Surveys in Mathematics and its Applications

ISSN 1842-6298 (electronic), 1843-7265 (print)
Volume 6 (2011), 165 – 174

CERTAIN THEOREMS ON TWO DIMENSIONAL LAPLACE TRANSFORM AND NON-HOMOGENEOUS PARABOLIC PARTIAL DIFFERENTIAL EQUATIONS

A. Aghili and B. Parsa Moghaddam

Abstract. In this work,we present new theorems on two-dimensional Laplace transformation.We also develop some applications based on these results. The two-dimensional Laplace transformation is useful in the solution of non-homogeneous partial differential equations. In the last section a boundary value problem is solved by using the double Laplace-Carson transform.

1 Introduction

R. S. Dahiya proved (1990) certain theorems involving the classical Laplace transform of N-variables and in the second part a non-homogeneous partial differential equation of parabolic type with some special source function was considered [4]. J. Saberi Najafi and R. S. Dahiya established (1992) several new theorems for calculating Laplace theorems of n-dimensions and in the second part application of those theorems to a number of commonly used special functions was considered, and finally, one-dimensional wave equation involving special functions was solved by using two dimensional Laplace transform [5]. Later the authors, established (2004, 2006, 2008) new theorems and corollaries involving systems of two - dimensional Laplace transform containing several equations [1, 2, 3].

Definition 1. *The generalization of the well-known Laplace transform*

$$L[f(t);s] = \int_0^\infty e^{-st} f(t)dt$$

to n-dimensional is given by

$$L_n[f(\bar{t});\bar{s}] = \int_0^\infty \int_0^\infty \dots \int_0^\infty exp(-\bar{s}.\bar{t}) f(\bar{t}) P_n(d\bar{t})$$

2010 Mathematics Subject Classification: Primary 44A30; Secondary 35L05.
Keywords: Two-dimensional Laplace transform; Two-dimensional Laplace-Carson transforms; Heat equation.

where $\bar{t} = (t_1, t_2, \ldots, t_n), \bar{s} = (s_1, s_2, \ldots, s_n), \bar{s}.\bar{t} = \sum_{i=1}^{n} s_i t_i$, *and* $P_n(d\bar{t}) = \prod_{k=1}^{n} dt_k$.

In addition to the notations introduced above, we will use the following throughout this article.
Let $\bar{t^v} = (t_1^v, t_2^v, \ldots, t_n^v)$ for any real exponent v and let $P_k(\bar{t})$ be the k-th symmetric polynomial in the components t_k of $\bar{t}$. Then
$(i) P_1(\bar{t^v}) = t_1^v + t_2^v + \ldots + t_n^v$
$(ii) P_n(\bar{t^v}) = t_1^v . t_2^v \ldots t_n^v$.
$P_0(\bar{t^v}) = 1$ *ect.*
Similarly for $\bar{s} = (s_1, s_2, \ldots, s_n), P_2(\bar{s}^v) = \sum_{i,j=1,i<j}^{n} s_i^v s_j^v$ and so on.
The inverse Laplace Transform is given by

$$L^{-1}\{F(\bar{s}); \bar{t}\} = (\frac{1}{2i\pi})^n \int_{a-i\infty}^{a+i\infty} \int_{d-i\infty}^{d+i\infty} \ldots \int_{c-i\infty}^{c+i\infty} e^{-\bar{s}\bar{t}} F(\bar{s}) P_n(\bar{s}) d\bar{s} \tag{1.1}$$

Remark 2. *In case of two dimensions, one has the following relationship*

$$f(x,y) = L^{-1}\{F(p,q); p, q\} = (\frac{1}{2i\pi})^2 \int_{a-i\infty}^{a+i\infty} \int_{d-i\infty}^{d+i\infty} e^{px+qy} F(p,q)\, dp\, dq \tag{1.2}$$

Example 3. *Using (1.2) to evaluate*

$$L_2^{-1}\{\frac{1}{2(p-\mu)(q+\sqrt{p+\lambda})}\}$$

λ, μ *are real numbers. By complex inversion formula for two dimensional Laplace transform one has the following relation,*

$$f(x,y) = (\frac{1}{2i\pi})^2 \int_{c-i\infty}^{c+i\infty} \int_{d-i\infty}^{d+i\infty} e^{px+qy} \{\frac{1}{2(p-\mu)(q+\sqrt{p+\lambda})}\} dp\, dq$$

or equivalently,

$$f(x,y) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{e^{px}}{2(p-\mu)} dp \{\frac{1}{2i\pi} \int_{d-i\infty}^{d+i\infty} \frac{e^{qy}}{q+\sqrt{p+\lambda}} dq\}.$$

