e^{-\tilde{h}z} a + q_2 e^{-hz})$$

of (4.3) are in C_-. There are numerous criteria of the stability of linear first order neutral type equations, cf. [28] and references therein. Let us suggest a new one. Assume that

$$q_0 + q_1 + q_2 < b. \tag{4.4}$$

Then due to (3.6) we have

$$|q_0 + q_1 e^{-\tilde{h}i\omega} a + q_2 e^{-hi\omega}| \le q_0 + q_1 + q_2 < |k(i\omega)| \;\; (\omega \in \mathbb{R}).$$

Hence due to the theorem of Rouché, all the zeros of $k_2(z)$ are in C_-. We thus arrive at our next result.

Corollary 42. *Let equation (3.3) have a positive root and condition (4.4) hold. Then equation (4.1) is L^2-a.s. in the class of nonlinearities (4.2).*

Now consider a higher order equations with the characteristic function

$$K(\lambda) = \prod_{j=1}^{n}(\lambda - \lambda a_j e^{-\tilde{h}_j\lambda} + b_j e^{-h_j\lambda}) \ \ (h_j, \tilde{h}_j, a_j, b_j = const \geq 0; \ j = 1, ..., n). \tag{4.5}$$

Let each of the equations

$$s = se^{\tilde{h}_j s} a_j + e^{hs} b_j \ \ (j = 1, 2, ..., n) \tag{4.6}$$

have at last one positive root. Then necessarily $a_j < 1$ $(j = 1, 2, ..., n)$. Taking into account that a product of the Laplace transforms of several functions corresponds to the convolution of these functions, we have due to Lemma 9.3.2 the following result:

Lemma 43. *Let each of equations (4.6) have a positive root. Then the Green function corresponding to the function $K(\lambda)$ defined by (4.5) is nonnegative.*

Now we can directly apply Theorem 9.1.2.

Example 44. *Consider equation (4.1) with $a = 0.3, b = 0.5, \tilde{h} = 0.1, h = 0.2$. So equation (3.3) has the form*

$$s = 0.3se^{0.1s} + 0.5e^{0.2s}. \tag{4.7}$$

Since

$$0.1e^{0.1} + 0.2e^{0.2} < 1,$$

due to (3.7), equation (4.7) has a positive root. Hence, by Corollary 9.4.1 equation (4.1) is L^2-a.s. in the class of nonlinearities (4.2), provided $q_0 + q_1 + q_2 < 0.5$.

Example 45. *Consider the equation*

$$\frac{d}{dt}[u(t) - \vartheta_1 u(t - \frac{2}{s})] = -\vartheta_2 u(t) + f(u(t), u(t - \frac{2}{s})) \ \ (t \geq 0), \tag{4.8}$$

with positive parameters ϑ_1, ϑ_2 and s, and the continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ assuming that

$$|f(w_1, w_2)| \leq q_1|w_1| + q_2|w_2| \ \ (w_1, w_2 \in \mathbb{R}), \tag{4.9}$$

where q_1 and q_2 are positive constants.

This equation is considered in the previous section, but now we investigate it under new conditions.

To apply Theorem 9.1.2 to (4.8) note that in the considered case

$$a = \vartheta_1, b = \vartheta_2, \tilde{h} = 2/s, h = 0.$$

Condition (3.7) with $c = 1$ takes the form

$$\vartheta_1 e^{2/s} + \vartheta_2 < 1. \tag{4.10}$$

Condition (4.4) takes the form

$$q_1 + q_2 < \vartheta_2. \tag{4.11}$$

Hence, by Corollary 9.4.1 equation (4.8) is L^2-a.s. in the class of nonlinearities (4.9), provided inequalities (4.10) and (4.11) hold.

10 Stability conditions via generalized norms

10.1 Preliminaries

In this section, the inequalities for real vectors or vector functions are understood in the coordinate-wise sense.

