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ON CERTAIN BASKAKOV-DURRMEYER TYPE OPERATORS

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Abstract. This paper is a study of the degree of approximation by the linear combinations of the derivatives of certain Durrmeyer type integral modification of the Baskakov operators in terms of the higher order modulus of smoothness.

1 Introduction

Let $H[0,\infty)=\{f : f \text{ is locally bounded on } (0,\infty) \text{ and } |f(t)|\leqslant M(1+t)^{\beta},\ M>0,\ \beta\in N\cup\{0\}\}$. Then, for $f\in H[0,\infty)$, the Durrmeyer modification of the classical Baskakov operators,

$$\widehat{V}(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x)f\left(\frac{k}{n}\right)$$

are defined as

$$V_n(f,x)=(n-1)\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}p_{n,k}(t)f(t)\,dt,\ \ n\in N,\ \ x\in[0,\infty).$$

An equivalence between local smoothness of functions and local convergence of Baskakov-Durrmeyer operators was given by Song Li [6]. Some results in simultaneous approximation by these operators were established in [7],[8].

In [3], [4] Gupta introduced an interesting modification of the Baskakov operators by combining the weight functions of beta operators and those of Baskakov operators so as to approximate Lebesgue integrable functions on $[0,\infty)$ and established asymptotic formulae and an estimation of error in simultaneous approximation. These operators are defined as follows:

$$B_n(f,x)=\sum_{k=0}^{\infty}p_{n,k}(x)\int_0^{\infty}b_{n,k}(t)f(t)\,dt,$$

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where $p_{n,k}(x) = \binom{n+k-1}{k} \frac{x^k}{(1+x)^{n+k}}$, $\quad b_{n,k}(t) = \frac{t^k}{B(k+1,n)(1+t)^{n+k+1}}$ and $B(k+1,n) = k!(n-1)!/(n+k)!$ is the beta integral.

The integral modification B_n of the Baskakov operators gives better results than the operators V_n and some approximation properties for the operators B_n become simpler in comparison to the operators V_n, (cf. [2],[3] and the references therein).

Throughout our work, let N denote the set of natural numbers, N^0 the set of non-negative integers, and $\|.\|_{C[a,b]}$, the sup-norm on $C[a,b]$, the space of continuous functions on $[a,b]$.

Let f be a real valued function over $\mathbb{R}$. The m th ($m \in N$) forward difference of the function f at the point x of step length δ is defined as

$$\Delta_\delta^m f(x) \overset{def}{\equiv} \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} f(x+j\delta).$$

As a convention we write $\Delta_\delta^0 f(x)$ as $f(x)$.

The mth order modulus of continuity $\omega_m(f,\delta,I)$ for a function continuous on an interval I is defined by

$$\omega_m(f,\delta) \overset{def}{\equiv} \sup_{0<|h|\leqslant\delta} \left\{ |\Delta_h^m f(x)| : x, x+mh \in I \right\},$$

where $\Delta_h^m f(x)$ is the mth forward difference with step length h.

It turns out the order of approximation by these operators is, at best $O(n^{-1})$, however smooth the function may be. In order to speed up the rate of convergence by the operators B_n, we consider linear combinations $B_n(f,k,x)$ of operators B_n defined as :

$$B_n(f,k,x) = \sum_{j=0}^{k} C(j,k) B_{d_j n}(f,x),$$

where

$$C(j,k) = \prod_{i=0, i\neq j}^{k} \frac{d_j}{d_j - d_i}, k \neq 0 \text{ and } C(0,0) = 1,$$

and $d_0, d_1, ...d_k$ are $(k+1)$ arbitrary but fixed distinct positive integers.

Section 2 of this paper contains some lemmas and their corollaries which we shall use in our main results. In section 3 we establish our main theorem. Further, the constant C is not the same at each occurrence.