If we calculate the integral in the curly bracket, then one has

$$f(x,y) = \frac{1}{2i\pi} \int_{c-i\infty}^{c+i\infty} \frac{e^{px-y\sqrt{p+\lambda}}}{2(p-\mu)} dp. \tag{1.3}$$

At this point, in order to avoid complex integration around a complicated key-hole contour,we use the appropriate integral representation for the exponential term as follows,

$$\frac{2}{\sqrt{\pi}} \int_0^\infty e^{-u^2 - \frac{(p+\lambda)y^2}{4u^2}} du = e^{-y\sqrt{p+\lambda}}$$

substitution of the above relation in (1.3) leads to the following

$$f(x,y) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\{\frac{1}{\sqrt{\pi}}\int_0^\infty e^{-u^2-\frac{(p+\lambda)y^2}{4u^2}}du\}\frac{e^{px}}{p-\mu}dp$$

changing the order of integration, yields

$$f(x,y) = \frac{1}{\sqrt{\pi}}\int_0^\infty e^{-u^2-\frac{\lambda y^2}{4u^2}}\{\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty}\frac{e^{p(x-\frac{y^2}{4u^2})}}{p-\mu}dp\}du$$

introducing the change of variable $p-\mu = s$, in the above integral, leads to

$$f(x,y) = \frac{e^{\mu x}}{\sqrt{\pi}}\int_0^\infty e^{-u^2-\frac{(\lambda+\mu)y^2}{4u^2}}\{\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty}\frac{e^{s(x-\frac{y^2}{4u^2})}}{s}ds\}du$$

finally, the value of the inner integral is equal to $H(x-\frac{y^2}{4u^2})$, therefore, we obtain

$$f(x,y) = \frac{e^{\mu x}}{\sqrt{\pi}}\int_{\frac{y}{2\sqrt{x}}}^\infty e^{-u^2-\frac{(\lambda+\mu)y^2}{4u^2}}du.$$

Interesting consequences of the two - dimensional Laplace transform will be given as follows.

Problem 4. *The heat transfer for cooling of a very thin semi - infinite rod into the surrounding medium satisfies,*

$$\frac{\partial u}{\partial t} = \kappa\frac{\partial^2 u}{\partial x^2} - hu \quad x,t>0 \quad h,\kappa\epsilon\Re^+ \tag{1.4}$$

with boundary conditions,

$$u(x,0)=0, \ \lim_{x\longrightarrow\infty} u(x,t)=0, \quad u(0,t)=t^\alpha e^{\beta t}$$

Solution 5. *Let $L\{u(x,t);p,q\} = U(p,q)$. Application of the two dimensional Laplace transform to (1.4) and after performing some easy calculations, we get [3]*

$$U(x,q) = \frac{\Gamma(\alpha+1)}{(q-\beta)^{\alpha+1}}e^{-x\sqrt{\frac{h+q}{\kappa}}}. \tag{1.5}$$

The inverse Laplace transform of the above equation is not available in tables and very difficult to evaluate (see [3]), but by using the following expressions and convolution theorem [6], we find the inverse as follows

$$L^{-1}\{\frac{1}{(q-\beta)^{\alpha+1}}\} = \frac{t^\alpha e^{\beta t}}{\Gamma(\alpha+1)}$$

$$L^{-1}\{e^{-x\sqrt{\frac{h+q}{\kappa}}}\} = \frac{x}{2t^{\frac{3}{2}}\sqrt{\kappa\pi}}$$

thus, we get

$$u(x,t) = \frac{xe^{\beta t}}{2\sqrt{\kappa\pi}}\int_0^t \xi^{-\frac{3}{2}}(t-\xi)^{\alpha}e^{-\frac{x^2}{4\kappa\xi}-(h+\beta)\xi}d\xi.$$

We shall evaluate only for some particular values of α, β, h *and* κ. [6]
Case (i)
For $\alpha = -\frac{5}{2}, h = \beta = 0$, *we get*

$$u(x,t) = \Gamma(-\frac{3}{2})\frac{x^4 - 12\kappa t x^2 + 12\kappa^2 t^2}{16\kappa^2 t^{\frac{9}{2}}\sqrt{\pi}}e^{-\frac{x^2}{4\kappa t}}. \tag{1.6}$$