Furthermore, let $\hat{\rho} := (\rho_1, ..., \rho_n)$ be a vector with positive coordinates $\rho_j < \infty$. We need the following set:

$$\tilde{\Omega}(\hat{\rho}) := \{v(t) = (v_j(t)) \in C([-\eta, \infty), \mathbb{C}^n) : \|v_j\|_{C([-\eta,\infty),\mathbb{C})} \le \rho_j; \;\; j = 1, ..., n\}.$$

If we introduce in $C([a, b], \mathbb{C}^n)$ the generalized norm as the vector

$$M_{[a,b]}(v) := (\|v_j\|_{C([a,b],\mathbb{C})})_{j=1}^n \;\; (v(t) = (v_j(t)) \in C([a, b], \mathbb{C}^n))$$

(see [23, Section 1.7] and references therein), then we can write down

$$\tilde{\Omega}(\hat{\rho}) := \{v \in C([-\eta, \infty), \mathbb{C}^n) : M_{[-\eta,\infty)}(v) \le \hat{\rho}\}.$$

Recall that $R(\tau) = (\tilde{r}_{jk}(\tau))_{j,k=1}^n$ and $R(\tau) = (r_{jk}(\tau))_{j,k=1}^n$ are an $n \times n$-matrix-valued functions defined on $[0, \eta]$, whose entries are real and have bounded variations. Again consider in $\mathbb{C}^n$ the problem

$$\dot{x} - \tilde{E}\dot{x} = Ex + Fx \;\; (t \ge 0), \tag{1.1}$$

$$x(t) = \phi(t) \in C^1(-\eta, 0) \; (-\eta \le t \le 0), \tag{1.2}$$

where F is a continuous causal mapping in $C(-\eta, \infty)$, E and $\tilde{E}$ are defined as in Subsection 3.2. A (mild) solution of problem (1.1), (1.2) is again is defined as a continuous function $x(t)$, such that

$$x(t) = z(t) + \int_0^t G(t - t_1) F x(t_1) dt_1 \; (t \ge 0), \tag{1.3}$$

and (1.2) holds. Here $G(t)$ is the fundamental solution of the linear equation

$$\dot{z} - \tilde{E}\dot{z} = Ez \tag{1.4}$$

and $z(t)$ is a solution of the problem (1.5), (1.2). Again use the operator

$$\hat{G}f(t) = \int_0^t G(t-t_1)f(t_1)dt_1 \ \ (f \in C(0,\infty)).$$

10.2 Stability conditions

Rewrite (1.1) in the form

$$\dot{x}_j(t) - \sum_{k=1}^n \int_0^\eta \dot{x}_j(t-s)d\tilde{r}_{jk}(s) = \sum_{k=1}^n \int_0^\eta x_j(t-s)dr_{jk}(s) + [Fx]_j(t) \quad (t \geq 0;\ j = 1, ..., n), \tag{2.1}$$

where $x(t) = (x_k(t))_{k=1}^n$, $[Fw]_j(t)$ mean the coordinates of the vector function $Fw(t)$ with a $w \in C([-\eta,\infty), \mathbb{C}^n)$. In addition,

$$\sup_j \sum_{k=1}^n var\ (\tilde{r}_{jk}) < 1. \tag{2.2}$$

It is assumed that F satisfies the following condition: there are nonnegative constants ν_{jk} $(j, k = 1, ..., n)$, such that for any

$$w(t) = (w_j(t))_{j=1}^n \in \tilde{\Omega}(\hat{\rho}),$$

the inequalities

$$\|[Fw]_j\|_{C([0,\infty),\mathbb{C})} \leq \sum_{k=1}^n \nu_{jk}\|w_k\|_{C([-\eta,\infty),\mathbb{C})} \ \ (j = 1, ..., n) \tag{2.3}$$

hold. In other words,

$$M_{[0,\infty)}(Fw) \leq \Lambda(F)M_{[-\eta,\infty)}(w) \ \ (w \in \tilde{\Omega}(\hat{\rho})), \tag{2.4}$$

where $\Lambda(F)$ is the matrix whose entries are ν_{jk}:

$$\Lambda(F) = (\nu_{jk})_{j,k=1}^n. \tag{2.5}$$

Lemma 46. *Let F be a continuous causal mapping in $C(-\eta,\infty)$ satisfying condition (2.6). Then*

$$M_{[0,T]}(Fw) \leq \Lambda(F)M_{[-\eta,T]}(w) \ \ (w \in \tilde{\Omega}(\hat{\rho}) \cap C([-\eta,T]), \mathbb{C}^n))$$

for all $T > 0$.

For the proof see [23, Lemma 10.7.1].

