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FIXED POINTS OF MAPPINGS WITH DIMINISHING PROBABILISTIC ORBITAL DIAMETERS

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Abstract. In this paper we prove a fixed point theorem for a pair of mappings with probabilistic diminishing orbital diameters on Menger spaces and introduce the notion of generalized joint diminishing probabilistic orbital diameters (gjdpod) for a quadruplet of mappings.

1 Introduction

The notion of 'diminishing orbital diameters' (dod) was introduced by Belluce and Kirk [1]. Subsequently, Fisher [3], Huang, Huang and Jeng [4], Liu [7], Ranganathan, Srivastva and Gupta [9], Singh [10], Wong [12] etc. obtained some more results in this settings.
Istrăţescu and Săcuiu [5] introduced the concept of non-expansive mappings and mapping with 'diminishing probabilistic orbital diameters' (dpod) on probabilistic metric spaces (PM-spaces). Singh and Pant [11] have shown that a non-expansive mapping on PM-space having dpod has a fixed point. They have also investigated that the condition of non-expansiveness of the mapping may be relaxed to the condition of the mapping being with relatively compact orbits.
In this paper we introduce the notion of dpod and gdpod for a pair of mappings and established a fixed point theorem. Subsequently, we introduced the concept of gjdpod for a quadruplet of mappings and prove a fixed point theorem. Some of the previously results of [7], [9], [10], [11] (in different settings) may be derived from our results.

1.1 Preliminaries

Definition 1. *[2].Let A be a non-empty subset of X. The function $D_A(.)$ defined by*

$$D_A(x) = \sup_{\varepsilon<x}\{\inf_{u,v\in A} F_{u,v}(\varepsilon)\}$$

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is called the probabilistic diameter of A.

Definition 2. *[2].The function $E_{A,B}(.)$ defined by*

$$E_{A,B}(\varepsilon) = \underset{x<\varepsilon}{lub}\, t\{\underset{u\in A}{glb}(\underset{v\in B}{lub}\, F_{u,v}(x)), (\underset{v\in B}{glb}\, \underset{u\in A}{lub}\, F_{u,v}(x))\}$$

is called the probabilistic distance between A and B.

Let $P : X \to X$ and $u \in X$, then $O_P\ (u) = (u, Pu, P^2u, \cdots ,)$ is called the orbit of u with respect to P and $\overline{O_P(u)}$ denotes the closure of $O_P(u)$.

Definition 3. *[5] Let P be a self map on a PM-space $X.P$ is said to have dpod at u if for $D_{O_P(P(u))}(\varepsilon) > 0$*

$$\underset{n\to\infty}{Lim}\, D_{O_P(P(u))}(\varepsilon) > D_{O_P(u)}(\varepsilon)$$

where H is a distribution function.

We now introduce the following definitions :

Definition 4. *A pair of mappings P, Q of a PM-space X is said to have diminishing probabilistic orbital diameters (dpod) if*

$$\underset{n\to\infty}{Lim}\, E_{O_P(P_n(u)),O_Q(Q_n(u))}(\varepsilon) > E_{O_P(u),O_Q(u)}(\varepsilon), \varepsilon > 0,$$

for all $u \in X$ with $E_{O_P(P_n(u)),O_Q(Q_n(u))}(\varepsilon) \neq H$.

Definition 5. *A pair of mappings P, Q on a PM-space X is said to have generalized diminishing probabilistic orbital diameters (gdpod) if*

$$\underset{n\to\infty}{Lim}\, E_{O_P(P_n(u)),O_Q(Q_n(v))}(\varepsilon) > E_{O_P(u),O_Q(v)}(\varepsilon), \varepsilon > 0,$$

for all $u \in X$ with $E_{O_P(P_n(u)),O_Q(Q_n(v))}(\varepsilon) \neq H$.

It is clear that (P, Q) has a dpod if (P, Q) has a gdpod. Also (P, P) has a dpod if and only if P has dpod.

2 Main Result

Theorem 6. *Let P and Q are continuous self mappings of a compact Menger space. If the pair (P,Q) has a gdpod on X, then for each $u,v \in X$, there exists some subsequences $\{P^{n_k}(u)\}$ of $P_n(u)$ and $\{Q^{n_k}(u)\}$ of $Q_n(u)$ converge to a common fixed point of P and Q.*

Proof. Let $u \in X, L(u)$ denotes the set of all points of X which are the limits of the subsequences $P_n(u)$ Since $L(u) \neq \phi$ because X is compact, $L(u)$ is mapped into itself by P. Also $L(u)$ is closed, so by Zorn's lemma there exists a minimal P-invariant non-empty subset $A \subset L(u)$ such that A is closed and mapped into itself by P.
Similarly we can find a minimal Q-invariant non-empty subset $B \subset L(v)$ such that B is closed and mapped into itself by Q. For $u_0 \in A$, $\overline{O_P(u_0)}$ is mapped into itself by P. Therefore minimality of A implies that $A = \overline{O_P(u_0)}$. Similarly for $v_0 \in B$, we have $B = \overline{O_Q(v_0)}$.
We now prove that $E_{A,B}(\varepsilon) = H$, $\varepsilon > 0$. Suppose $E_{A,B}(\varepsilon) \neq H$, $\varepsilon > 0$. Since P,Q has a dpod, we have

$$E_{A,B}(\varepsilon) = E_{O_P(u_0),O_Q(v_0)}(\varepsilon) < \underset{n\to\infty}{Lim}\, E_{O_P(P_n(u_0)),O_Q(Q_n(v_0))}(\varepsilon)$$

