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AN INTRODUCTION TO THE CHEEGER PROBLEM

Enea Parini

Abstract. Given a bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, the Cheeger problem consists of finding a subset E of Ω such that its ratio perimeter/volume is minimal among all subsets of Ω. This article is a collection of some known results about the Cheeger problem which are spread in many classical and new papers.

1 Introduction

In 1970, Jeff Cheeger established in his work [9] the following inequality:

$$\lambda_1(\Omega) \geq \left(\frac{h_1(\Omega)}{2}\right)^2,$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain, $\lambda_1(\Omega)$ is the first eigenvalue of the Laplacian under Dirichlet boundary conditions, and $h_1(\Omega)$ is defined as

$$h_1(\Omega) := \inf_{E \subset \overline{\Omega}} \frac{P(E;\mathbb{R}^n)}{V(E)}.$$

Here $P(E;\mathbb{R}^n)$ is the perimeter of E in distributional sense (see [14]) measured with respect to $\mathbb{R}^n$, while $|E|$ is the n-dimensional Lebesgue measure of E. $h_1(\Omega)$ is called *Cheeger constant* of Ω, and a set $C \subset \overline{\Omega}$ such that

$$\frac{P(C;\mathbb{R}^n)}{|C|} = h_1(\Omega)$$

is a *Cheeger set*. The task of determining the Cheeger constant of a given domain and of finding a Cheeger set has been considered by many authors. Since the related results are spread in many classical and new papers, it makes sense to collect them in this introductory survey.

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The paper is structured as follows: after introducing the functions of bounded variation in Section 1, we study existence and regularity properties of Cheeger sets (Sections 3 and 4). In Section 5 uniqueness and nonuniqueness issues are discussed, while in Section 6 we treat a quantitative isoperimetric estimate. Finally, we discuss some applications of the Cheeger problem.

2 Functions of bounded variation

Let $\Omega \subset \mathbb{R}^n$ be an open set. The *total variation* in Ω of a function $u \in L^1(\Omega)$ is defined as

$$|Du|(\Omega) := \sup \left\{ \int_\Omega u \,\mathrm{div}\varphi \,\middle|\, \varphi \in C_c^1(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}.$$

A function u such that $|Du|(\Omega) < +\infty$ is said to be of *bounded variation*. The space of the functions of bounded variation will be denoted by $BV(\Omega)$. It turns out that $BV(\Omega)$ endowed with the norm

$$\|u\|_{BV} := \|u\|_1 + |Du|(\Omega)$$

is a Banach space. A set $E \subset \mathbb{R}^n$ has *finite perimeter* in Ω if its characteristic function χ_E belongs to $BV(\Omega)$, so that

$$P(E;\Omega) := |D\chi_E|(\Omega) < +\infty.$$

If Ω has Lipschitz boundary, then a set E of finite perimeter in Ω has also finite perimeter in $\mathbb{R}^n$, and

$$P(E;\mathbb{R}^n) = P(E;\Omega) + \mathcal{H}^{n-1}(\partial\Omega \cap \partial E),$$

where $\mathcal{H}^{n-1}$ stands for the $(n-1)$-dimensional Hausdorff measure in $\mathbb{R}^n$. In particular,

$$P(\Omega;\mathbb{R}^n) = \mathcal{H}^{n-1}(\partial\Omega).$$

Similarly, if $u \in BV(\Omega)$, then $u \in BV(\mathbb{R}^n)$ (extending it to zero outside Ω), and

$$|Du|(\mathbb{R}^n) = |Du|(\Omega) + \int_{\partial\Omega} |u| \, d\mathcal{H}^{n-1}.$$

We will make use of the following results.

Proposition 2.1. *[14, Theorem 1.9] Let $\{u_k\}$ be a sequence of functions in $BV(\Omega)$ converging in $L^1_{loc}(\Omega)$ to a function u. Then*

$$|Du|(\Omega) \leq \liminf_{k\to\infty} |Du_k|(\Omega).$$

Proposition 2.2. *[14, Theorem 1.19] Let $\Omega \subset \mathbb{R}^n$ be a domain with Lipschitz boundary, and let $\{u_k\}$ be a sequence of functions in $BV(\Omega)$ such that*

$$\|u_k\|_{BV} \leq M$$

for some $M > 0$. Then there exists a subsequence $\{u_{k_j}\}$ and a function $u \in BV(\Omega)$ such that $u_{k_j} \to u$ in $L^1(\Omega)$.