It is also assumed that the entries $G_{jk}(t,s)$ of the fundamental solution $G(t)$ of equation (1.5) satisfy the conditions

$$\gamma_{jk} := \sup_{t\geq 0} \int_0^\infty |G_{jk}(t,s)|ds < \infty. \tag{2.6}$$

Denote by $\hat{\gamma}$ the matrix with the entries γ_{jk}:

$$\hat{\gamma} = (\gamma_{jk})_{j,k=1}^n.$$

Theorem 47. *Let the conditions (2.3) and (2.6) hold. If, in addition, the spectral radius $r_s(Q)$ of the matrix $Q = \hat{\gamma}\Lambda(F)$ is less than one, then the zero solution of system (2.1) is stable. Moreover, if a solution z of the linear problem (1.4), (1.2) satisfies the condition*

$$M_{[-\eta,\infty)}(z) + Q\hat{\rho} \leq \hat{\rho}\,, \tag{2.7}$$

then the solution $x(t)$ of problem (2.1), (1.2) satisfies the inequality

$$M_{[-\eta,\infty)}(x) \leq (I-Q)^{-1} M_{[-\eta,\infty)}(z). \tag{2.8}$$

Proof. Take a finite $T > 0$ and define on $\Omega_T(\hat{\rho}) = \tilde{\Omega}(\hat{\rho}) \cap C(-\eta, T)$ the mapping Φ by

$$\Phi w(t) = z(t) + \int_0^t G(t-t_1)[Fw](t_1)dt_1 \;\; (0 \leq t \leq T; w \in \Omega_T(\hat{\rho})),$$

and

$$\Phi w(t) = \phi(t) \text{ for } -\eta \leq t \leq 0.$$

Then by (2.4),

$$M_{[-\eta,T]}(\Phi w) \leq M_{[-\eta,T]}(z) + \hat{\gamma}\Lambda(F) M_{[-\eta,T]}(w).$$

According to (2.7) Φ maps $\Omega_T(\hat{\rho})$ into itself. Taking into account that Φ is compact we prove the existence of solutions. Furthermore,

$$M_{[-\eta,T]}(x) = M_{[-\eta,T]}(\Phi x) \leq M_{[-\eta,T]}(z) + QM_{[-\eta,T]}(x).$$

So

$$M_{[-\eta,T]}(x) \leq (I-Q)^{-1} M_{[-\eta,T]}(z).$$

Hence letting $T \to \infty$, we obtain (2.8), completing the proof. □

Note that since $Q \geq 0, \hat{\rho} \geq 0$ and $r_s(Q) < 1$, we have $Q\hat{\rho} \leq \hat{\rho}$.

The Lipschitz condition

$$M_{[0,\infty)}(Fw - Fw_1) \le \Lambda(F) M_{[-\eta,\infty)}(w - w_1) \ \ (w_1, w \in \tilde{\Omega}(\hat{\rho})) \tag{2.9}$$

together with the Generalized Contraction Mapping theorem also allows us to prove the existence and uniqueness of solutions.

Note that one can use the well-known inequality

$$r_s(Q) \le \max_j \sum_{k=1}^n \hat{Q}_{jk}, \tag{2.10}$$

where $\hat{Q}_{jk}$ are the entries of Q. About this inequality, as well as about other estimates for the matrix spectral radius see [30, Section 16].

10.3 Systems with diagonal linear parts

Consider the system

$$\dot{x}_j - a_j \dot{x}(t - \tilde{h}_j) + b_j x(t - h_j) = F_j(x) \tag{3.1}$$

where $b_j, h_j, \tilde{h}_j$ are positive constants and $0 < a_j = const < 1$. Let each of the equations

$$s = s e^{\tilde{h}_j s} a_j + e^{h_j s} b_j \tag{3.2}$$

have a positive root. Then as it is shown in Subsection 3.5 the fundamental solution $G_j(t)$ of the equation

$$\dot{z}_j - a_j \dot{z}(t - \tilde{h}_j) + b_j z(t - h_j) = 0, \tag{3.3}$$

is positive

$$\int_0^\infty G_j(t) dt = \frac{1}{b_j}. \tag{3.4}$$

In the considered case $G_{jj}(t) = G_j(t), G_{jk} = 0, j \ne k$, $\gamma_{jj} = 1/b_j$. Thus under condition (2.3), we have $Q = (\nu_{jk}/b_j)_{j,k=1}^n$. So (2.10) takes the form