2 Preliminaries

Lemma 1. *[3] If the function $\mu_{n,m}(x)$, $m \in N^0$ are defined as*

$$\mu_{n,m}(x) = \sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} b_{n,k}(t)(t-x)^m \, dt.$$

Then

$$\mu_{n,0}(x) = 1 \;\; \mu_{n,1}(x) = \frac{1+x}{n-1}$$

and

$$\mu_{n,2}(x) = \frac{2x^2(n+1) + 2x(n+2) + 2}{(n-1)(n-2)}.$$

Consequently, for each $x \in [0,\infty)$, $\mu_{n,m}(x) = O\big(n^{-[m+1]/2}\big)$. If $K > 2$, then for sufficiently large n, we have

$$\mu_{n,2}(x) \leqslant \frac{K\varphi^2(x)}{n}, \; \varphi(x) = \sqrt{x(1+x)}. \tag{2.1}$$

Lemma 2. *For the function $p_{n,k}(x)$, there holds the result*

$$x^r \frac{d^r p_{n,k}(x)}{dx^r} = \sum_{\substack{2i+j \leqslant r \\ i,j \geqslant 0}} n^i (k-nx)^j q_{i,j,r}(x) p_{n,k}(x),$$

where $q_{i,j,r}(x)$ are certain polynomials in x independent of n and k.

Proof. The proof is easy to prove, hence details are omitted. □

Lemma 3. *For the functions $A_{m,n}(x)$ given by*

$$A_{m,n}(x) \equiv \sum_{\nu=0}^{\infty} \left(\frac{\nu}{n} - x\right)^m p_{n,\nu}(x),$$

we have $A_{0,n}(x) = 1$, $A_{1,n}(x) = 0$ and there holds the recurrence relation

$$n\, A_{m+1,n}(x) = \varphi^2(x) \left[A'_{m,n}(x) + n\, m\, A_{m-1,n}(x)\right], \tag{2.2}$$

where $m \geqslant 1, x \in [0,\infty)$ and $\varphi^2(x) = x(1+x)$.

Consequently, for all $x \in [0,\infty)$, $A'_{m,n}(x) = O\big(n^{-[(m+1)/2]}\big)$.

Proof. The proof follows by straightforward calculations, hence omitted. □

Lemma 4. *Let δ and γ be any two positive real numbers. Then, for any $s>0$ and $0<a_1<b_1<\infty$, we have,*

$$\sup_{x\in[a_1,b_1]}\left| n\sum_{k=0}^{\infty} p_{n,k}(x) \int\limits_{|t-x|\geqslant\delta} b_{n,k}(t)\, t^{\gamma}\, dt\right| = O\big(n^{-s}\big).$$

Making use of Taylor's expansion, Schwarz inequality for integration and then for summation and Lemma 1, the proof of this lemma easily follows, hence the details are omitted.

Lemma 5. *[2] If f is r times differentiable on $[0,\infty)$, $f^{(r)}$ is locally integrable in the Lebesgue sense on $[0,\infty)$ and $f^{(r)}=O(t^{\alpha})$ as $t\to\infty$ for some $\alpha>0$. Then, for $r=0,1,2,...$ and $n>\alpha+r$ we have*

$$B_n^{(r)}(f;x)=\prod_{s=0}^{r-1}\frac{n+s}{n-s-1}\sum_{k=0}^{\infty}p_{n+r,k}(x)\int\limits_0^{\infty} b_{n-r,k+r}(t) f^{(r)}(t)\, dt.$$

Let $f\in C[a,b]$ and $[a_1,b_1]\subset[a,b]$. Then, for sufficiently small $\delta>0$, the Steklov mean $f_{\delta,m}$ of m-th order corresponding to f is defined as follows:

$$f_{\delta,m}(t)=\delta^{-m}\int\limits_{-\delta/2}^{\delta/2}\dots\int\limits_{-\delta/2}^{\delta/2}\Big(f(t)+(-1)^{m-1}\Delta^m_{\sum_{i=1}^m t_i} f(t)\Big)\prod_{i=1}^m dt_i, t\in[a_1,b_1].$$

Lemma 6. *For the function $f_{\delta,m}$, we have*

1. *[(a)]*
2. $\|f^{(r)}_{\delta,m}\|_{C[a_1,b_1]}\leqslant C\delta^{-r}\omega_m(f,\delta,[a,b]), r=1,2,...,m;$
3. $\|f-f_{\delta,m}\|_{C[a_1,b_1]}\leqslant C\omega_m(f,\delta,[a,b]);$
4. $\|f_{\delta,m}\|_{C[a_1,b_1]}\leqslant C\|f\|_{C[a,b]};$

where C is a certain constant that depends on m but is independent of f and δ.