Proposition 2.3. *[14, Theorem 1.23] Let $u \in BV(\Omega)$, and define*

$$E_t := \{x \in \Omega \,|\, u(x) > t\}.$$

Then,

$$|Du|(\Omega) = \int_{-\infty}^{+\infty} P(E_t; \Omega)\, dt.$$

3 Existence of a Cheeger set

In the following, $\Omega \subset \mathbb{R}^n$ will be a bounded domain with Lipschitz boundary. The perimeter of a set will be always measured with respect to $\mathbb{R}^n$, so that we will write

$$P(E) := P(E; \mathbb{R}^n).$$

We recall that the Cheeger constant is defined as

$$h_1(\Omega) := \inf_{E \subset \Omega} \frac{P(E)}{|E|},$$

with the convention that

$$\frac{P(E)}{|E|} = +\infty$$

whenever $|E| = 0$.

Proposition 3.1. *For every bounded domain $\Omega \subset \mathbb{R}^n$ with Lipschitz boundary, there exists at least one Cheeger set.*

Proof. Let us define

$$\widetilde{h}_1(\Omega) := \inf_{v \in BV(\Omega) \setminus \{0\}} \frac{|Dv|(\mathbb{R}^n)}{\|v\|_1}. \tag{3.1}$$

By definition, $\widetilde{h}_1(\Omega) \leq h_1(\Omega)$. Moreover, applying the direct method of the Calculus of Variations, the existence of a function $u \in BV(\Omega)$, $u \not\equiv 0$, such that

$$\frac{|Du|(\mathbb{R}^n)}{\|u\|_1} = \widetilde{h}_1(\Omega)$$

follows readily from Propositions 2.1 and 2.2. Since $|D|u||(\mathbb{R}^n) \leq |Du|(\mathbb{R}^n)$ (see [2, Exercise 3.12]), we can consider without loss of generality $u \geq 0$. Define

$$E_t := \{x \in \Omega \,|\, u(x) > t\}.$$

From Proposition 2.3 and Cavalieri's principle, we have

$$\begin{aligned} 0 = |Du|(\mathbb{R}^n) - \widetilde{h}_1(\Omega)\|u\|_1 &= \int_0^{+\infty} [P(E_t) - \widetilde{h}_1(\Omega)|E_t|]\, dt \\ &\geq \int_0^{+\infty} [P(E_t) - h_1(\Omega)|E_t|]\, dt \geq 0. \end{aligned}$$

It follows that for almost every $t \in \mathbb{R}$ (in the sense of the Lebesgue measure on $\mathbb{R}$),

$$P(E_t) - \widetilde{h}_1(\Omega)|E_t| = 0. \tag{3.2}$$

Since $u \not\equiv 0$, there must exist $s \in \mathbb{R}$ such that $|E_s| > 0$ and for which (3.2) holds. This yields at once

$$\widetilde{h}_1(\Omega) = h_1(\Omega)$$

as well as the existence of a Cheeger set for Ω. □

Remark 3.2. From the proof of Proposition 3.1, it follows that if u is a minimizer for $\widetilde{h}_1(\Omega)$, then almost every level set of u with positive Lebesgue measure is a Cheeger set for Ω. In fact, by [6, Theorem 2] this is actually true for *all* its level sets of positive Lebesgue measure.

Proposition 3.3. *Let $\Omega \subset \mathbb{R}^n$ have a boundary of class Lipschitz. Then*

$$h_1(\Omega) = \inf_{\substack{E \subset\subset \Omega \\ \partial E \text{ smooth}}} \frac{P(E)}{|E|}.$$

This is a straightforward consequence of the following proposition.