Case (ii)
For $\alpha = -\frac{1}{2}, \beta = \kappa = 1, h = 0$, *we have*

$$u(x,t) = \frac{e^t}{\sqrt{t}}\{e^{-\frac{x^2}{4t}} - x\int_0^{\infty}\frac{J_1(\sqrt{u^2-x^2})}{\sqrt{u^2-x^2}}e^{-\frac{u^2}{4t}}du\}. \tag{1.7}$$

Case (xii)
For $\alpha = \frac{n-3}{2}, h = \beta = 0$, *we obtain*

$$u(x,t) = \frac{\Gamma(\frac{n-1}{2})}{\sqrt{2^n t^{n+1}\pi}}He_n(\frac{x}{\sqrt{2\kappa t}})e^{-\frac{x^2}{4\kappa t}} \quad n = 0,1,2,\dots \tag{1.8}$$

where $He_n(x)$ *is Hermite polynomial.*

2 THE MAIN THEOREM

Theorem 6. *Suppose*
(xi) $L_2\{f(x,y); x,y\} = F(p,q)$
(xii) $L_2\{f(\sqrt{x},\sqrt{y}); x,y\} = \varphi(p,q)$ *then*

$$L_2^{-1}\{F(\sqrt{p},\sqrt{q})\} = \frac{1}{16\pi}(xy)^{-\frac{3}{2}}\varphi(\frac{1}{4x},\frac{1}{4y}) \tag{2.1}$$

Proof. We start with the operational relation

$$Ł\{x^{-\frac{1}{2}}\frac{\partial^n}{\partial s^n}(e^{-\frac{s^2}{4x}}); x\} = (-1)^n\sqrt{\pi}p^{\frac{n-1}{2}}e^{-s\sqrt{p}} \tag{2.2}$$

for $n = 1$, we have

$$Ł\{\frac{1}{2}x^{-\frac{3}{2}}se^{-\frac{s^2}{4x}}; x\} = \sqrt{\pi}e^{-s\sqrt{p}}$$

$$Ł\{\frac{1}{2}y^{-\frac{3}{2}}se^{-\frac{t^2}{4y}};y\} = \sqrt{\pi}e^{-t\sqrt{q}}$$

by multiplying the above equations to get

$$Ł\{\frac{1}{4}(xy)^{-\frac{3}{2}}tse^{-\frac{s^2}{4x}-\frac{t^2}{4y}};x,y\} = \pi e^{-s\sqrt{p}-t\sqrt{q}}$$

at this point, we multiply both sides by $f(s,t)$ and integrate with respect to s and t over the first quadrant, to obtain

$$Ł_2\{\frac{1}{4}(xy)^{-\frac{3}{2}}\int_0^\infty\int_0^\infty tse^{-\frac{s^2}{4x}-\frac{s^2}{4y}}f(s,t)dsdt;x,y = \int_0^\infty\int_0^\infty \pi e^{-s\sqrt{p}-t\sqrt{q}}f(s,t)dsdt$$

we make the change of variables $s^2=u$ and $t^2=v$ on the left hand side to get

$$Ł_2\{\frac{1}{16}(xy)^{-\frac{3}{2}}\int_0^\infty\int_0^\infty e^{-\frac{u}{4x}-\frac{v}{4y}}f(\sqrt{u},\sqrt{v})dudv;x,y = \pi\int_0^\infty\int_0^\infty e^{-s\sqrt{p}-t\sqrt{q}}f(s,t)dsdt \tag{2.3}$$

finally by using (1.2) on the left hand side and (2.3) on the right hand side we have

$$L_2^{-1}\{F(\sqrt{p},\sqrt{q})\} = \frac{1}{16\pi}(xy)^{-\frac{3}{2}}\varphi(\frac{1}{4x},\frac{1}{4y}).$$

□

Example 7. *Let* $f(x,y)=e^{-xy}$ *, then*

$$F(p,q) = -e^{pq}Ei(-pq)$$

$$\varphi(p,q) = \frac{4(\sqrt{4pq-1}-arctan(\sqrt{4pq-1}))}{|4pq-1|\sqrt{4pq-1}}$$

using (2.1) and simplifying, leads to

$$L_2^{-1}\{-e^{\sqrt{pq}}Ei(-\sqrt{pq})\} = \frac{2\sqrt{xy}arctan(\frac{\sqrt{1-4xy}}{2\sqrt{xy}}))-\sqrt{1-4xy}}{|4xy-1|\sqrt{xy(1-4xx)}}.$$