Following [5, Theorem 18.17] or [9, pp.163-165] the proof of the above lemma easily follows hence the details are omitted.

Lemma 7. *[1] If for $r\in N^0$, we define*

$$T_{r,n,m}(x)=\sum_{k=0}^{\infty}p_{n+r,k}(x)\int\limits_0^{\infty} b_{n-r,k+r}(t)(t-x)^m\, dt,$$

then,

$$T_{r,n,0}(x)=1, T_{r,n,1}(x)=\frac{1+r+x(1+2r)}{(n-r-1)},$$

$$T_{r,n,2}(x)=\frac{2(2r^2+4r+n+1)x^2+2(2r^2+5r+2+n)x+(r^2+3r+2)}{(n-r-1)(n-r-2)}$$

and there holds the recurrence relation:

$$(n-r-m-1)T_{r,n,m+1}(x)=x(1+x)\{T'_{r,n,m}(x)+2mT_{r,n,m-1}(x)\} + \{(m+r+1)(1+2x)-x\}T_{r,n,m}(x),\ n>r+m+1.$$

Consequently, $T_{r,n,m}(x)=O(n^{-[(m+1)/2]})$, *where* $[\beta]$ *is the integer part of* β.

Remark 8. *From above lemma by induction on* m *it follows that*

$$T_{r,n,2m}(x)=\sum_{i=0}^{m} q_{i,m,n}(x)\left[\frac{\phi^2(x)}{n}\right]^{m-i} n^{-2i},$$

$$T_{r,n,2m+1}(x)=(1+2x)\sum_{i=0}^{m} s_{i,m,n}(x)\left[\frac{\phi^2(x)}{n}\right]^{m-i} n^{-2i-1},$$

where $q_{i,m,n}(x)$ *and* $s_{i,m,n}(x)$ *are polynomials in* x *of fixed degree with coefficients that are bounded uniformly for all* n.

Lemma 9. *Let* $f\in H[0,\infty)$ *and be bounded on every finite subinterval of* $[0,\infty)$ *admitting a derivative of order* $2k+r+2$ *at a fixed point* $x\in(0,\infty)$. *Let* $f(t)=O(t^\alpha)$ *as* $t\to\infty$ *for some* $\alpha>0$, *then, we have*

$$\lim_{n\to\infty} n^{k+1}\left[B_n^{(r)}(f,k,x)-f^{(r)}(x)\right]=\sum_{i=r}^{2k+r+2} f^{(i)}(x)Q(i,k,r,x), \tag{2.3}$$

and

$$\lim_{n\to\infty} n^{k+1}\left[B_n^{(r)}(f,k+1,x)-f^{(r)}(x)\right]=0, \tag{2.4}$$

where $Q(i,k,r,x)$ *are certain polynomials in* x.

Further, the limits in (2.3) and (2.4) hold uniformly in $[a,b]$, *if* $f^{(2k+r+2)}$ *exists and is continuous on* $(a-\eta,b+\eta)\subset(0,\infty),\eta>0$.

Proof. By the Taylor's expansion, we have

$$f(t)=\sum_{i=0}^{2k+r+2}\frac{f^{(i)}(x)}{i!}(t-x)^i+\epsilon(t,x)(t-x)^{2k+r+2},$$

where $\epsilon(t,x) \to 0$ as $t \to x$. Applying the linear combinations $B_n(f,k,.)$ on both side of above expansion and using Lemma 6, for sufficiently, large n we have

$$
\begin{aligned}
& n^{k+1}\left[B_n^{(r)}(f,k,x) - f^{(r)}(x)\right] \\
&= n^{k+1}\left[\sum_{j=0}^{k} C(j,k) B_{d_j n}^{(r)}(f,x) - f^{(r)}(x)\right] \\
&= n^{k+1}\left[\sum_{i=0}^{2k+r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{k} C(j,k) \prod_{s=0}^{r-1} \frac{nd_j + s}{nd_j - s - 1} \sum_{l=0}^{\infty} p_{d_j n + r, l}(x)\right. \\
&\times \left.\int_0^{\infty} b_{nd_j - r, l+r}(t) \frac{\partial^r}{\partial t^r}(t-x)^i \, dt - f^{(r)}(x)\right] \\
&+ n^{k+1}\left[\sum_{j=0}^{k} C(j,k) B_{d_j n}^{(r)}\left(\epsilon(t,x)(t-x)^{2k+r+2}, x\right)\right] \\
&=: E_1 + E_2, \text{ say.}
\end{aligned}
$$