Proposition 3.4 ([23], Theorem 2)**.** *Let $\Omega \subset \mathbb{R}^n$ have a boundary of class Lipschitz, and let $E \subset \Omega$ be a set of finite perimeter. Then there exists a sequence of sets of finite perimeter $\{E_k\}$ such that:*

(i) $E_k \subset\subset \Omega$ *for every* k*;*

(ii) $\chi_{E_k} \to \chi_E$ *in* $L^1_{loc}(\mathbb{R}^n)$ *as* $k \to \infty$*;*

(iii) $P(E_k) \to P(E)$ *as* $k \to \infty$*.*

Proof (of Proposition 3.3). Let C be a Cheeger set for Ω. Then there exists a sequence $\{E_k\}$ of sets of finite perimeter satisfying (i), (ii) and (iii) in Proposition 3.4. By classical results, each E_k can be in its turn be approximated in a similar way by a sequence of sets compactly contained in Ω, but not necessarily in E_k, and with smooth boundary (see [14, Theorem 1.24]). Hence the claim follows. □

However, a Cheeger set can not be compactly contained in Ω, as the following proposition states.

Proposition 3.5. *Let C be a Cheeger set for Ω. Then, $\partial C \cap \partial\Omega \neq \emptyset$.*

Proof. Suppose, by contradiction, that $C \subset\subset \Omega$. Then it would be possible to find a $t > 1$ such that the set

$$tC := \{x \in \mathbb{R}^n \,|\, t^{-1}x \in C\}$$

is still contained in Ω. But then

$$\frac{P(tC)}{|tC|} = \frac{t^{n-1}P(C)}{t^n|C|} = \frac{1}{t}\frac{P(C)}{|C|} < \frac{P(C)}{|C|},$$

a contradiction to the definition of Cheeger set. Hence, the boundary of C must intersect the boundary of Ω. □

4 Regularity of Cheeger sets

Let C be a Cheeger set for Ω, and set $V_0 := |C|$. Then, C will be in particular a set which minimizes the perimeter among all the subsets of Ω with volume V_0. Hence, some classical regularity results find application.

Proposition 4.1. *Let C be a Cheeger set for Ω. Then $\partial C \cap \Omega$ is analytic, possibly except for a closed singular set whose Hausdorff dimension does not exceed $n - 8$.*

Proof. If $V_0 = |\Omega|$, then $C = \Omega$ and $\partial C \cap \Omega = \emptyset$, so that there is nothing to prove. If $V_0 < |\Omega|$, the result is stated in [15, Theorem 1] (one has to set $\Gamma = \emptyset$ in the notation used there). The idea of the proof is the following: let E be a set of finite perimeter in Ω, $x \in \partial E$, $r > 0$ such that $B_r(x) \subset \Omega$. We define

$$\psi(x, r) := |D\chi_E|(B_r(x)) - \inf\{|D\chi_F|(B_r(x)) \,|\, F\Delta E \subset\subset B_r(x)\}$$

The quantity ψ gives a measure of how far the set E is from being a perimeter-minimizing set (without volume constraints). A result of Tamanini ([27, Lemma 3]) states that, if E is a set of finite perimeter with $\psi(x, r) \leq Cr^{n-1+2\alpha}$ for some $x \in \partial E$ and all $0 < r < R$ with given constants C, R and $0 < \alpha < 1$, then the *tangent cone* to ∂E in x, as defined in [14, Theorem 9.3], is area-minimizing. This is what actually happens in this case, since it can be proved (see [16]) that for a set minimizing perimeter under a volume constraint we have

$$\psi(x, r) \leq Cr^n$$

for a constant $C > 0$, for each $x \in \partial E$ and for all sufficiently small $r > 0$. The properties of area minimizing tangent cones, which can be found in [14, Chapter

9], allow us to reason in a way similar to [22] and finally state the claim. The dimension $n-8$ appearing in the theorem is linked to the following fact: $x \in \partial E$ is a regular point if and only if the tangent cone in x is a half-space. In $\mathbb{R}^n$, $n \leq 7$, the only possible area minimizing tangent cones are half-spaces, while in $\mathbb{R}^8$ there exist nontrivial area minimizing cones such as the so-called *Simon's cone* (see [4]). □

Another important property of Cheeger sets is the constancy of the mean curvature of $\partial C \cap \Omega$; the result is stated for instance in [13, Theorem 1.22].