Example 8. *Let* $f(x,y)=\sin(\alpha xy)$*, then*

$$F(p,q) = -\frac{1}{\alpha}\{Ci(\frac{pq}{\alpha})\cos(\frac{pq}{\alpha})+Si(\frac{pq}{\alpha})\sin(\frac{pq}{\alpha})\}$$

$$\varphi(p,q) = \frac{2\alpha\pi}{(4pq+\alpha^2)^{\frac{3}{2}}}$$

using (2.1) and simplifying, we get

$$L_2\{-\frac{\alpha^2}{(1+4\alpha^2xy)^{\frac{3}{2}}};x,y\} = Ci(\frac{\sqrt{pq}}{\alpha})\cos(\frac{\sqrt{pq}}{\alpha})+Si(\frac{\sqrt{pq}}{\alpha})\sin(\frac{\sqrt{pq}}{\alpha}).$$

Remark 9. *In order to calculate the inverse Laplace transform of* $p^{\frac{i}{2}}q^{\frac{j}{2}}F(\sqrt{p},\sqrt{q})$ *for all values of* i *and* j *, it is sufficient to use relation (2.2) for different value of* n *by applying the same procedure as in Theorem 6. In the following we give two theorems.*

Theorem 10. *Let*
(xi) $L_2\{f(x,y);x,y\}=F(p,q)$
(xii) $L_2\{y^{-\frac{1}{2}}f(\sqrt{x},\sqrt{y});x,y\}=\varphi(p,q)$
(xiii) $L_2\{y^{\frac{1}{2}}f(\sqrt{x},\sqrt{y});x,y\}=\psi(p,q)$ *then*

$$L_2^{-1}\{q^{\frac{1}{2}}F(\sqrt{p},\sqrt{q})\}=\frac{1}{32\pi}x^{-\frac{3}{2}}y^{-\frac{5}{2}}\psi(\frac{1}{4x},\frac{1}{4y})-\frac{1}{16\pi}(xy)^{-\frac{3}{2}}\varphi(\frac{1}{4x},\frac{1}{4y}). \tag{2.4}$$

Proof. To get (2.4), we start with (2.2) for $n=1,2$. The rest of the proof is similar to the proof of Theorem 6. □

Example 11. *Let* $f(x,y)=\sin(\alpha xy)$*, then*

$$F(p,q)=-\frac{1}{\alpha}\{Ci(\frac{pq}{\alpha})\cos(\frac{pq}{\alpha})+Si(\frac{pq}{\alpha})\sin(\frac{pq}{\alpha})\}$$

$$\varphi(p,q)=\frac{2\alpha\sqrt{\pi}}{\sqrt{p}(4pq+\alpha^2)}$$

$$\psi(p,q)=\frac{8\sqrt{\pi p}}{(4pq+\alpha^2)^2}$$

using (2.4) and simplifying,leads to

$$L_2\{\frac{\alpha^2(4\alpha^2xy-1)}{\sqrt{\pi y}(4\alpha^2xy+1)^2};x,y\}=\sqrt{q}\{Ci(\frac{\sqrt{pq}}{\alpha})\cos(\frac{\sqrt{pq}}{\alpha})+Si(\frac{\sqrt{pq}}{\alpha})\sin(\frac{\sqrt{pq}}{\alpha})\}.$$

Theorem 12. *Let*
(xi) $L_2\{f(x,y);x,y\}=F(p,q)$
(xii) $L_2\{yf(\sqrt{x},\sqrt{y});x,y\}=\varphi(p,q)$
(xiii) $L_2\{f(\sqrt{x},\sqrt{y});x,y\}=\psi(p,q)$ *then*

$$L_2^{-1}\{qF(\sqrt{p},\sqrt{q})\}=\frac{1}{64\pi}x^{-\frac{3}{2}}y^{-\frac{7}{2}}\varphi(\frac{1}{4x},\frac{1}{4y})-\frac{3}{32\pi}x^{-\frac{3}{2}}y^{-\frac{5}{2}}\psi(\frac{1}{4x},\frac{1}{4y}). \tag{2.5}$$

Proof. To get (2.5), we start with (2.2) for $n=1,2$. The rest of the proof is similar to the proof of Theorem 6. □

Example 13. *Let* $f(x,y)=\ln(xy)$*, then*

$$\varphi(p,q)=\frac{1-2\gamma-\ln(pq)}{2pq^2}$$

$$\psi(p,q) = -\frac{2\gamma + \ln(pq)}{pq}$$

using (2.5) and simplifying,we get

$$L_2^{-1}\{\frac{\sqrt{q}(4\gamma + \ln(pq))}{2\sqrt{q}}\} = \frac{1}{4\pi}x^{-\frac{1}{2}}y^{-\frac{3}{2}}(\ln(16xy) - 2(\gamma+1)).$$

In the following example, we give an application of two dimensional Laplace transform and complex inversion formula for calculating a series related to Lageurre Polynomials.