This implies

$$E_{\overline{O_P(u_0),O_Q(v_0)}}(\varepsilon) < \underset{n\to\infty}{Lim}\, E_{\overline{O_P(P_n(u_0)),O_Q(Q_n(v_0))}}(\varepsilon) = E_{A,B}(\varepsilon)$$

contradiction. Hence $E_{A,B}(\varepsilon) = H$, which implies that $A = B = (w)$ (say). Then it is clear that w is a common fixed point of P and Q. If z is another fixed point of P and $z \neq w$ Then we have

$$\underset{n\to\infty}{Lim}\, E_{O_P(P_n(z)),O_Q(Q_n(w))}(\varepsilon) > E_{O_P(z),O_Q(w)}(\varepsilon), \varepsilon > 0,$$

or $E_{z,w}(\varepsilon) > E_{z,w}(\varepsilon)$,
a contradiction. Hence w is a unique fixed point of P. Similarly, we may show that w is a unique fixed point of Q.
This completes the proof of the theorem □

Remark 7. *If in the above theorem condition gdpod is replaced by the condition dpod, then it no longer assures the existence of a common fixed point for P and Q. (see [7])*

Corollary 8. *Let P be a continuous self mapping of a compact Menger space X. If (P,P) has a gdpod, then P has a unique fixed point. Furthermore, for each $u \in X$, there exists some subsequences of $P^n(u)$ converge to a unique fixed point of P.*

Pant, Dimri and Chandola [8] have introduced the concept of joint sequence of iterates for a quadruplet of mappings as follows:

Definition 9. *[8] Let $B = (P, Q, S, T)$ be a quadruplet of self mappings on a PM-space X. For u_0 in X, let $Tu_n = QPu_{n-1}$, if n is odd and $Tu_n = SQu_{n-1}$ if n is even, then the sequence*

$$J_B(u_0) = \{u_0, Pu_0, QPu_0, SQPu_0, TSQPu_0, \cdots\}$$

is called the joint sequence of iterates of B at u_0

We now introduce the notion of gjdpod for a quadruplet of mappings in PM-space
Let $\delta_u(\varepsilon) = \underset{n\to\infty}{Lim}\, D_{J_{B_n}(u)}(\varepsilon)$.

Definition 10. *B will be called to have gjdpod at u if for $D_{J_{B_n}(u)}(\varepsilon) \neq H, \varepsilon > 0$,*

$$\delta_u(\varepsilon) > D_{J_{B(u)}}(\varepsilon).$$

Theorem 11. *Let X be a compact Menger space and $B = (P, Q, S, T)$ be a quadruplet of continuous self mappings on X such that B have gjdpod on X. Then for each $u_0 \in X$, a subsequence of $J_B(u_0)$ converges to a common fixed point of P, Q, S and T.*

Proof. For $u_0 \in X$, let $A(u_0)$ denote the set of all points of X which are limit of subsequences of the sequence $J_B(u_0)$. Since X is compact, $A(u_0) \neq \phi$ Also $A(u_0)$ is closed and mapped into itself by P, Q, S and T. Let some subsequence of $J_B(u_0)$ converge to a point u in X, so $u \in A(u_0)$. Further P, Q, S and T are continuous, therefore $J_B(u_0) \subset A(u_0)$. By Zorn's Lemma, there exists a minimal nonempty subset $K \subset A(u_0)$ such that K is closed and mapped into itself by P, Q, S and T. Also for $q_0 \in K$, $J_B(q_0)$ is mapped into itself by P, Q, S and T. Therefore minimality of K implies that $K = \overline{J_B(q_0)}$. Suppose $D_K(\varepsilon) \neq H, \varepsilon > 0$. Since B have gjdpod, then we have

$$\delta_{q_0}(\varepsilon) > D_{J_{B(q_0)}}(\varepsilon).$$

This implies that $D_{\overline{J_{B_n}(q_0)}}(\varepsilon) > D_{\overline{J_B(q_0)}}(\varepsilon)$, for some integer n. Thus

$$D_{J_{B_n}(q_0)}(\varepsilon) > D_{J_{B}(q_0)}(\varepsilon), \varepsilon > 0$$

This shows that $J_{B_n}(q_0)$ is a proper subset of K, contradicting the minimality of K. Hence $D_K(\varepsilon) = H, \varepsilon > 0$ Thus K consists of a single point q_0. So we have $P(q_0) = Q(q_0) = R(q_0) = S(q_0) = q_0$. Therefore q_0 is the common fixed point of P, Q, S and T □

Remark 12. *With $Q = S = T = I$ (Identity mapping), the notion of gjdpod is same as dpod and then result of Kirk (Th. A, [6]) follow.*

Remark 13. *If any two of P, Q, S, T are taken as identity maps then gjdpod reduces to jdpod and the result of Singh and Pant (Th. 4, [11]) is obtained as corollary.*

Remark 14. *It is not necessary that any continuous mapping P in Theorem 11 has dpod on X, since in such a case it might be possible to obtain a family B of continuous self mappings on X such that $B \cup P$ has a gjdpod (see, for illustration [9]).*

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