Now,

$$
\begin{aligned}
E_1 &= n^{k+1}\left[\left\{\sum_{i=r+1}^{2k+r+2} \frac{f^{(i)}(x)}{i!} \sum_{j=0}^{k} C(j,k) \prod_{s=0}^{r-1} \frac{(nd_j + s)}{nd_j - s - 1} \sum_{l=0}^{\infty} p_{d_j n + r, l}(x)\right.\right. \\
&\times \left.\int_0^{\infty} b_{nd_j - r, l+r}(t) \frac{\partial^r}{\partial t^r}(t-x)^i \, dt\right\} \\
&+ \left.\left\{f^{(r)}(x) \sum_{j=0}^{k} C(j,k) \prod_{s=0}^{r-1} \frac{(nd_j + s)}{nd_j - s - 1} - f^{(r)}(x)\right\}\right] \\
&=: E_{1,1} + E_{1,2} \text{ say.}
\end{aligned}
$$

In view of the identity

$$
C(j,k) d_j^{-m} = \begin{cases} 1; \; m = 0 \\ 0; \; m = 1, 2, \ldots k, \end{cases} \tag{2.5}
$$

we get

$$
\begin{aligned}
E_{1,2} &= n^{k+1} f^{(r)}(x) \sum_{j=0}^{k} C(j,k) \left\{\prod_{s=0}^{r-1} \frac{(1 + s/nd_j)}{1 - (s+1)/nd_j} - 1\right\} \\
&= 0 \text{ as } n \to \infty.
\end{aligned}
$$

Next, by Lemma 7 and identity (2.5), we get

$$\begin{aligned} E_{1,1} &= n^{k+1}\sum_{i=r+1}^{2k+r+2}\frac{f^{(i)}(x)}{i!}\frac{i!}{(i-r)!}\sum_{j=0}^{k}C(j,k)\left\{\prod_{s=0}^{r-1}\frac{(1+s/nd_j)}{1-(s+1)/nd_j}\right\}T_{r,d_jn,i-r}(x)\\ &= \sum_{i=r+1}^{2k+r+2}f^{(i)}(x)Q(i,k,r,x)+o(1),\ n\to\infty. \end{aligned}$$

Thus,

$$\begin{aligned} E_1 &= n^{k+1}\left(f^{(r)}(x)+\sum_{i=r+1}^{2k+r+2}f^{(i)}(x)Q(i,k,r,x)\right)\\ &= n^{k+1}\sum_{i=r}^{2k+r+2}f^{(i)}(x)Q(i,k,r,x), \end{aligned}$$

as $n\to\infty$. Now, we proceed to show that $E_2\to 0$ as $n\to\infty$. For this, it is sufficient to prove that $I=n^{k+1}B_n^{(r)}\left(\epsilon(t,x)(t-x)^{2k+r+2},x\right)\to 0$ as $n\to\infty$.

Using Lemma 2, we get

$$\begin{aligned} |I| &\leqslant n^{k+1}\sum_{\nu=0}^{\infty}p_{n,\nu}^{(r)}(x)\int_0^{\infty}b_{n,\nu}(t)\left|\epsilon(t,x)(t-x)^{2k+r+2}\right|dt\\ &\leqslant n^{k+1}M(x)\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\sum_{\nu=0}^{\infty}p_{n,\nu}(x)\left|\nu-nx\right|^j\times\\ &\times\int_0^{\infty}b_{n,\nu}(t)\left|\epsilon(t,x)(t-x)^{2k+r+2}\right|dt, \end{aligned}$$