Example 14. *Show that*

$$\sum_{n=0}^{\infty} T_a^{(n)}(x)L_n(y)\lambda^n = \frac{1}{\Gamma(a+1)}y^a e^{-\lambda} I_0(2\sqrt{\lambda x}) \tag{2.6}$$

$T_a^{(n)}(x)$ *is sonine polynomial and defined by*

$$T_a^{(n)}(x) = \frac{(-1)^n}{\Gamma(a+n+1)}L_a^{(n)}(x)$$

.

Solution 15. *It is well known that*

$$L\{L_n(x), p\} = \frac{1}{p}(1-\frac{1}{p})^n$$

$$L\{T_a^{(n)}(y), q\} = \frac{(1-q)^n}{n!q^{n+a+1}}$$

taking two - dimensional Laplace transform of the left hand side, leads to the following

$$L_2\{\sum_{n=0}^{\infty} L_n(x)T_a^{(n)}(y)\lambda^n, p, q\} = \int_0^{\infty}\int_0^{\infty}(\sum_{n=0}^{\infty} L_n(x)T_a^{(n)}(y)\lambda^n e^{-px-qy})dxdy$$

changing the order of summation and double integration to get,

$$L_2\{\sum_{n=0}^{\infty} L_n(x)T_a^{(n)}(y)\lambda^n, p, q\} = \sum_{n=0}^{\infty}(\int_0^{\infty}\int_0^{\infty} L_n(x)T_a^{(n)}(y)\lambda^n e^{-px-qy}dxdy)$$

the value of the inner integral is

$$\sum_{n=0}^{\infty}\lambda^n(\int_0^{\infty}\int_0^{\infty} L_n(x)T_a^{(n)}(y)e^{-px-qy}dxdy) = \sum_{n=0}^{\infty}\frac{\lambda^n}{n!}\frac{1}{p}(1-\frac{1}{p})^n\frac{(1-q)^n}{q^{n+a+1}} = \frac{1}{pq^{a+1}}e^{-\frac{\lambda(q-1)(p-1)}{pq}}.$$

using complex inversion formula for two - dimensional Laplace transform to obtain,

$$L_2^{-1}\{\frac{1}{pq^{a+1}}e^{-\frac{\lambda(q-1)(p-1)}{pq}}\} = \frac{1}{\Gamma(a+1)}y^a e^{-\lambda}I_0(2\sqrt{\lambda x}).$$

Definition 16. *The two-dimensional Laplace - Carson transform of function $f(x,y)$ is defined by Ditkin and Prudnikov [6] as follows:*

$$F(p,q) = pq\int_0^\infty\int_0^\infty e^{-px-qy}f(x,y)dxdy$$

and symbolically is denoted by $F(p,q)\doteqdot f(x,y)$ where the symbol $\doteqdot$ is called "operational". The correspondence between $f(x,y)$ and $F(p,q)$ may be interpreted as transformation which transform the function $f(x,y)$ into the function $F(p,q)$.

3 Solution to non-homogeneous impulsive parabolic type P.D.E

Problem 17. *Let*

$$u_{xx} - u_y = 2\delta(y)\sin(x) - \delta(x)\sin(y) \;\; x,y>0 \tag{3.1}$$

with boundary conditions

$$u(0,y) = y^\alpha, \quad u(x,0) = \sin(x), \quad \lim_{x\longrightarrow\infty} u(x,y) = 0 \tag{3.2}$$

Solution 18. *The transformed equation takes the following form*

$$U(p,q) = \frac{pq}{(p^2+1)(q^2+1)} + pH(q) + \frac{p^2\Gamma(\alpha+1)}{q^\alpha(p^2-q)} + \frac{pq}{(p^2+1)(q^2-q)}$$

where $H(q)$ is a Laplace-Carson transform of $u_x(0,y)$. At this point, we take the inverse of each term with respect to p only, then we obtain

$$u(x,q)\doteq\frac{\sqrt{q}}{q+1}\{\sinh(\sqrt{q}x)-\sqrt{q}\sin(x)\}+\frac{\sinh(\sqrt{q}x)}{\sqrt{q}}H(q)+\frac{\Gamma(\alpha+1)}{q^\alpha}\cosh(\sqrt{q}x)+\frac{\sqrt{q}}{q^2+1}\sinh(\sqrt{q}x)$$