Proposition 4.2. *The mean curvature of $\partial C \cap \Omega$ is constant at every regular point, and equal to $\frac{1}{n-1} \cdot h_1(\Omega)$.*

Proof. The fact that the mean curvature is constant at every regular point of $\partial C \cap \Omega$ follows from [15, Theorem 2]. To show that it is exactly equal to $h_1(\Omega)$, take a regular point $x_0 \in \partial C \cap \Omega$. Then there exist a ball B, an open interval I and a function $f \in C^\infty(B; I)$ such that, if we set $F = B \times I$, then $x_0 \in B$ and $E \cap F$ is the epigraph of $-f$. Take now $g \in C_c^2(B; I)$, and set

$$E_t = (E \setminus F) \cup epi\,(-(f + tg))$$

where $t \in (-\varepsilon, \varepsilon)$, with ε so small that E_t is still contained in Ω. As E is a Cheeger set, it follows that the functional

$$I(t) = P(E_t) - h_1(\Omega)|E_t|$$

satisfies $I(0) = 0$, and $I(t) \geq 0$ for $t \in (-\varepsilon, \varepsilon)$. So we have

$$\begin{aligned} 0 \leq I(t) - I(0) = & \int_B \sqrt{1 + |D(f + tg)|^2} - h_1(\Omega) \int_B (f + tg) \\ & - \int_B \sqrt{1 + |Df|^2} + h_1(\Omega) \int_B f = J(t) - J(0) \end{aligned}$$

for every $t \in (-\varepsilon, \varepsilon)$, where

$$J(t) := \int_B \sqrt{1 + |D(f + tg)|^2} - h_1(\Omega) \int_B (f + tg)$$

It follows $J'(0) = 0$, which means, after integrating by parts,

$$-\int_B \operatorname{div} \left(\frac{Df}{\sqrt{1 + |Df|^2}} \right) g = h_1(\Omega) \int_B g$$

and since this relation is valid for every $g \in C_c^2(B; I)$, the theorem is finally proved. □

A Cheeger set enjoys also boundary regularity. More precisely, the following result holds.

Proposition 4.3. *[15, Theorem 3] Let C be a Cheeger set for Ω, and let $x \in \partial\Omega$ be such that $\partial\Omega \cap B_r(x)$ is of class C^1 for some $r > 0$. Then there exists a $\rho \in (0, r)$ such that $\partial C \cap B_\rho(x)$ is also of class C^1.*

In particular, this implies that ∂C and $\partial\Omega$ must meet tangentially at regular points of $\partial\Omega$.

5 Uniqueness and nonuniqueness

A relevant question is whether there can exist more than one Cheeger set for a given domain Ω. This is not the case if Ω is convex. A first result in this direction concerns planar convex domains. Given two sets $A, B \subset \mathbb{R}^n$, we define

$$A \oplus B := \{x \in \mathbb{R}^n \,|\, x = a + b,\ a \in A,\ b \in B\}.$$

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a convex domain. Then there exists a unique Cheeger set C for Ω. Moreover, C is convex, has boundary of class $C^{1,1}$, and*

$$C = C_R \oplus B_R,$$

where

$$C_R = \{x \in \Omega \,|\, dist(x; \partial\Omega)\} \leq R,$$

B_R is the disc of radius R, and R is such that $|C_R| = \pi R^2$.

Proof. Let H_Ω be the union of all discs with largest radius contained in Ω. If C is a Cheeger set for Ω, it follows from [12, Theorem 33] that $|C| \geq |H_\Omega|$. It is then possible to apply [26, Theorem 3.32] to state the uniqueness and the regularity result. The characterization of C as union of balls of suitable radius has been established in [19, Theorem 1]. □

The result was generalized to higher dimensional domains some years later.

Proposition 5.2. *[1, Theorem 1] Let $\Omega \subset \mathbb{R}^n$ be a convex domain. Then there exists a unique Cheeger set C for Ω. Moreover, C is convex and has boundary of class $C^{1,1}$.*

In general, if $n \geq 3$ it does not hold true that the Cheeger set of a convex domain is the union of balls of suitable radius (see [18, Remark 13]).