where $M(x)=\sup_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}\left|q_{i,j,r}(x)\right|/x^r$. Applying the Schwarz inequality, we get

$$\begin{aligned} |I| \leqslant\ & n^{k+1}M(x)\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\left\{\sum_{\nu=0}^{\infty}p_{n,\nu}(x)(\nu-nx)^{2j}\right\}^{1/2}\\ &\times\left\{\sum_{\nu=0}^{\infty}p_{n,\nu}(x)\left(\int_0^{\infty}b_{n,\nu}(t)\left|\epsilon(t,x)(t-x)^{2k+r+2}\right|dt\right)^2\right\}^{1/2}. \end{aligned}$$

Since, $\epsilon(t,x)\to 0$ as $t\to x$, for a given $\varepsilon>0$ there exists a $\delta>0$ such that $\left|\epsilon(t,x)\right|<\varepsilon$, whenever $0<|t-x|<\delta$, and for $|t-x|\geqslant\delta$ there exists a constant C such that $\left|\epsilon(t,x)\right|<C|t-x|^{\beta}$, where β is an integer $\geqslant 2k+r+2$.

As, $\int\limits_0^\infty b_{n,\nu}(t)\,dt = 1$, we get

$$\begin{aligned} K &= \left(\int\limits_0^\infty b_{n,\nu}(t)\left|\epsilon(t,x)(t-x)^{2k+r+2}\right| dt\right)^2 \\ &\leqslant \left(\int\limits_0^\infty b_{n,\nu}(t)\,dt\right)\left(\int\limits_0^\infty b_{n,\nu}(t)\left(\epsilon(t,x)(t-x)^{2k+r+2}\right)^2 dt\right) \\ &\leqslant \int\limits_{0<|t-x|<\delta} b_{n,\nu}(t)\,\varepsilon^2(t-x)^{4k+2r+4}\,dt + \int\limits_{|t-x|\geqslant\delta} b_{n,\nu}(t)\,C^2(t-x)^{4k+2r+2\beta+4}\,dt. \end{aligned}$$

Next, by Lemma 3 and Lemma 7, we have

$$\begin{aligned} |I| &\leqslant n^{k+1}M(x)\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^{i+j}O\big(n^{-j/2}\big)O\big(n^{-(2k+r+2)/2}\big)\left\{\varepsilon^2+O\big(n^{-\beta/2}\big)\right\}^{1/2} \\ &\leqslant O(1)\left\{\varepsilon^2+O\big(n^{-\beta/2}\big)\right\}^{1/2} \\ &\leqslant \varepsilon O(1). \end{aligned}$$

Since, ε is arbitrary, it follows that $I \to 0$ as $n \to \infty$. The assertion (2.4) follows along similar lines by using identity 2.5 for $k+1$ in place of k.

The last assertion follows, due to the uniform continuity of $f^{(2k+r+2)}$ on $[a,b] \subset R_+$ (enabling δ to become independent of $x \in [a,b]$) and the uniformity of $o(1)$ terms in the estimate of $E_{1,1}$ and $E_{1,2}$ (in fact, it is a polynomial in x). □

3 Simultaneous Approximation

Theorem 10. *Let $f \in H[0,\infty)$ for some $C > 0$ and $r \in N$. If $f^{(r)}$ exists and is continuous on $(a-\eta, b+\eta), \eta > 0$ then for sufficiently large n,*

$$\left\|B_n^{(r)}(f,k,) - f^{(r)}\right\|_{C[a,b]} \leqslant C\,n^{-(k+1)}\left\{\|f\|_{C[a,b]} + \omega_{2k+2}\left(f^{(r)}; n^{-1/2}; (a-\eta, b+\eta)\right)\right\},$$

where C is independent of f and n.