or

$$u(x,q)\doteq\frac{1}{2}e^{\sqrt{q}x}[\frac{\sqrt{q}}{q+1}+\frac{H(q)}{\sqrt{q}}+\frac{\Gamma(\alpha+1)}{q^\alpha}+\frac{\sqrt{q}}{q^2+1}]$$
$$-\frac{1}{2}e^{-\sqrt{q}x}[\frac{\sqrt{q}}{q+1}+\frac{H(q)}{\sqrt{q}}-\frac{\Gamma(\alpha+1)}{q^\alpha}+\frac{\sqrt{q}}{q^2+1}]-\frac{q}{q+1}\sin(x). \tag{3.3}$$

Since, the limit of $u(x,y)$ is bounded as x tends to infinity, we must have the following relationship

$$\frac{\sqrt{q}}{q+1}+\frac{H(q)}{\sqrt{q}}+\frac{\Gamma(\alpha+1)}{q^\alpha}+\frac{\sqrt{q}}{q^2+1}=0 \tag{3.4}$$

by replacing (3.4) in (3.3) we get

$$u(x,q)\doteq\frac{\Gamma(\alpha+1)}{q^\alpha}e^{-\sqrt{q}x}-\frac{q}{q+1}\sin(x).$$

Inverting the above relation with respect to and using the fact that

$$q^{-\alpha}e^{-\sqrt{q}x}\doteq\frac{2^{\alpha+\frac{1}{2}}}{\sqrt{\pi}}y^{\alpha}e^{-\frac{x^2}{4y}}D_{-(2\alpha+1)}(\frac{x}{\sqrt{2y}})$$

we get finally

$$u(x,y)=\frac{2^{\alpha+\frac{1}{2}}\Gamma(\alpha+1)}{\sqrt{\pi}}y^{\alpha}e^{-\frac{x^2}{4y}}D_{-(2\alpha+1)}(\frac{x}{\sqrt{2y}})-e^{-y}\sin(x)$$

where $D_v(x)$ is Struve's function [6].

Conclusion 19. *The two-dimensional Laplace transform provides a powerful method for analyzing linear systems. It is heavily used in solving differential and integral equations. The main purpose of this work is to develop a method of computing Laplace transform pairs of two-Dimensions from known one-Dimensional Laplace transform and making continuous effort in expanding the transform tables and in designing algorithms for generating new inverses and direct transform from known ones. It is clear that the theorems of that type described here can be further generated for other type of functions and relations. These relations can be used to calculate new Laplace transform pairs.*

Acknowledgement. The authors are grateful to the referee of his/her remarks.

References

[1] A. Aghili and B. Salkhordeh Moghaddam, *Laplace transform pairs of n-dimensions and a Wave equation*, Int. Math. Forum, **5** (2004), 377-382. MR2036717. Zbl pre 05376491.

[2] A. Aghili and B. Salkhordeh Moghaddam, *Multi-Dimensional Laplace Transform and systems of partial differential equations*, Int. Math. Forum, **1** (2006), 21-24. MR2256663 (2007h:44001). Zbl 1121.44005.

[3] A. Aghili and B. Salkhordeh Moghaddam, *Laplace transform pairs of N-dimensions and second order linear differential equations with constant coefficients*, Annales Mathematicae et informaticae **35** (2008), 3-10. MR2475861. Zbl pre05770422.

[4] R. S. Dahiya and M. Vinayagamoorty, *Laplace Transform pairs of N - dimensions and heat conduction problem*, Math. Comput. Modelling, **13** No. 10 (1990), 35-50. MR1091930(92g:44002).

[5] R. S. Dahiya and J. Saberi-Nadjafi, *Theorems on N - dimensional Laplace transforms and their applications*, 15th annual Conference of Applied Mathematics, Univ. of Central Oklahoma. Electronic Journal of Differential Equations, Conference 02, 1999, 61-74. MR1737591(2001g:44011).

[6] V. A. Ditkin and A. P. Prudnikov, *Calcul operationnelle*, Mir Publication, Moscow, 1985.

Arman Aghili
Department of Mathematics,
University of Guilan,
P.O.Box 1841, Rasht, Iran.
e-mail: armanaghili@yahoo.co

Behrouz Parsa Moghaddam
Department of Mathematics,
Islamic Azad University,
P.O.Box 1616, Lahijan, Iran.
e-mail: parsa.math@yahoo.com