If Ω is not convex, one can not expect in general uniqueness of the Cheeger set, as shown by simple examples such as the "barbell domain" (see [19]). We observe that the star-shapedness of Ω is not a sufficient condition for uniqueness of the Cheeger

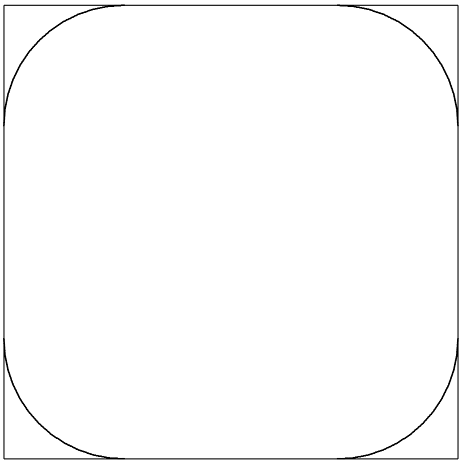

Figure 1: The Cheeger set for a square.

set; indeed, there exist L-shaped domains which admit infinitely many Cheeger sets (see [24]). However, an interesting result states that if Ω is a domain admitting more than one Cheeger set, then it is possible to find a set $\widetilde{\Omega}$ arbitrarily close to Ω and admitting only one Cheeger set. Here is the precise statement.

Proposition 5.3. *[7, Theorem 1] Let $\Omega \subset \mathbb{R}^n$ be an open set with finite volume. Then, for any compact set $K \subset \Omega$ there exists a bounded open set $\widetilde{\Omega}$ such that $K \subset \widetilde{\Omega} \subset \Omega$ and $\widetilde{\Omega}$ has a unique Cheeger set.*

Another property of the class of Cheeger sets is the fact that it is stable under countable union: if $\{C_n\}$ is a sequence of Cheeger sets for Ω, then also $C := \bigcup_n C_n$ is a Cheeger set ([6, Theorem 3]). This allows to define the notion of *maximal Cheeger set* ([5, Proposition 1.1]), which is a Cheeger set C such that, if $\widetilde{C}$ is another Cheeger set, then $\widetilde{C} \subset C$. The maximal Cheeger set is always unique. Similarly one can define the notion of *minimal Cheeger set* ([7, Lemma 2.5]); in this case, there may be more than one minimal Cheeger set, but they are always finitely many.

6 Quantitative isoperimetric estimates

A celebrated result of De Giorgi ([10]) states that, if E is a set of finite perimeter in $\mathbb{R}^n$, and E^* is a ball such that $|E^*| = |E|$, then $P(E^*) \leq P(E)$, with equality holding if and only if E is itself a ball. This implies that

$$h_1(\Omega) \geq h_1(\Omega^*).$$

In fact, if C is a Cheeger set for Ω, then Ω^* contains a ball C^* with the same volume as C. Hence,

$$h_1(\Omega) = \frac{P(C)}{|C|} \geq \frac{P(C^*)}{|C^*|} \geq h_1(\Omega^*).$$

The equality sign holds if and only if Ω is a ball. However, by means of a so-called *quantitative isoperimetric inequality*, it is possible to say that if the difference $h_1(\Omega) - h_1(\Omega^*)$ is small, then Ω must be somehow "near" to be a ball. More precisely, one defines the *Fraenkel asymmetry* of a set Ω as

$$A(\Omega) := \inf \left\{ \frac{|\Omega \,\Delta\, B|}{|\Omega|} \;\middle|\; B \text{ is a ball with } |B| = |\Omega| \right\}.$$

Observe that $A(\Omega) = 0$ if and only if Ω is a ball. Then the following result holds.

Proposition 6.1. *[11] Let $A(\Omega)$ be defined as above. Then,*

$$h_1(\Omega) \geq h_1(\Omega^*) \left[1 + \frac{A(\Omega)^2}{C} \right],$$

where $C = C(n) > 0$ depends only on the dimension n.

7 Applications of the Cheeger problem

Besides the well-known Cheeger's inequality mentioned in the introduction, the Cheeger problem appears in several mathematical contexts. One example is the study of plate failure under stress (see [20]). If Ω represents the shape of a planar plate subject to a constant uniform pressure p, we want to determine the minimal value of p for which the plate breaks down; here we do not consider bending or buckling effects. Let $E \subset \Omega$; the vertical force acting on E will be equal to $p|E|$, while the opposing force exerted on E by the portion of the plate surrounding it can be supposed to have the form $\sigma P(E)$, where $\sigma > 0$ is a constant. Hence, failure will not occur if for every subdomain $E \subset \Omega$ one has

$$p|E| \leq \sigma P(E).$$

This is equivalent to ask that

$$\frac{p}{\sigma} \leq \inf_{E \subset \Omega} \frac{P(E)}{|E|} = h_1(\Omega) \Leftrightarrow p \leq \sigma h_1(\Omega).$$

Thus, failure will occur for $p = \sigma h_1(\Omega)$ along a Cheeger set for Ω.