$$r_s(Q) \le \max_j \frac{1}{b_j} \sum_{k=1}^n \nu_{jk}. \tag{3.5}$$

For instance, let

$$(F_j x)(t) = f_j(x_1(t), x_1(t - \tilde{h}), x_1(t - h), ..., x_n(t), x_n(t - \tilde{h}), x_n(t - h))$$

where the functions $f_j : \mathbb{R}^{3n} \to \mathbb{R}$ are scalar continuous and

$$(3.6) \qquad |f_j(z_{01}, z_{11}, z_{21}, ..., z_{0n}, z_{1n}, z_{2n})| \leq \sum_{m=0}^{2} \sum_{k=1}^{n} q_{jmk}|z_{mk}|$$

$$(q_{jmk} = const \geq 0;\ z_{mk} \in \mathbb{R}; m = 0, 1, 2;\ j, k = 1, ..., m;\ t \geq 0).$$

Hence condition (2.3) follows with

$$\nu_{jk} = \sum_{m=0}^{2} q_{jmk}.$$

11 Input-to-state Stability

In this section we establish explicit conditions that provide the input-to-state stability for the considered systems. systems is rather rich. The input-to-state and input-output stability of nonlinear retarded systems with causal mappings was investigated considerably less than the one for systems without delay. In papers [13] and [16], bounded input-to-bounded output stability conditions for multivariable retarded systems was derived via the Karlson inequality. In the paper [17], the author has derived a criterion for the L^2-input-to-state stability of one-contour retarded systems with causal mappings, that is, for systems governed by scalar functional differential equations. At the same time, to the best of our knowledge the input-to-state stability of nonlinear neutral type delay systems especially with causal mappings was not investigated in the available literature. In this paper we improve and generalize the main result from [17].

For a positive $\eta < \infty$, and an input $u \in L^2(0, \infty) = L^2([0, \infty), \mathbb{C}^n)$, consider in $\mathbb{C}^n$ the problem

$$(1.1) \qquad \dot{x}(t) - \int_0^\eta d\tilde{R}(\tau)\dot{x}(t-\tau) = \int_0^\eta dR(\tau)x(t-\tau) + [F(x)](t) + u(t),$$

$$(1.2) \qquad x(t) = 0 \text{ for } -\eta \leq t \leq 0,$$

where $x(t)$ is the state, $R(s) = (r_{ij}(s))_{i,j=1}^n$ and $\tilde{R}(s) = (\tilde{r}_{ij}(s))_{i,j=1}^n$ are real $n \times n$-matrix-valued functions defined on $[0, \eta]$, whose entries have bounded variations $var(r_{ij})$ and $var(\tilde{r}_{ij})$. Recall that $Var(R) = (var(r_{ij}))_{i,j=1}^n$ and $V(R) = \|Var(R)\|_n$. It is assumed that

$$(1.3) \qquad V(\tilde{R}) < 1.$$

Let F be a continuous causal mapping in $L^2(-\eta, \infty)$ satisfying the following condition: there is a constant q, such that

$$(1.4) \qquad \|Fw\|_{L^2(0,\infty)} \leq q\|w\|_{L^2(-\eta,\infty)} \ \ (w \in L^2(-\eta, \infty)).$$

A (mild) solution of problem (1.1), (1.2) is a continuous function $x(t)$ defined on $[0,\infty)$, such that

$$x(t) = \int_0^t G(t-t_1)([Fx](t_1)+u(t_1))dt_1 \ (t \geq 0),$$

with the zero initial condition. As above $G(t)$ is the fundamental solution of the linear equation

$$\dot{x}(t) - \int_0^\eta d\tilde{R}(s)\dot{x}(t-s) - \int_0^\eta dR(s)x(t-s) = 0 \ (t \geq 0).$$

The solution existence is proved in [25]. The uniqueness of solutions is assumed.

We will say that equation (1.1) is input-to-state L^2-stable, if for any $\epsilon > 0$, there is a $\delta > 0$, such that $\|u\|_{L^2(0,\infty)} \leq \delta$ implies $\|x\|_{L^2(R_+)} \leq \epsilon$ for any solution of problem (1.1), (1.2).

Recall that $\theta_d(K)$ is defined in Subsection 3.1. The following result has been proved in [25]:

Let all the characteristic values of $K(.)$ be in C_-. Let the conditions (1.3), (1.4) and $q\theta_d(K) < 1$ hold. Then (1.1) is input-to-state L^2-stable.

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