Proof. Using $f^{(r)}_{\delta,2k+2} = \big(f^{(r)}\big)_{\delta,2k+2}$ and linearity of the operators $B_n^{(r)}(f,k,)$, we can write

$$\begin{aligned} I &= \left\| B_n^{(r)}\left(f,k,\right) - f^{(r)} \right\|_{C[a,b]} \\ &\leqslant \left\| B_n^{(r)}\left(f - f_{\delta,2k+2},k,\right) \right\|_{C[a,b]} + \left\| B_n^{(r)}\left(f_{\delta,2k+2},k,\right) - f_{\delta,2k+2}^{(r)} \right\|_{C[a,b]} \\ &+ \left\| f^{(r)} - f_{\delta,2k+2}^{(r)} \right\|_{C[a,b]} \\ &:= E_1 + E_2 + E_3, \text{ say.} \end{aligned}$$

Hence by property (b) of the Steklov mean, we get

$$E_3 \leqslant C\,\omega_{2k+2}\left(f^{(r)},\delta,(a-\eta,b+\eta)\right).$$

Next, applying Lemma 9 and Lemma 6, for each $m = r, r+1, ..., 2k+2+r$, it follows that

$$\begin{aligned} E_2 &\leqslant C\,n^{-(k+1)} \sum_{m=r}^{2k+2+r} \|f_{\delta,2k+2}^{(m)}\|_{C[a,b]} \\ &\leqslant C\,n^{-(k+1)} \left(\|f_{\delta,2k+2}\|_{C[a,b]} + \|f_{\delta,2k+2}^{(2k+2+r)}\|_{C[a,b]} \right) \\ &\leqslant C\,n^{-(k+1)} \left(\|f_{\delta,2k+2}\|_{C[a,b]} + \|(f^{(r)})_{\delta,2k+2}^{2k+2}\|_{C[a,b]} \right). \end{aligned}$$

Hence, by property (a) and (c) of Steklov mean we have

$$E_2 \leqslant C\,n^{-(k+1)} \left\{ \|f\|_{C[a,b]} + \omega_{2k+2}\left(f^{(r)},\delta,(a-\eta,b+\eta)\right) \right\}.$$

Let $f - f_{\delta,2k} = \phi$.

From the smoothness of the function $f - f_{\delta,2k}, k = 1,2,...2k+2$, we can write

$$\begin{aligned} \phi(t) &= \sum_{m=0}^{r} \frac{\phi^{(m)}(x)}{m!}(t-x)^m + \frac{\phi^{(r)}(\xi) - \phi^{(r)}(x)}{r!}(t-x)^r \psi(t) \\ &+ \theta(t,x)\left(1-\psi(t)\right), \end{aligned}$$

where ξ lies between t and x, and ψ is the characteristic function of the interval $(a-\eta, b+\eta)$. Moreover, by direct calculations it can be proved that $\theta(t,x)$ tends to 0 as t tends to x. For $t \in (a-\eta, b+\eta)$ and $x \in [a,b]$, we write

$$\phi(t) = \sum_{m=0}^{r} \frac{\phi^{(m)}(x)}{m!}(t-x)^m + \frac{\phi^{(r)}(\xi) - \phi^{(r)}(x)}{r!}(t-x)^r,$$

and in the case $t \in (0,\infty) \setminus (a-\eta, b+\eta)$, $x \in [a,b]$ we define

$$\theta(t,x) = \phi(t) - \sum_{m=0}^{r} \frac{\phi^{(m)}(x)}{m!}(t-x)^m.$$

Again, from the linearity of the operators $B_n^{(r)}(f,k,)$, we get

$$
\begin{aligned}
B_n^r(\phi(t),x) &= \sum_{m=0}^{r}\frac{\phi^{(m)}(x)}{m!}B_n^r((t-x)^m,x)\\
&+ \frac{B_n^r\Big(\big(\phi^{(r)}(\xi)-\phi^{(r)}(x)\big)(t-x)^r\psi(t),x\Big)}{r!}+\\
&+ B_n^r\Big(\theta(t,x)\left(1-\psi(t)\right),x\Big)\\
&:= J_1+J_2+J_3, \quad \text{say.}
\end{aligned}
$$

By using Lemma 1, we get

$$
J_1 = \sum_{m=0}^{r}\frac{\phi^{(m)}(x)}{m!}B_n^r((t-x)^m,x) = C\sum_{m=0}^{r}\frac{\phi^{(m)}(x)}{m!}n^{-[(m+1)/2]}
$$