Another application concerns the asymptotic behaviour of the first eigenvalue of the p-Laplacian for $p \to 1$, as shown in [18]. Define for $p > 1$

$$\lambda_1(p; \Omega) := \inf_{v \in W_0^{1,p}(\Omega) \setminus \{0\}} \frac{\int_\Omega |\nabla v|^p}{\int_\Omega |v|^p}.$$

One can easily show that the infimum is actually attained, and that a minimizer is a weak solution of the equation

$$\begin{cases} -\Delta_p u &= \lambda |u|^{p-2}u \quad \text{in } \Omega, \\ u &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\lambda = \lambda_1(p;\Omega)$ and $\Delta_p u = \mathrm{div}(|\nabla u|^{p-2}\nabla u)$ is the p-Laplacian. On one hand, it is possible to generalize Cheeger's inequality to the p-Laplacian as follows (see [21, Appendix]):

$$\lambda_1(p;\Omega) \geq \left(\frac{h_1(\Omega)}{p}\right)^p.$$

On the other hand, one can show ([18, Corollary 6]) that

$$\limsup_{p\to 1} \lambda_1(p;\Omega) \leq h_1(\Omega),$$

which finally yields

$$\lim_{p\to 1} \lambda_1(p;\Omega) = h_1(\Omega).$$

Moreover, the first eigenfunctions converge in $L^1(\Omega)$ to a minimizer of (3.1), and hence to a function whose level sets are Cheeger sets for Ω. Consequently, if Ω admits only one Cheeger set C, then the first eigenfunctions converge to a suitably scaled characteristic function of C.

We also mention the interpretation given by Gilbert Strang in [25] in the context of maximal flow-minimal cut problems. Given a bounded, planar domain Ω, and given two functions $F, c : \Omega \to \mathbb{R}$, we want to find the maximal value of $\lambda \in \mathbb{R}$ such that there exists a vector field $v : \Omega \to \mathbb{R}^2$ satisfying

$$\begin{cases} \mathrm{div}\ v &= \lambda F \\ |v| &\leq c. \end{cases}$$

The problem can be interpreted as follows: given a source or sink term F, we want to find the maximal flow in Ω under the capacity constraint given by c. It turns out that if $F \equiv 1$ and $c \equiv 1$, then the maximal value of λ is equal to the Cheeger constant of Ω, while the boundary of a Cheeger set is the associated minimal cut. This kind of results have found an interesting application in medical image processing (see [3]).

The Cheeger problem can be extended by considering its weighted version. More precisely, given a function $g \in C^1(\overline{\Omega})$ with $g \geq g_0$ for a constant $g_0 > 0$, one defines the *weighted total variation* of a function $u \in L^1(\Omega)$:

$$|Du|_g(\Omega) := \sup\left\{ \int_\Omega u\, \mathrm{div}(g\varphi) \,\middle|\, \varphi \in C^1_c(\Omega;\mathbb{R}^n),\ \|\varphi\|_\infty \leq 1 \right\}.$$

Then one tries to find

$$h_1^{f,g}(\Omega) := \inf_{u\in BV_g(\Omega)} \frac{|Du|_g(\mathbb{R}^n)}{\int_\Omega fu},$$

where $f \in L^\infty(\Omega)$ with $f \geq f_0$ for a constant $f_0 > 0$, and $BV_g(\Omega)$ is the space of functions with finite weighted total variation. This problem was introduced in [17] in connection to landslide modelling. Extentions of the Cheeger problem involving anisotropic norms and anisotropic total variation turned out to be useful in image processing (see [8]).

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Enea Parini
Mathematisches Institut, Universität zu Köln
Weyertal 86-90
D-50931 Köln, Germany.
e-mail: eparini@math.uni-koeln.de