Again, using the Lemma 6 for the functions $f^{(r)}-f^{(r)}_{\delta,2k+2}$, $r=1,2,...2k+2$. Hence, for sufficiently large n, we have

$$
\begin{aligned}
|J_1| &\leqslant C\left\|f-f_{\delta,2k+2}\right\|_{C[a,b]}+o\left(n^{-1}\right)\\
&\leqslant C\omega_{2k+2}\Big(f,\delta,(a-\eta,b+\eta)\Big)+o\left(n^{-1}\right).
\end{aligned}
$$

Next, using Lemma 2 and Schwarz inequality for integration and then for summation we get

$$
\begin{aligned}
J_2 &\leqslant \frac{2}{r!}\|f^{(r)}-f^{(r)}_{\delta,2k+2}\|_{C[a,b]}B_n^r(\psi(t)|t-x|^r,x)\\
&\leqslant \frac{2}{r!}\|f^{(r)}-f^{(r)}_{\delta,2k+2}\|_{C[a,b]}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\frac{|q_{i,j,r}(x)|}{x^r}\sum_{\nu=1}^{\infty}p_{n,\nu}(x)|\nu-nx|^j\times\\
&\qquad \times\int_0^{\infty}b_{n,\nu}(t)\psi(t)|t-x|^r\,dt\\
&\leqslant \frac{2}{r!}\|f^{(r)}-f^{(r)}_{\delta,2k+2}\|_{C[a,b]}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}}n^i\frac{|q_{i,j,r}(x)|}{x^r}\sum_{\nu=1}^{\infty}p_{n,\nu}(x)|\nu-nx|^j\times\\
&\qquad \times\left(\int_0^{\infty}b_{n,\nu}(t)\,dt\right)^{1/2}\left(\int_0^{\infty}b_{n,\nu}(t)(t-x)^{2r}\,dt\right)^{1/2}
\end{aligned}
$$

$$
\begin{aligned}
&\leqslant C\,\|f^{(r)}-f^{(r)}_{\delta,2k+2}\|_{C[a,b]}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\left(\sum_{\nu=1}^{\infty}p_{n,\nu}(x)(\nu-nx)^{2j}\right)^{1/2}\times\\
&\qquad\times\left(\sum_{\nu=1}^{\infty}p_{n,\nu}(x)\int_0^{\infty}b_{n,\nu}(t)(t-x)^{2r}\right)^{1/2}\\
&\leqslant C\,\|f^{(r)}-f^{(r)}_{\delta,2k+2}\|_{C[a,b]}\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^iO\big(n^{j/2}\big)O\big(n^{-r/2}\big)\\
&\leqslant C\,\|f^{(r)}-f^{(r)}_{\delta,2k+2}\|_{C[a,b]}.
\end{aligned}
$$

Since $t\in(0,\infty)\setminus(a-\eta,b+\eta)$, we can choose a $\delta>0$ in such a way that $|t-x|\geqslant\delta$ for all $x\in I$. Thus, by Lemma 2, we obtain

$$
|J_3|\leqslant\sum_{\substack{2i+j\leqslant r\\ i,j\geqslant 0}} n^i\frac{|q_{i,j,r}(x)|}{x^r}\sum_{\nu=1}^{\infty}p_{n,\nu}(x)|\nu-nx|^j\int_{|t-x|\geqslant\delta}b_{n,\nu}(t)|\theta(t,x)|\,dt.
$$

For $|t-x|\geqslant\delta$, we can find a constant $C>0$ such that $|\theta(t,x)|\leqslant C\,t^{\gamma}$. Finally using Schwarz inequality for integration and then for integration, Lemma 3, and Lemma 4, it easily follows that $J_3=O\big(n^{-s}\big)$ for any $s>0$.

Combining the estimates J_1-J_3, we obtain

$$
\begin{aligned}
E_1&\leqslant C\,\|f^{(r)}-f^{(r)}_{\delta,2k+2}\|_{C[a,b]}\\
&\leqslant C\,\omega_{2k+2}\left(f^{(r)},\delta,(a-\eta,b+\eta)\right)\ \text{(in view of (b) of Steklov mean).}
\end{aligned}
$$

Finally, taking $\delta=n^{-1/2}$ the theorem is concluded. □

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